DIRECT SUMS OF NONSINGULAR INDECOMPOSABLE INJECTIVE MODULES

MAMORU KUTAMI and KIYOICHI OSHIRO

Throughout this paper R is an associative ring with identity and all R-modules are unitary R-modules.

The main purpose of this paper is to study direct sums of nonsingular indecomposable injective R-modules. Let Q be the maximal ring of left quotients of $R/G(R)$, where $G(R)$ is the Goldie torsion submodule of R. It is easy to check that a nonsingular R-module is annihilated by $G(R)$, and it is also nonsingular as an $R/G(R)$-module. [2, Theorem 2.2] says that every nonsingular injective R-module has a left Q-module structure compatible with the $R/G(R)$-module structure. The following easy lemma is frequently used in this paper: A nonsingular injective R-module is indecomposable as an R-module iff it is simple as a Q-module.

In Theorem 2.4, we give a simple proof of the following result ([3], [8]): Every complete decomposition of any completely decomposable nonsingular R-module complements direct summands. Further the following more general result is shown in Theorem 2.5: Let M be a completely decomposable R-module, and N a direct summand of M. If N is nonsingular, then N is quasi-injective.

In Theorem 3.2, we prove that a nonsingular locally injective R-module M is completely decomposable iff the ascending chain condition holds for the annihilator left ideals of elements of M. This result yields several known characterizations and some new ones for Q to be a semisimple artinian ring.

Finally it is shown in Corollary 4.2 that the ascending chain condition holds for irreducible left ideals of R iff every locally injective submodule of any completely decomposable R-module M is a direct summand of M.

1. Preliminaries. Following Yamagata [6] we call an R-module M locally injective if for each x in M there exists a submodule N which contains x and is isomorphic to $E_R(Rx)$, the injective hull of Rx.

Let $M=\sum_{i\in I}\oplus M_i$ be a direct sum of R-modules $\{M_i\}_{i\in I}$, and N a direct summand of M. N has *the exchange property in* $M=\sum_{i\in I}\oplus M_i$ if there exists a submodule M_i' of M_i for each i such that $M=N\oplus(\sum_{i\in I}\oplus M_i')$.

An R-module is said to be completely decomposable if it is written as a direct sum of indecomposable injective submodules. We call such a decomposition complete. A completely decomposable module M complements direct summands if every direct summand of M has the exchange property in any complete decomposition of M.

For a subset N of an R-module, we set $\mathscr{A}(N)=\{(0:a)_R \mid a\in N\}$, where $(0:a)_R=\{r\in R \mid ra=0\}$. Let N and M be R-modules with $N\subseteqq M$. We use $N\subseteqq_e M$ to denote that M is an essential extension of N. For a given R-module M, we denote its singular submodule and its Goldie torsion submodule by $Z_R(M)$ and $G(M)$, respectively. Note that $Z_R(M/Z_R(M))=G(M)/Z_R(M)$, $G(R)$ is a two-sided ideal of R and $R/G(R)$ is a left nonsingular ring.

Now it is easy to verify that if M is a nonsingular R-module, then

(1) $G(R)M=0$ and therefore M becomes a left $R/G(R)$-module by a usual way,

(2) M is also nonsingular as an $R/G(R)$-module and

(3) M is injective as an R-module iff it is injective as an $R/G(R)$-module.

Thus [2, Theorem 2. 2] says that a nonsingular injective R-module has a unique Q-module structure compatible with the $R/G(R)$-module structure, where Q is the maximal ring of left quotients of $R/G(R)$. Therefore if M is a nonsingular R-module, then we have $M\subseteqq QM\subseteqq E_R(M)$.

It is well known (e. g. [2, Theorem 3. 12]) that every finitely generated nonsingular Q-module is both projective and injective. Using injectivity of every cyclic nonsingular Q-module, we can easily show the following result which is a key lemma in this paper.

Lemma 1. 1. *Let M be a nonsingular injective R-module. Then M is indecomposable as an R-module iff M is simple as a Q-module.*

A left ideal I of R is said to be a closed left ideal provided it has no proper essential extension in R. If I is a closed left ideal of R containing $G(R)$, then $I/G(R)$ is clearly a closed left ideal of $R/G(R)$. The converse of this fact is also true:

Lemma 1. 2. *If I is a left ideal of R containing $G(R)$ such that $I/G(R)$ is a closed left ideal of $R/G(R)$, then I is a closed left ideal of R.*

Proof. Let J be a left ideal of R with $I\subseteqq_e J$. Then J/I is singular as an R-module and so is $(J/G(R))/(I/G(R))$. Hence by [2, Proposition 1. 28], $(J/G(R))/(I/G(R))$ is a singular $R/G(R)$-module. However $J/G(R)$

is nonsingular as an $R/G(R)$-module. It follows that $I/G(R) \subseteq_e J/G(R)$, whence we get $I=J$ as required.

2. Indecomposable nonsingular injective modules. In this section we also use Q to stand for the maximal ring of left quotients of $R/G(R)$ as before.

By examining the proof of [2, Proposition 6. 18] we have

Lemma 2. 1. *Let M be an R-module which is a direct sum of R-modules $\{M_i\}_{i\in I}$, and N a direct summand of M. Then for each i, there is a submodule M_i' of M_i such that $N\cap(\sum_{i\in I}\oplus M_i')=0$ and $N\oplus(\sum_{i\in I}\oplus M_i') \subseteq_e M$.*

Lemma 2. 2. *If M is an R-module such that $G(M)$ is a direct summand of M, then, for any direct summand N of M, $G(N)$ is a direct summand of N.*

Proof. Let $M=G(M)\oplus F=N\oplus L$. Then $G(M)=G(N)\oplus G(L)$, which implies that $N=G(N)\oplus(N\cap(F\oplus G(L))$.

Lemma 2. 3. *Let $M=G\oplus F$ for some submodules G and F. Then any direct summand N of M containing G is the form $N=G\oplus F'$ for some direct summand F' of F. (In fact $F'=N\cap F$).*

Proof. Let $M=G\oplus F=N\oplus L$. Then $N=G\oplus(N\cap F)$ and so $M=G\oplus(N\cap F)\oplus L$. From this we also have $F=(N\cap F)\oplus(F\cap(G\oplus L))$. Hence $N\cap F$ is a direct summand of F.

Theorem 2. 4. *For a completely decomposable nonsingular R-module M, every complete decomposition of M complements direct summands.*

Proof. Let $M=\sum_{i\in L}\oplus M_i$ be any complete decomposition of M, N any direct summand of M, and $M=N\oplus L$. Then $QM=QN\oplus QL$ because $E_R(N)\cap E_R(L)=0$. However, $QM=\sum_{i\in I}\oplus QM_i=\sum_{i\in I}\oplus M_i=M$ and so we have $N=QN$. Since each M_i is simple as a Q-module by Lemma 1. 1, M is completely reducible as a Q-module. Hence there exists a subset J of I such that $M=N\oplus(\sum_{i\in J}\oplus M_j)$.

Remark. This theorem is also given as a consequence of Harada

[3, 6. 5. 1] or Yamagata [8, Corollary 4. 3]. However their results can be obtained from the following theorem as its corollaries.

Theorem 2.5. *Let M be a completely decomposable R-module, and N a direct summand of M. If N is nonsingular, then N is quasi-injective.*

Proof. Since any indecomposable injective R-module is either Goldie torsion or nonsingular, we see from Lemmas 1. 1, 2. 2 and Theorem 2. 4 that N is a direct sum of indecomposable injective modules and is completely reducible as a Q-module. Now, to show the assertion, let N' be an R-submodule of N, and f an R-homomorphism $N' \longrightarrow N$. Then the mapping $f' : QN' \longrightarrow QN = N$ given by $\sum_i q_i n_i \longrightarrow \sum_i q_i f(n_i)$, $q_i \in Q$, $n_i \in N'$, is a Q-homomorphism and QN' is a direct summand of N. Therefore we can extend f to an R-homomorphism $N \longrightarrow N$ as required.

Theorem 2.6. *Let M be an R-module with $M = G(M) \oplus (\sum_{i \in I} \oplus M_i)$ where each M_i is indecomposable injective. Then, for a direct summand N of M, the following statements hold:*

(1) *There exists a subset J of I and submodules M'_j $(j \in J)$ of M such that $M_j \cong M'_j$ $(j \in J)$* and $N = G(N) \oplus (\sum_{j \in J} \oplus M'_j)$.

(2) *There exists a subset K of I for which*

$$M = G(M) \oplus (\sum_{j \in J} \oplus M'_j) \oplus (\sum_{k \in K} \oplus M_k).$$

Proof. (1) Let $M = N \oplus N'$. By Lemma 2. 2, $N = G(N) \oplus H$ and $N' = G(N') \oplus H'$ for some submodules $H \subseteq N$ and $H' \subseteq N'$. Since $M = G(M) \oplus H \oplus H'$, we have $\sum_{i \in I} \oplus M \cong H \oplus H'$. Applying Theorem 2. 4, there exists a subset J of I and submodules M'_j of H with $M'_j \cong M_j (j \in J)$ such that $H = \sum_{j \in J} \oplus M'_j$. Consequently we have $N = G(N) \oplus (\sum_{j \in J} \oplus M'_j)$.

(2) Since $M = G(M) \oplus H \oplus H'$, by Lemma 2. 3 $G(M) \oplus H = G(M) \oplus F$ for some direct summand F of $\sum_{i \in I} \oplus M_i$. According to Theorem 2. 4, there exists a subset K of I for which $\sum_{i \in I} \oplus M_i = F \oplus (\sum_{k \in K} \oplus M_k)$. Thus we get $M = G(M) \oplus H \oplus (\sum_{k \in K} \oplus M_K)$.

Remark. In order to study the problem when complete decompositions of modules complement direct summands, Theorem 2. 6 essentially reduces it to the case of Goldie torsion modules (cf. [6]).

3. Nonsingular locally injective modules. Q also denotes the maximal ring of left quotients of $R/G(R)$.

The following lemma is easily verified, so we omit its proof.

Lemma 3.1. *If M is a nonsingular R-module, then the following conditions are equivalent*:

(a) *M is locally injective as an R-module.*

(b) *M is locally injective as an $R/G(R)$-module.*

(c) $M = QM$.

Theorem 3.2. *If M is a locally injective R-module, then the following conditions are equivalent*:

(a) *M does not contain proper essential locally injective submodules.*

(b) *Any locally injective submodule of M is a direct summand of M.*

(c) *M does not contain proper essential submodules which are direct sums of injective modules.*

(d) *Any submodule of M which is a direct sum of injective submodules is a direct summand of M.*

Furthermore, in case M is nonsingular, the following conditions are also equivalent to each of the above conditions (a) *through* (d).

(e) *M is a direct sum of indecomposable injective modules.*

(f) *The ascending chain condition holds for elements in $\mathscr{A}(M)$.*

Proof. (a)$\Longrightarrow$(b). Let N be a locally injective submodule of M. We can choose a maximal independent family $\{M_i\}_{i\in I}$ of injective submodules of M such that $N\cap(\sum_{i\in I}\oplus M_i)=0$. Since $N\oplus(\sum_{i\in I}\oplus M_i)$ is an essential locally injective submodule of M, we have $M=N\oplus(\sum_{i\in I}\oplus M_i)$.

(b)$\Longrightarrow$(d) and (d)$\Longrightarrow$(c) are trivial.

(c)$\Longrightarrow$(a). Let N be a locally injective essential submodule of M. Then we can find injective submodules $\{N_i\}_{i\in I}$ of N such that $\sum_{i\in I}\oplus N_i\subseteq_e N$. By (c) $\sum_{i\in I}\oplus N_i = M$ and so $M=N$.

(f)$\Longrightarrow$(a). Let N be a locally injective essential submodule of M, and suppose that $M\neq N$. Then, by (f), $\mathscr{A}(M-N)$ has a maximal member $(0:e)_R$ under inclusion. Since $N\subseteq_e M$, $Re\cap N\neq 0$ and by the local injectivity of N there exists an injective submodule F of N such that $0\neq Re\cap F$. Hence $M=F\oplus L$ for some submodule L. We express e as $e=f+k$, $f\in F$, $k\in L$. Then $(0:e)_R\subseteq(0:k)_R$. But k can not be in N and so we have $(0:e)_R=(0:k)_R$. Since $F\cap Re\neq 0$, there exists r in R such that $0\neq re\in F$. Inasmuch as $0=(rf-re)+$

$rk \in F \oplus L$, we see that $rk = 0$ and $re = 0$, a contradiction. Thus $M = N$.

(d) $\Longrightarrow$ (f). Suppose that there exists an infinite subset $\{x_i \mid i = 1, 2, \cdots\}$ of M such that $(0 : x_1)_R \subsetneqq (0 : x_2)_R \subsetneqq \cdots$. Then for each i the canonical map $Rx_i \longrightarrow Rx_{i+1}$ induces the canonical map $Qx_i \longrightarrow Qx_{i+1}$ which is not an isomorphism. Since each Qx_{i+1} is projective as a Q-module, the sequence $Qx_i \longrightarrow Qx_{i+1} \longrightarrow 0$ splits. As a result, there exists an infinite subset $\{y_i \mid i = 1, 2, \cdots\}$ of non-zero elements in Qx_1 such that $\{Qy_i \mid i = 1, 2, \cdots\}$ is an independent family. By (d), $\sum_{i=1}^{\infty} \oplus Qy_i$ is a direct summand of Qx_1. But this is impossible. Therefore the ascending chain condition must hold for elements in $\mathscr{A}(M)$.

(f) $\Longrightarrow$ (e). First we claim that M has the essential socle as a Q-module. To see this, let $0 \neq x \in M$. Then $Qx \cong Qe_1$ for some idempotent e_1 in Q since Qx is projective as a Q-module. We may show that Qe_1 contains a simple Q-module. If Qe_1 is not simple, then there exist non-zero idempotents e_2, f_2 in Qe_1 such that $e_1 = e_2 + f_2$, $e_2 f_2 = f_2 e_2 = 0$ and $Qe_1 = Qe_2 \oplus Qf_2$. Since $(0 : e_1)_Q \subsetneqq (0 : e_2)_Q$, it follows that $(0 : e_1)_R \subsetneqq (0 : e_2)_R$. If Qe_2 is not simple, similarly we can take a non-zero idempotent e_3 in Qe_2 such that $(0 : e_2)_R \subsetneqq (0 : e_3)_R$. By repeating this argument, we obtain a subset $\{e_1, e_2, \cdots\}$ of Qe_1 such that $(0 : e_1)_R \subsetneqq (0 : e_2) \subsetneqq \cdots$. But by (f) this sequence must terminate. Thus Qe_1 contains a simple Q-submodule as claimed.

As we have shown above, (f) implies (a) and from (a) we see that M just coincides with its socle, because the socle of M is a locally injective submodule of M. We write $M = \sum_{i \in I} \oplus Qx_i$ as a direct sum of simple Q-submodules. Then $Qx_i = E_R(Rx_i)$ and so by Lemma 1. 1 it is indecomposable. Consequently M is a direct sum of indecomposable injective R-submodules.

(e) $\Longrightarrow$ (f). Assume that the ascending chain condition does not hold for elements in $\mathscr{A}(M)$, and let $\{x_i \mid i = 1, 2, \cdots\}$ be an infinite subset of M such that $(0 : x_1)_R \subsetneqq (0 : x_2)_R \subsetneqq \cdots$. Inasmuch as $(0 : x_1)_Q \subsetneqq (0 : x_2)_Q \subsetneqq \cdots$, we can obtain an independent infinite family of non-zero Q-submodules of Qx_1. However Qx_1 is completely reducible as a Q-module and so it does not contain such an infinite family of non-zero submodules. Therefore (f) holds.

Using Theorem 3. 2, we show the following result.

Theorem 3.3. *The following conditions are equivalent*:

(a) Q *is a semisimple artinian ring.*

(b) *The ascending chain condition holds for elements in* $\mathscr{A}(Q)$.

(c) *For every nonsingular injective R module M, the ascending chain condition holds for elements in* $\mathscr{A}(M)$.

(d) *For every nonsingular locally injective R-module M, the ascending chain condition holds for elements in* $\mathscr{A}(M)$.

(e) *Every nonsingular injective R-module is a direct sum of indecomposable injective submodules.*

(f) *Every nonsingular locally injective R-module is a direct sum of indecomposable injective submodules.*

(g) *Every nonsingular locally injective R-module is injective.*

(h) *Any direct sum of nonsingular injective R-modules is also injective.*

(i) *Every nonsingular locally injective R-module is a direct sum of injective submodules.*

Proof. Since (e) and (f) in Theorem 3.2 are equivalent, we obtain the equivalences (a)$\Longleftrightarrow$(b), (c)$\Longleftrightarrow$(e) and (d)$\Longleftrightarrow$(f).

(a)$\Longrightarrow$(f). Any nonsingular locally injective R-module M is a Q-module by Lemma 3.1 and so by (a) it is a completely reducible Q-module. As we have shown in the proof of Theorem 3.2, M is a direct sum of indecomposable injective R-modules.

The implications (d)$\Longrightarrow$(c)$\Longrightarrow$(b), (f)$\Longrightarrow$(i) and (a)$\Longrightarrow$(h)$\Longrightarrow$(g) are trivial.

(g)$\Longrightarrow$(a). Let A be a left ideal of Q. Since A is a locally injective nonsingular R-module, A is injective as an R-module and so is as a Q-module. Hence A is a direct summand of Q as a Q-module.

(i)$\Longrightarrow$(a). Since every left ideal of Q is a locally injective nonsingular R-module, we see from (i) that every left ideal of Q is a direct sum of principal left ideals. Therefore every left ideal of Q is projective as a Q-module, whence Q is a left hereditary ring. Since Q is left self-injective, this yields that Q is a semisimple artinian ring (see [4]).

Remarks. (1) The equivalence of (a), (e) and (h) was shown by Teply [5, Theorem 1.2].

(2) It is easy to see that $\mathscr{A}(M) \subseteq \mathscr{A}(Q)$ for every nonsingular R-module M.

(3) $\mathscr{A}(Q)$ coincides with the family of all closed left ideals of R containing $G(R)$. (Therefore (a)$\Longleftrightarrow$(b) in Theorem 3.3 is nothing but a well-known result in case $Z_R(R)=0$.) To see this, let $x \in Q$. Since Q is regular, $xx'x = x$ for some x' in Q. Putting $e = xx'$, we see that

$e=e^2$, $(0: x)_R=(0: e)_R$ and $(0: e)_R/G(R)=Q(1-e)\cap(R/G(R))$. Since $Q(1-e)\cap(R/G(R))$ is a closed left ideal of $R/G(R)$ (see [1, p. 112]), by Lemma 1.2 $(0:e)_R$ is a closed left ideal of R. Conversely let I be a closed left ideal of R containing $G(R)$. Then $I/G(R)$ is a closed left ideal of $R/G(R)$. Hence $I/G(R)=Qe\cap(R/G(R))$ for some idempotent e in Q, and thus $I=(0:1-e)_R$.

4. Indecomposable injective modules.

Theorem 4.1. *Let M be a direct sum of injective R-modules $\{M_i\}_{i\in I}$, and consider the following conditions:*

(a) *The ascending chain condition holds for elements in $\cup_{i\in I}\mathscr{A}(M_i)$.*

(b) *M satisfies any of the conditions* (a) *through* (d) *in Theorem* 3.2.

(c) *Every direct summand of M has the exchange property in the decomposition $M=\sum_{i\in I}\oplus M_i$.*

(d) *Every direct summand of M is a direct sum of injective modules. Then we have* (a)$\Longrightarrow$(b)$\Longrightarrow$(c)$\Longrightarrow$(d).

Proof. (a)$\Longrightarrow$(b). By an argument similar to that used in the proof of (f)$\Longrightarrow$(a) in Theorem 3.2, we shall show that M does not contain proper essential locally injective submodules. Assume that M contains a proper essential locally injective submodule N. Since $(M-N)\cap(\cup_{i\in I}M_i)$ is not empty, $\mathscr{A}((M-N)\cap(\cup_{i\in I}M_i))$ has a maximal member, say $(0:x)_R$, under inclusion. Since $N\subseteq_e M$ and N is locally injective, there exists an injective submodule F of N such that $F\cap Rx\neq 0$. Since F is injective, as is well-known (e.g. [2, Proposition 6.18]), F has the exchange property in $M=\sum_{i\in I}\oplus M_i$. Hence there are submodules $M_i'\subseteq M_i$ for all i such that $M=F\oplus(\sum_{i\in I}\oplus M_i)$. Let $x=f+x_{i_1}+\cdots+x_{i_n}$, $f\in F$, $x_{i_j}\in M_{i_j}$, $j=1,2,\cdots,n$. Take any $rx(\neq 0)\in F\cap Rx$. Then $0=(rf-rx)+rx_{i_1}+\cdots+rx_{i_n}$ and so $rx_{i_1}=\cdots=rx_{i_n}=0$. This implies that $(0:x)_R\subsetneqq(0:x_{i_j})$ for all j and hence by the maximality of $(0:x)_R$ we must have $x_{i_j}\in N$ for all j. However this contradicts the fact that $x\notin N$. Thus M does not contain proper essential locally injective submodules.

(b)$\Longrightarrow$(c). If N is a direct summand of M, then by Lemma 2.1 there are submodules M_i' of M_i for all i such that $N\oplus(\sum_{i\in I}\oplus E(M_i'))\subseteq_e M$. Hence, by Theorem 3.2(a), we have $M=N\oplus(\sum_{i\in I}\oplus E(M_i'))$.

(c)$\Longrightarrow$(d). This is trivial.

Finally we show the following result.

Corollary 4. 2. *The following conditions are equivalent:*

(a) *The ascending chain condition holds for irreducible left ideals of R.*

(b) *Every locally injective submodule of any completely decomposable R-module is a direct summand.*

Proof. In [7 ; 8], Yamagata gave several conditions which are equivalent to (a). One of these was

(*) Any complete decomposition of any completely decomposable R-module complements direct summands.

However both (a)$\Longrightarrow$(b) and (b)$\Longrightarrow$(*) follow from Theorem 4. 1, and so (a) and (b) are equivalent.

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REFERENCES

[1] C. FAITH: Lectures on Injective Modules and Quotient Rings, Lecture Notes in Math. **49**, Springer-Verlag, Berlin, 1967.

[2] K. R. GOODEARL: Ring Theory: Nonsingular Rings and Modules, Marcel Dekker, New York, 1976.

[3] M. HARADA: Applications of factor categories to completely indecomposable modules, Publ. Dep. Math. Lyon **11** (1974), 19—104.

[4] B. OSOFSKY: Rings all of whose finitely generated modules are injective, Pacific J. Math. **14** (1964), 645—650.

[5] M. L. TEPLY: Torsionfree injective modules, Pacific J. Math. **28** (1969), 441—453.

[6] K. YAMAGATA: Nonsigular rings and Matlis' problem, Sci. Rep. T. K. D. (**A**), **11** (1971), 114—121.

[7] K. YAMAGATA: A note on a problem of Matlis, Proc. Japan Acad. **49** (1973), 145—147.

[8] K. YAMAGATA: The exchange property and direct sums of indecompcsable injective modules, Pacific J. Math. **55** (1974), 301—317.

DEPARTMENT OF MATHEMATICS
YAMAGUCHI UNIVERSITY

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