ON THE p-RATIONALITY OF LIFTED CHARACTERS

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1. Introduction. Let G be a finite group, and H a normal subgroup of G. If χ is an irreducible complex character of H which is G-stable, then there exist a central extension

$$1 \longrightarrow Z \longrightarrow \hat{G} \stackrel{f}{\longrightarrow} G \longrightarrow 1$$

and an irreducible character $\hat{\chi}$ of $\hat{G}$ satisfying the following conditions:

(1) Z is a cyclic group whose order divides $|H|^2$.

(2) There is a normal subgroup $\hat{H}$ of $\hat{G}$ such that $f^{-1}(H)=\hat{H}Z=\hat{H}\times Z$ and $\hat{\chi}(\hat{h})=\chi(f(\hat{h}))$ for all $\hat{h}\in\hat{H}$.

The above fact appears essentially as the first step in the proof of Theorem 2D of Fong [3]. We now assume that H is a p'-subgroup for some prime number p and let $|G||H|^2=p^a m$, $(p, m)=1$.

The objective of this note is to prove the following

Theorem. *Under the notation and the assumption as above, we can choose* $\hat{\chi}$ *so that it is realizable over the field of m-th roots of unity.*

This is included in the statement (1.1) (ii) in Chap. X of Feit [2], whose proof seems to us, however, obscure. Indeed, we guess from the context that "*a suitable scalar*" on the fourth paragraph of p. 523 of [2] might be chosen from the field of $|H|^2$-th roots of unity, which does not seem so obvious. Our argument below will ensure that the scalar may be chosen, at least, from the field of m-th roots of unity. In the final step of it, we need some number theoretical facts.

One of the reasons we are interested in the above result is that it gives an easy proof to one of the statements in Theorem 1.2 of Issacs [5]. Namely, every Brauer character of a p-solvable group is lifted to a p-rational ordinary one, which of course strengthens Fong-Swan Theorem (see Chap X, (2.1) of [2]).

2. Proof of Theorem. Let K be the field of m-th roots of unity. In it, we fix a prime divisor $\mathfrak{p}$ of p and denote by $\mathfrak{o}$ the ring of $\mathfrak{p}$-integers.

As H is a p'-group, we can construct a projective representation $S: G \longrightarrow \mathrm{GL}(n, \mathfrak{o})$ satisfying the following conditions:

(1) S_H is a representation affording χ.

(2) $S(h)S(g) = S(hg)$ for all $h \in H$ and $g \in G$.

Let α be the 2-cocycle of G determined by S. From the above, α may be regarded as a 2-cocycle of $\bar{G} = G/H$. Let $\{1 = g_1, \cdots, g_r\}$ be a transversal of H in G. Let a_i be one of the n-th roots of det $S(g_i)$ for each i $(1 \leq i \leq r)$.

We define $c: \bar{G} \longrightarrow K(a_1, \cdots, a_r)$ by $c(\bar{x}) = a_i^{-1}$ if $\bar{x} = \bar{g}_i$, and the 2-cocycle α' of G by $\alpha'(x, y) = c(\bar{x})c(\bar{y})c(\overline{xy})^{-1}\alpha(x, y)$ for $x, y \in G$, where $\bar{g}$ denotes the image of g by the natural map $G \longrightarrow \bar{G}$.

Then, α' is the 2-cocycle of G determined by the projective representation S' over $K(a_1, \cdots, a_r)$, which is defined as

$$S'(g) = a_i^{-1}S(g) \text{ if } g \in G \text{ and } \bar{g} = \bar{g}_i.$$

We can easily see that $\alpha'(x, y)$ is a $|H|^2$-th root of unity and $\alpha'(x, h) = a'(h, y) = 1$ for all $x, y \in G$ and $h \in H$ (to be precise, $\alpha'(x, y)$ is a $\chi(1)d$-th root of unity, where d is the order of the group $\{\det S(h) \mid h \in H\}$).

Following to the method of Schur, we construct a central extension

$$1 \longrightarrow Z \longrightarrow \hat{G} \xrightarrow{f} G \longrightarrow 1$$

in order to lift S' to a representation of $\hat{G}$ (see §51 of [1]). Here, we let Z be the multiplicative group generated by $\{\alpha'(x, y) \mid x, y \in G\}$. From the construction, we have that $|Z|$ divides $|H|^2$ and there exists a transversal $J = \{\hat{x} \mid x \in \hat{G}\}$ of Z in G such that $f(\hat{x}) = x$ an $\hat{x}\hat{y} = \alpha'(x, y)\widehat{xy}$ for all $x, y \in G$.

If we put $T(z\hat{x}) = zS'(x)$ for $z \in Z$ and $\hat{x} \in J$, then T is a representation of $\hat{G}$. Let $\hat{\chi}$ be the character afforded by T. Our purpose is to prove that $\hat{\chi}$ is realizable over K. For this, it is sufficient to show that each a_i lies in K.

Let L be the field of $p^a m$-th roots of unity. Clearly, L is a splitting field for $\hat{G}$, so that $\hat{\chi}(\hat{g})$ lies in L for all $\hat{g} \in \hat{G}$. If $\hat{x} \in J$ and $f(\hat{x}) = x \in Hg_i$, then $\hat{\chi}(\hat{x}) = a_i^{-1} \mathrm{tr}\, S(x)$. Here we note that the set $\{S(hg_i) = S(h)S(g_i) \mid h \in H\}$ spans the full matrix ring $M(n, K)$ over K, since $\{S(h) \mid h \in H\}$ does it. In particular, there exists some $x \in Hg_i$ such that tr $S(x) \neq 0$ and then a_i must belong to L from the above.

On the other hand, we have that $L \cap K(a_i) = K$. In fact, $\mathfrak{p}$ is completely ramified in L/K, while it is unramified in $K(a_i)/K$, as $X^n - a_i^n$ is separable mod $\mathfrak{p}$ (remember that $a_i^n = \det S(g_i)$ is a unit in $\mathfrak{o}$). Thus we conclude that $a_i \in K$ for all i $(1 \leq i \leq r)$ and the proof is complete.

We conclude this note with the next

Remark. *Let G be a p-solvable group of order $|G| = p^a h$, $(p, h) = 1$. Then every p-Brauer character is lifted to an ordinary one which is realizable over the field of h-th roots of unity.*

To see this, it is sufficient to show that every p-rational character of G is realizable over the field of h-th roots of unity. But this is an easy consequence of Corollary 2E of Fong [4].

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