

# ON SOME SUMS INVOLVING FAREY FRACTIONS

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## 1. Introduction

It is our aim in this paper to give some refinements of theorems proved by Hall [4] and Lehner-Newman [5] on some sums involving Farey fractions on the one hand, and determine passingly the values of some series appearing as the constant terms in our asymptotic formulas on the other.

First of all we shall fix the following notations and preserve them throughout this paper.

Let  $F_n (n \in \mathbf{N})$  be the Farey series of order  $n$ , that is,  $F_n$  be the aggregate of irreducible fractions between 0 and 1 with denominators  $\leq n$ , arranged in ascending order of magnitude:  $F_n = \{h/k \mid 0 \leq h \leq k \leq n, (h, k) = 1\}$ . For any term  $h/k (< 1)$  of  $F_n$ , we denote by  $h'/k'$  its successor in  $F_n$ , and by  $Q_n$  the set of all pairs  $(k, k')$  of the denominators of thus adjacent terms:  $Q_n = \{(k, k') \mid h'/k' \text{ follows } h/k \text{ in } F_n\}$ . For any function  $f: \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{C}$  write

$$(1) \quad S_n = \sum_{(k, k') \in Q_n} f(k, k').$$

We now state briefly Lehner-Newman's [5] and Hall's [4] results which we shall need or improve in this paper, and further refer to our refinements of them.

Using the fact that  $Q_n = \{(a, b) \mid 0 < a, b \leq n, (a, b) = 1, n + 1 \leq a + b \leq 2n - 1\}$  and interpreting the sum  $S_n$  as the one taken over all coprime pairs  $(a, b)$  satisfying the conditions above, they obtained the useful formula ( $r \geq 2$ )

$$(2) \quad S_r - S_{r-1} = \sum_{\substack{k=1 \\ (k, r)=1}}^r \{f(k, r) + f(r, k) - f(k, r - k)\},$$

and as a result the sum formula (which we shall refer to as the Lehner-Newman sum formula)

$$(3) \quad S_n = f(1, 1) + \sum_{r=2}^n \sum_{\substack{k=1 \\ (k, r)=1}}^r \{f(k, r) + f(r, k) - f(k, r - k)\}.$$

The motivation of Lehner and Newman for considering these types of sums seems to be found in the attempt to determine the value of the series

$$\sum_{r=1}^{\infty} r^{-2} \sum_{\substack{k=1 \\ (k,r)=1}}^r (r+k)^{-1},$$

which has been proved by Gupta [2] and by themselves to be equal to  $3/4$ .

As an application into another direction of (3) they have proved the following

**Theorem (Lehner-Newman).** *For the function*

$$(4) \quad f(x, y) = x^a y^b \quad (0 \leq a, b \in \mathbf{R}),$$

*it holds that*

$$S_n = c_{a,b} n^{a+b+2} + O(n^{a+b+1} \log n),$$

$$\text{with } c_{a,b} = \frac{6}{\pi^2} \left\{ \frac{1}{(1+a)(1+b)} - \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)} \right\}.$$

We shall prove that the error term can be replaced by

$$(5) \quad O(n^{a+b+1} \log^{2/3} n (\log \log n)^{1+\varepsilon})$$

for any  $\varepsilon > 0$ , in the special case when  $0 \leq a, b \in \mathbf{Z}$ . Our result is not only a refinement of Hall's asymptotic formula but also a generalization since he considered the case when the exponents  $a, b$  are equal, while ours are arbitrary integers  $\geq 0$ ; and Lehner-Newman's are any real numbers  $\geq 0$ , which is also a case lying within our reach of improvement via the Euler-Maclaurin sum formula as in [5] though we shall confine ourselves to the special case as the essence is the same.

We now turn to the statement of Hall's results and our strengthening of them. He considered sums defined by ( $2 \leq m \in \mathbf{N}$ )

$$(6) \quad S_n(m) = \sum_{(k,k') \equiv 0_n} (k k')^{-m},$$

and has proved

**Theorem (Hall).** *The following asymptotic formulas are valid :*

$$(7) \quad S_n(2) = 12\pi^{-2} n^{-2} \{ \log n + \gamma + 2^{-1} - \zeta(2)^{-1} \zeta'(2) \} + O(n^{-3} \log^2 n),$$

$$(8) \quad S_n(m) = 2\zeta(m)^{-1} \zeta(m-1) n^{-m} + O(n^{-m-1} \log^\theta n), \quad (m \geq 3),$$

where  $\theta$  is 0 or 1 according as  $m \geq 4$  or  $m = 3$ .

Our refinement concerning Hall's theorem will be that the error can be reduced to

$$(9) \quad O(n^{-3} \log^{5/3} n (\log \log n)^{1+\epsilon})$$

for any  $\epsilon > 0$  in case  $m = 2$ , while we shall not be able to obtain refinements in case  $m \geq 3$ , when all that we can do will be that we are convinced that his theorem is best possible in so far as the error is estimated as  $O(n^{-m-1} \log^{\theta} n)$ . We shall, however, be able to determine the values of the series

$$(10) \quad c(m) = \sum_{r=2}^{\infty} r^{-m} \sum_{\substack{k=1 \\ (k,r)=1}}^r \sum_{l=1}^{m-1} \binom{m}{l} k^{-m+l} (r-k)^{-l},$$

which will be proved to have the value 1.

We shall primarily be concerned with the study of cases when  $m = 2$  and  $m = 3$  because the former is of more importance and other cases can be treated similarly to the latter.

In both cases of refining the error by  $\log^{1/3-\epsilon} n$  the sum

$$\sum_{d \leq x} \mu(d) d^{-1} P_1(x/d),$$

where  $P_1(\alpha)$  is the ordinary Bernoulli function defined by  $P_1(\alpha) = \{\alpha\} - 1/2$  ( $\{\alpha\}$  is the fractional part of  $\alpha : \alpha - [\alpha]$ ), will play the vital role, to which we shall essentially appeal in the latter case, and, in the former case, on the equivalent of which, namely, the sharper estimate for the average order of Euler's  $\varphi$ -function, we shall base our arguments.

The author expresses his great thanks to Professor K. Shiratani for useful advices and suggestions in treating the (partial) refinement of Lehner-Newman's theorem, whose article [8] he owes greatly some of his calculations to. He also greatly appreciates Professor S. Uchiyama for his kindly informing him of Hall's paper and for suggesting him the possibility of improving some of the theorems of Hall. He owes also a lot to his book [9] not a few of his calculations throughout the paper.

### 2. Basic formulas and estimates

We shall collect in this section some formulas and estimates which we shall require in what follows.

Defining the Bernoulli numbers  $B_m$  as coefficients appearing in the expansion :

$$\frac{t}{e^t - 1} = 1 + \sum_{m=1}^{\infty} \frac{B_m}{m!} t^m, \quad B_0 = 1,$$

we have the summation formula [1], [8]

$$(11) \quad \sum_{h=1}^{N-1} h^a = \frac{(N+B)^{a+1} - B^{a+1}}{a+1} \quad (0 \leq a \in \mathbf{Z}).$$

Let  $\sigma(r)$  denote the sum of divisors of  $r$  :  $\sigma(r) = \sum_{d|r} d$ , then it holds that [3]

$$(12) \quad S(N) = \sum_{r=1}^N \sigma(r) = 12^{-1} \pi^2 N^2 + O(N \log N).$$

Denoting by  $\varphi(r) = \sum_{\substack{k < r \\ (k,r)=1}} 1$  Euler's function, we have a sharper estimate due to Салтыков [7] and Walfisz [10] than used in [5] :

$$(13) \quad \phi(N) = \sum_{r=1}^N \varphi(r) = 3\pi^{-2} N^2 + O(N \log^{2/3} N (\log \log N)^{1+\varepsilon})$$

for any  $\varepsilon > 0$ , or, equivalently, we have the Салтыков-Walfisz estimate [7], [10] ( $x \geq 3$ )

$$(14) \quad \begin{aligned} U_x &= \sum_{d \leq x} \mu(d) d^{-1} P_1(x/d) \\ &= O(\log^{2/3} x (\log \log x)^{1+\varepsilon}) \quad (\forall \varepsilon > 0). \end{aligned}$$

As particular cases we have

$$(12') \quad S(N) = O(N^2),$$

$$(13') \quad \phi(N) = \{2 \zeta(2)\}^{-1} N^2 + O(N \log N).$$

Lastly, we need the Stirling formula valid for  $x \geq 2$  :

$$(15) \quad \log [x] ! = [x] \log [x] - [x] + 2^{-1} \log [x] + O(1).$$

### 3. Proof of the estimate (5)

Substituting (4) in (2), we obtain as in [5]

$$(16) \quad S_r - S_{r-1} = r^b \sum_{\substack{k=1 \\ (k,r)=1}}^r k^a + r^a \sum_{\substack{k=1 \\ (k,r)=1}}^r k^b - \sum_{\substack{k=1 \\ (k,r)=1}}^r k^a (r-k)^b.$$

Let us consider the sum  $S^{(1)}(r) = \sum_{\substack{k=1 \\ (k,r)=1}}^r k^a$  firstly. Observing the formula

$$(17) \quad \sum_{d|r} \mu(d) = \begin{cases} 1, & \text{if } r = 1, \\ 0, & \text{if } r \neq 1, \end{cases}$$

we argue as in [5] to obtain

$$S^{(1)}(r) = \sum_{k=1}^r \left\{ \sum_{d|(k,r)} \mu(d) \right\} k^a = \sum_{d|r} \mu(d) d^a \sum_{h=1}^{r/d} h^a.$$

Applying the sum formula (11), we have

$$\begin{aligned} \sum_{h=1}^{r/d} h^a &= \frac{1}{a+1} \sum_{k=1}^{a+1} \binom{a+1}{k} B_{a+1-k} (r/d+1)^k \\ &= \frac{1}{a+1} \sum_{k=1}^{a+1} \binom{a+1}{k} B_{a+1-k} \sum_{l=0}^k \binom{k}{l} r^a f_l(r), \end{aligned}$$

where we have put  $f_l(r) = \sum_{d|r} \mu(d)(r/d)^{l-a}$ . As for  $f_l(r)$  we see the followings :

$$(18) \quad f_{a+1}(r) = \varphi(r), f_a(r) = 0 \quad (\text{since } r \geq 2),$$

whence we are led to the expression giving the dominant term explicitly

$$\begin{aligned} S^{(1)}(r) &= \frac{r^a \varphi(r)}{a+1} \\ &+ \frac{r^a}{a+1} \left\{ \sum_{k=1}^a \binom{a+1}{k} B_{a+1-k} \sum_{l=0}^k \binom{k}{l} f_l(r) + \sum_{l=0}^a \binom{a+1}{l} f_l(r) \right\}, \end{aligned}$$

so that

$$\begin{aligned} \sum_{r=1}^n r^b S^{(1)}(r) &= \frac{1}{a+1} \sum_{r=1}^n r^{a+b} \varphi(r) \\ &+ \frac{1}{a+1} \left\{ \sum_{k=1}^a \binom{a+1}{k} B_{a+1-k} \sum_{l=0}^k \binom{k}{l} \sum_{r=1}^n r^{a+b} f_l(r) \right. \\ &\left. + \sum_{l=0}^a \binom{a+1}{l} \sum_{r=1}^n r^{a+b} f_l(r) \right\}. \end{aligned}$$

For  $l = a$ , we have seen that  $f_a(1) = 1, f_a(r) = 0 (r \neq 1)$ , so the sum corresponding to  $f_a(r)$  is  $O(1)$ . For  $l \leq a - 1, f_l(r)$  has the form  $\sum_{d|r} \mu(d) (d/r)^k = f(r)$ , say, with  $(a \geq) k \in N$ . Therefore we infer that

$$\begin{aligned} \sum_{r=1}^n r^{a+b} f(r) &\leq \sum_{r=1}^n r^{a+b-1} \sum_{d|r} (d/r)^{k-1} d \\ &\leq \sum_{r=1}^n r^{a+b-1} \sigma(r). \end{aligned}$$

We apply the partial summation to obtain

$$\begin{aligned} \sum_{r=1}^n r^{a+b-1} \sigma(r) &= (1-a-b) \int_1^n S(x) x^{a+b-2} dx + S(n) n^{a+b-1} \\ &= O(n^{a+b+1}), \end{aligned}$$

because of (12').

Next, let us calculate the sum

$$S^{(2)}(r) = \sum_{\substack{k=1 \\ (k,r)=1}}^r k^a (r-k)^b.$$

Arguing in the same way as before, we have

$$\begin{aligned} S^{(2)}(r) &= \sum_{d|r} \sum_{h=1}^{r/d} \mu(d) (hd)^a (r-hd)^b \\ &= \sum_{l=0}^b (-1)^l \binom{b}{l} r^{b-l} \sum_{d|r} \mu(d) d^{a+l} \sum_{h=1}^{r/d} h^{a+l} \\ &= \sum_{l=0}^b \frac{(-1)^l \binom{b}{l}}{a+l+1} \sum_{m=1}^{a+l+1} \binom{a+l+1}{m} B_{a+l+1-m} \sum_{k=0}^m \binom{m}{k} f_{k,l}(r), \end{aligned}$$

where  $f_{k,l}(r) = \sum_{d|r} \mu(d) (r/d)^{k-a-l}$ .

The term corresponding to  $k-a-l=1$ , i. e., to  $m=k=a+l+1$  is

given by  $r^{a+b} \varphi(r) \sum_{l=0}^b \frac{(-1)^l \binom{b}{l}}{a+l+1}$ , which proves to be the dominant term.

Applying the same reasoning as that used in evaluating  $\sum_{r=1}^n r^b S^{(1)}(r)$ , we deduce that

$$\sum_{r=1}^n S^{(2)}(r) = c'_{a,b} \sum_{r=1}^n r^{a+b} \varphi(r) + O(n^{a+b+1}),$$

with 
$$c'_{a,b} = \sum_{l=0}^b \frac{(-1)^l \binom{b}{l}}{a+l+1}.$$

Substituting these results when we sum the terms (16) over  $r=1, \dots, n$ , we have

$$(19) \quad S_n = c''_{a,b} \sum_{r=1}^n r^{a+b} \varphi(r) + O(n^{a+b+1}),$$

with 
$$c''_{a,b} = \frac{1}{a+1} + \frac{1}{b+1} - c'_{a,b}.$$

Now, let us transform the main term by using the sharper estimate

(13). Applying the partial summation again, we infer that

$$\begin{aligned} \sum_{r=1}^n r^{a+b} \varphi(r) &= - (a + b) \int_1^n \psi(x) x^{a+b-1} dx + \psi(n) n^{a+b} \\ &= - \frac{3}{\pi^2} \frac{a+b}{a+b+2} n^{a+b+2} + \frac{3}{\pi^2} n^{a+b+2} \\ &\quad + O\left(\int_1^n x^{a+b} \log^{2/3} x (\log \log x)^{1+\epsilon} dx\right). \\ &= \frac{6}{\pi^2} \frac{n^{a+b+2}}{a+b+2} + O(n^{a+b+1} \log^{2/3} n (\log \log n)^{1+\epsilon}). \end{aligned}$$

There remains to prove the coincidence of  $\frac{6}{\pi^2} c''_{a,b}$  with  $c_{a,b}$ , which is seen as follows :

$$\begin{aligned} \frac{\Gamma(a+1) \Gamma(b+1)}{\Gamma(a+b+1)} &= B(a+1, b+1) = \int_0^1 t^a (1-t)^b dt = \sum_{l=0}^b (-1)^l \binom{b}{l} \int_0^1 t^{a+l} dt \\ &= \sum_{l=0}^b (-1)^l \binom{b}{l} \frac{1}{a+l+1} = c'_{a,b}. \end{aligned}$$

Thus we have proved a refinement of the one mentioned in §1 in the special case.

**Theorem 1.** *Notations being the same as in the Lehner-Newman theorem, for integers  $a, b \geq 0$ , it holds that*

$$S_n = c_{a,b} n^{a+b+2} + O(n^{a+b+1} \log^{2/3} n (\log \log n)^{1+\epsilon}), \text{ for any } \epsilon > 0.$$

**4. Investigation of the case of  $m = 2$**

Putting  $f(x, y) = (xy)^{-2}$  in the formula (3), we have

$$\begin{aligned} S_n &= S_n(2) = \sum_{r=2}^n \sum_{\substack{k=1 \\ (k,r)=1}}^r \{2k^{-2}r^{-2} - k^{-2}(r-k)^{-2}\} \\ (20) \quad &= 1 - \sum_{r=2}^n 4r^{-3} \sum_{\substack{k=1 \\ (k,r)=1}}^r k^{-1} \\ &= 1 - 4 \sum_{r=2}^n r^{-3} S'_r, \end{aligned}$$

say, where we have used the decomposition into partial fractions twice and the fact that the condition  $(k, r) = 1$  is equivalent to  $(r - k, r) = 1$ .

We now proceed to the computation of the sum  $S'_r$ . Noting the identity (17) and applying the formula

$$(21) \quad \sum_{k=1}^N k^{-1} = \log N + \gamma + 2^{-1} N^{-1} + O(N^{-2}),$$

which can be proven by means of the Euler sum formula, to it, we deduce that

$$\begin{aligned} S'_r &= \sum_{k=1}^r k^{-1} \left\{ \sum_{d|(k,r)} \mu(d) \right\} = \sum_{d|r} \mu(d) d^{-1} \sum_{h=1}^{r/d} h^{-1} \\ &= \sum_{d|r} \mu(d) d^{-1} \log(r/d) + \gamma \sum_{d|r} \mu(d) d^{-1} \\ &\quad + 2^{-1} r^{-1} \sum_{d|r} \mu(d) + O(r^{-2} \sum_{d|r} d |\mu(d)|) \\ &= \sum_{d|r} \mu(d) d^{-1} \log(r/d) + \gamma r^{-1} \varphi(r) + O(r^{-2} \sigma(r)), \end{aligned}$$

whence it follows that

$$(22) \quad \sum_{r=2}^n S'_r r^{-3} = c(2) - S_n^{(1)} - S_n^{(2)} + O\left(\sum_{r=n+1}^{\infty} r^{-5} \sigma(r)\right),$$

where we have put  $c(2) = \sum_{r=2}^{\infty} S'_r r^{-3}$ ,  $S_n^{(1)} = \sum_{r=n+1}^{\infty} r^{-3} \sum_{d|r} \mu(d) d^{-1} \log(r/d)$ ,

and  $S_n^{(2)} = \sum_{r=n+1}^{\infty} r^{-4} \varphi(r)$ .

The  $O$ -term can readily be seen by an application of the partial summation and the trivial estimate (12') to be of order  $n^{-3}$ , which is negligible compared with our error term expected to be of order  $n^{-3} \log^{5/3} n$  ( $\log \log n$ ) $^{1+\epsilon}$  for any  $\epsilon > 0$ .

We now treat the sum  $S_n^{(2)}$  which is much easier than  $S_n^{(1)}$ . Applying the partial summation, together with the use of (13'), we have

$$(23) \quad \begin{aligned} S_n^{(2)} &= 4 \int_{n+1}^{\infty} \psi(x) x^{-5} dx - (n+1)^{-4} \psi(n) \\ &= \{2 \zeta(2)\}^{-1} n^{-2} + O(n^{-3} \log n). \end{aligned}$$

Here again the error is negligible compared with that written above.

Now we are in a position to be engaged in the consideration of the sum  $S_n^{(1)}$ . In order to transform it we shall first treat the sum  $T_r$  defined by ( $r \geq 2$ )

$$(24) \quad T_r = \sum_{k=2}^r \sum_{d|k} \mu(d) d^{-1} \log(k/d) = \sum_{a=1}^r \mu(a) d^{-1} \sum_{1 \leq r/d} \log l.$$

By the Stirling formula (15) we infer that



$$\begin{aligned} \sum_{k \leq x} \log k &= \log [x] ! \\ &= x \log x - \{x\} \log x - x + 2^{-1} \log x + O(1) \\ &= x \log x - x - P_1(x) \log x + O(1), \end{aligned}$$

so that

$$(25) \quad T_r = r (\log r) \sum_{d=1}^r \mu(d) d^{-2} - r \sum_{d=1}^r \mu(d) d^{-2} \log d - r \sum_{d=1}^r \mu(d) d^{-2} - (\log r) \sum_{d=1}^r \mu(d) d^{-1} P_1(r/d) + \sum_{d=1}^r \mu(d) d^{-1} P_1(r/d) \log d + O(\log r).$$

We now have the following formulas for finite sums appearing in the above, without the factor  $P_1(r/d)$  [9] :

$$(26) \quad \sum_{d=1}^r \mu(d) d^{-2} = \zeta(2)^{-1} + O(r^{-1}),$$

$$(27) \quad \sum_{d=1}^r \mu(d) d^{-2} \log d = \sum_{d=1}^{\infty} \mu(d) d^{-2} \log d + O(r^{-1} \log r).$$

Recalling the identity  $-\mu(d) \log d = \sum_{d'|d} \mu(d') \Lambda(dd'^{-1})$ , we see that (27) reduces to

$$(27') \quad \text{LHS of (27)} = \zeta(2)^{-2} \zeta'(2) + O(r^{-1} \log r).$$

There remains to deal with the terms containing the factor  $P_1(r/d)$ . By partial summation, together with (14), we have

$$(28) \quad \begin{aligned} \sum_{d=1}^r \mu(d) d^{-1} P_1(r/d) \log d &= \int_1^r U_x x^{-1} dx + U_r \log r \\ &= O(\log^{5/3} r (\log \log r)^{1+\epsilon}). \end{aligned}$$

Substituting (14), (26), (27') and (28) into (25), we conclude that

$$(29) \quad \begin{aligned} T_r &= \zeta(2)^{-1} r \log r - \zeta(2)^{-1} \{ \zeta(2)^{-1} \zeta'(2) + 1 \} r \\ &\quad + O(\log^{5/3} r (\log \log r)^{1+\epsilon}). \end{aligned}$$

With the aid of (29) and partial summation we can now transform  $S_n^{(1)}$  to obtain

$$(30) \quad \begin{aligned} S_n^{(1)} &= 3 \int_{n+1}^{\infty} T_x x^{-1} dx - T_{n+1} n^{-3} \\ &= 3 \zeta(2)^{-1} \{ 2^{-1} n^{-2} \log n + 4^{-1} n^{-2} - 2^{-1} n^{-2} (\zeta(2)^{-1} \zeta'(2) + 1) \} \\ &\quad - \zeta(2)^{-1} n^{-2} \{ \log n - \zeta(2)^{-1} \zeta'(2) - 1 \} + O(n^{-3} \log^{5/3} n (\log \log n)^{1+\epsilon}) \\ &= \zeta(2)^{-1} n^{-2} \{ 2^{-1} \log n - 2^{-1} \zeta(2)^{-1} \zeta'(2) + 4^{-1} \} \\ &\quad + O(n^{-3} \log^{5/3} n (\log \log n)^{1+\epsilon}). \end{aligned}$$

Substitution of (23) and (30) into (22) leads us to the conclusion that

$$(31) \quad \sum_{r=2}^n S_r r^{-3} = c(2) + \zeta(2)^{-1} n^{-2} \{-2^{-1} \log n + 2^{-1} \zeta(2)^{-1} \zeta'(2) - 4^{-1} - 2^{-1} \gamma\} \\ + O(n^{-3} \log^{5/3} n (\log \log n)^{1+\epsilon}),$$

with the constant  $c(2)$  defined in (22). By (20) and (31) we have eventually

$$(32) \quad S_n(2) = 1 - 4c(2) + 2\zeta(2)^{-1} n^{-2} \{\log n - \zeta(2)^{-1} \zeta'(2) + 2^{-1} + \gamma\} \\ + O(n^{-3} \log^{5/3} n (\log \log n)^{1+\epsilon}).$$

On the other hand we have by Hall's theorem (7) an asymptotic formula for  $S_n(2)$  without the constant factor, so, letting  $n \rightarrow \infty$ , we are convinced that  $1 - 4c(2)$  must be equal to 0. We have thus proved

**Theorem 2.** *For any  $\epsilon > 0$  and  $n \geq 2$  it holds that*

$$\sum_{(k, k') \in Q_n} (kk')^{-2} = 12\pi^{-2} n^{-2} \{\log n + \gamma + 2^{-1} - \zeta(2)^{-1} \zeta'(2)\} \\ + O(n^{-3} \log^{5/3} n (\log \log n)^{1+\epsilon}).$$

We have in passing obtained the value of a series, which is given by

**Corollary 1.** *We have*

$$\sum_{r=2}^{\infty} r^{-3} \sum_{\substack{k=1 \\ (k, r)=1}}^r k^{-1} = 4^{-1}.$$

## 5. The case of $m \geq 3$ , in particular of $m = 3$

We now turn to the investigation of the case  $m \geq 3$ . We have first of all

$$(33) \quad S_n = S_n(m) = 1 - c(m) + 2m \sum_{r=n-1}^{\infty} r^{-m+1} \sum_{\substack{k=1 \\ (k, r)=1}}^r k^{-m+1} \\ + 2m \sum_{r=n+1}^{\infty} r^{-m-1} \sum_{\substack{k=1 \\ (k, r)=1}}^r k^{-m+2} (r-k)^{-1} \\ + O\left\{ \sum_{r=n+1}^{\infty} r^{-m} \sum_{\substack{k=1 \\ (k, r)=1}}^r \sum_{l=2}^{m-2} \binom{m}{l} k^{l-m} (r-k)^{-l} \right\} \\ = 1 - c(m) + 2m S(1; m) + 2m S(2; m) \\ + O\left\{ \sum_{r=n+1}^{\infty} r^{-m} \sum_{\substack{k=1 \\ (k, r)=1}}^r k^{-a} (r-k)^{-b} \right\},$$

say.

In the error term we have summands of type  $k^{-a}(r-k)^{-b}$  with  $2 \leq a$ ,  $b \in N$ ,  $a + b = m$ , which are decomposed into partial fractions in the following manner :

$$\begin{aligned} k^{-a}(r-k)^{-b} &= r^{-2} \{k^{2-a}(r-k)^{-b} + k^{-a}(r-k)^{2-b} + 2k^{1-a}(r-k)^{1-b}\} \\ &\leq r^{-2} \{k^{2-a}(r-k)^{-b} + k^{-a}(r-k)^{2-b}\} + r^{-3} \{k^{-1} + (r-k)^{-1}\}. \end{aligned}$$

Hence, summing over  $k$  and  $r$ , we have

The error term

$$(34) \quad = O\left(\sum_{r=n+1}^{\infty} r^{-m-2} O(1) + \sum_{r=n+1}^{\infty} r^{-m-3} O(\log r)\right) = O(n^{-m-1}).$$

If  $m \geq 4$ , then we discuss in the same vein, noting the inequality  $k^{-m+2}(r-k)^{-1} \leq r^{-1} k^{2-m} + r^{-2} \{k^{-1} + (r-k)^{-1}\}$ , to have

$$(35) \quad S(2; m) = O(n^{-m-1}).$$

If that is not the case, we have

$$\begin{aligned} (36) \quad 2^{-1} S(2; 3) &= \sum_{r=n+1}^{\infty} r^{-5} \sum_{\substack{k=1 \\ (k,r)=1}}^r k^{-1} \\ &= \sum_{r=n+1}^{\infty} r^{-5} \left\{ \sum_{d|r} \mu(d) d^{-1} \log(r/d) + \gamma r^{-1} \varphi(r) \right\} + O(n^{-5}) \\ &\quad \text{(by (21))} \\ &= 5 \int_{n+1}^{\infty} T_x x^{-6} dx - T_n (n+1)^{-5} + O(n^{-4}) \quad \text{(by (13'), (25))} \\ &= 4^{-1} \zeta(2)^{-1} n^{-4} \log n + O(n^{-4}) \quad \text{(by (29)).} \end{aligned}$$

We now go back to the general case and set on the evaluation of the sum  $S(1; m)$ . As before, we have

$$\begin{aligned} (37) \quad \sum_{\substack{k=1 \\ (k,r)=1}}^r k^{-m+1} &= \sum_{d|r} \mu(d) d^{-m+1} \sum_{h=1}^{r/d} h^{-m+1} \\ &= \zeta(m-1) \sum_{d|r} \mu(d) d^{-m+1} - \sum_{d|r} \mu(d) d^{-m+1} \sum_{r/d < h} h^{-m+1}. \end{aligned}$$

Now we have by Euler's summation formula,

$$(38) \quad \sum_{r/d < h} h^{-m+1} = (m-2)^{-1} (r/d)^{-m+2} + O((r/d)^{-m+1}).$$

Combination of (37) and (39) yields

$$(39) \quad \begin{aligned} &\text{LHS of (37)} \\ &= \zeta(m-1) \sum_{d|r} \mu(d) d^{-m+1} - (m-2)^{-1} r^{-m+1} \varphi(r) + O(r^{-m+1} \tau(r)), \end{aligned}$$

where  $\tau(r) = \sum_{d|r} 1$ .

Consequently, we have

$$(40) \quad S(1; m) = \zeta(m-1) \sum_{r=n+1}^{\infty} r^{-m-1} \sum_{d|r} \mu(d) d^{-m+1} \\ + O \left\{ \sum_{r=n+1}^{\infty} r^{-2m} \varphi(r) + \sum_{r=n+1}^{\infty} r^{-2m} \tau(r) \right\}.$$

Using the estimates  $\sum_{r=n+1}^{\infty} r^{-2m} \varphi(r) = O(n^{-2m+2})$ , and  $\sum_{r=n+1}^{\infty} r^{-2m} \tau(r) = O(n^{-2m+1} \log n)$ , we see that thus appearing errors can be absorbed in  $O(n^{-m-1})$ .

We proceed as to the first summand in (40), say,  $= s(1; m)$ , as follows: Putting  $V_N = \sum_{r=1}^N \sum_{d|r} \mu(d) d^{-m+1}$ , we have successively

$$\begin{aligned} V_N &= \sum_{d=1}^N \mu(d) d^{-m+1} [N/d] \\ &= N \sum_{d=1}^N \mu(d) d^{-m} + O(1) \\ &= N \zeta(m)^{-1} + O(1), \end{aligned}$$

whence it follows that

$$(41) \quad s(1; m) = \zeta(m-1) \left\{ (m+1) \int_{n-1}^{\infty} V_x x^{-m} dx - V_n (n+1)^{-m-1} \right\} \\ = \zeta(m-1) \zeta(m)^{-1} m^{-1} n^{-m} + O(n^{-m-1}).$$

Substituting (41) into (40), we obtain the same formula for  $S(1; m)$  as the RHS of (41); hence by this formula and (33) (if  $m = 3$ , also by (36)) we conclude that

$$(42) \quad S_n(m) = 1 - c(m) + 2\zeta(m-1) \zeta(m)^{-1} n^{-m} \\ + 3\theta \zeta(2)^{-1} n^{-4} \log n + O(n^{-m-1}),$$

where  $\theta = 0$  or  $1$  according as  $m \geq 4$  or  $m = 3$ .

By the same reasoning as before we have the value of  $c(m)$ . We have thus verified

**Theorem 3.** *For any natural numbers  $m \geq 3$  and  $n \geq 2$  we have an asymptotic formula*

$$(43) \quad \sum_{(k, k') \in Q_n} (kk')^{-m} = 2 \zeta(m)^{-1} \zeta(m-1) n^{-m} \\ + 3\theta \zeta(2)^{-1} n^{-4} \log n + O(n^{-m-1}),$$

where  $\theta$  is defined as above.

Further we have determined as a by-product the values of certain series:

**Corollary 1'.** We have for  $m \geq 2$  and  $n \geq 2$

$$\sum_{r=2}^{\infty} r^{-m} \sum_{\substack{k=1 \\ (k, r)=1}}^r \sum_{l=1}^{m-1} \binom{m}{l} k^{-m+l} (r-k)^{-l} = 1.$$

**Remark 1.** We could obtain a more detailed asymptotic formula than in Theorem 3 if only we took the terms appearing in the Euler-Maclaurin sum formula to the higher order of derivatives instead of having them absorbed in the error as in (38). But, if we do so, we cannot remove those terms as the presence of the term  $n^{-4} \log n$  in (43) shows, so that Hall's estimate is best possible in the sense mentioned in § 1.

**Remark 2.** As is seen by the procedure employed in evaluating (33), we can also treat the sum of type  $\sum_{(k, k') \in Q_n} k^{-a} k'^{-b}$ , with  $2 \leq a, b \in N$ ; we have, however, restricted ourselves to the case  $a = b$  for the sake of simplicity.

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(Received February 27, 1978)