ON THE p'-SECTION SUM IN A FINITE GROUP RING

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Let G be a finite group, and p a prime number. Let $C_1=\{1\}, C_2, \cdots, C_r$ be all the p-regular classes of G. We denote by S_i the p'-section containing C_i, namely $S_i=\{\sigma\in G \mid \sigma'\in C_i\}$, where σ' denotes the p'-component of the element σ of G $(i=1, 2, \cdots, r)$. In particular, S_1 is the set consisting of all the p-elements of G. Let k be a field of characteristic p, and J the Jacobson radical of the group ring $k[G]$.

Recently, the author is informed by T. Okuyama that in 1955 R. Brauer stated the following without proof.

Theorem (Brauer [2]). *Let $\hat{S}_j=\sum\sigma$, where the summation is taken over all $\sigma\in S_j$. Then there holds that $\cap_{j=1}^{r}(0:\hat{S}_j)=J$.*

We recall here the previous paper [6]. There, we showed that $(0:\hat{S}_1)\supset J$, while the inclusion $\cap_{j=1}^{r}(0:\hat{S}_j)\subset J$ is an easy deducement of Proposition 1 in [6]. So that, we should like to provide a new proof of the above Theorem along with the arguments used in the proofs of these results.

As a consequence of the Theorem, we have that if e is a primitive idempotent of $k[G]$, then $k[G]\hat{S}_je$ is the socle of $k[G]e$, provided $\hat{S}_je\neq 0$. The condition will be described by the value on C_j of the Brauer character afforded by $k[G]e$. On the other hand, Okuyama's proof of the Theorem is different from ours. There, the condition $\hat{S}_je\neq 0$ is discussed in connection with the coefficients a_τ's which appear in the expression $e=\sum_{\tau\in G}a_\tau\tau$, $a_\tau\in k$. We refer to it at the end of this paper.

In the proof of the Theorem, from the beginning, we may assume that k is a splitting field for G. In addition to the notations introduced above, we shall use the following.

Let $\mathfrak{p}$ be a prime divisor of p in an algebraic number field containing the $|G|$-th roots of unity, and ν the exponential valuation associated with p multiplied by a factor to make $\nu(p)=1$. We assume henceforth k is the residue class field of ν. If α is a $\mathfrak{p}$-integer, then $\bar{\alpha}$ denotes the residue class of α in k. Let $\{\eta_1, \eta_2, \cdots, \eta_r\}$ and $\{\phi_1, \phi_2, \cdots, \phi_r\}$ be the set of the principal indecomposable Brauer characters of G and the set of the irreducible Brauer characters of G respectively, in which we arrange the indices so that $(\eta_j, \phi_j)=\delta_{ij}$ for all i, j $(i, j=1, 2, \cdots, r)$. The k-algebra $k[G]/J$ is isomorphic to a direct sum of full matrix algebras over k; $k[G]/J\simeq\sum_{i=1}^{r} M(n_i, k)$. We assume that under the isomorphism the simple component corresponding to the irreducible k-character $\bar{\phi}_i$ is mapped onto $M(n_i, k)$ and so $n_i=\phi_i(1)$. If I is a subset of $k[G]$, then $(0:I)$ denotes the set of the right annihilators of I in $k[G]$. Finally, we put $\lambda(\sum_{\sigma\in G} a_\sigma\sigma)=a_1$, where $a_\sigma\in k$ and 1 denotes the identity of G.

Now we enter into the proof of the Theorem. Let S_j be a fixed p'-section and $\sigma\in C_j$. There holds that $\nu(\eta_i(\sigma))\geq\nu(|C_G(\sigma)|)$ for all η_i ([3], (84.14)). After a suitable change of indices if necessary, we may assume that the first $\eta_1, \eta_2, \cdots, \eta_t$ are all that enjoy the equality sign in the above. We put $\eta_i'(\sigma)=\eta_i(\sigma)/\mathfrak{p}^n$, where $|C_G(\sigma)|=p^n h$, $(p, h)=1$.

From the orthogonality relation

$$\sum_{i=1}^{r}\eta_i(\sigma)\phi_i(\tau^{-1})=\begin{cases}|C_G(\sigma)| & \text{if } \tau \text{ is conjugate to } \sigma\\ 0 & \text{otherwise}\end{cases}$$

and that $\overline{\phi_i(\tau)}=\overline{\phi_i(\tau')}$ for any element τ of G, we get (reducing mod $\mathfrak{p}$)

$$(*)\qquad \sum_{i=1}^{t}\overline{\eta'(\sigma)}\,\overline{\phi_i(\tau^{-1})}=\begin{cases}\bar{h} & \text{if } \tau\in S_j\\ 0 & \text{otherwise.}\end{cases}$$

Let $U_j=\sum_e k[G]e+J$, where e runs over the primitive idempotents such that $k[G]e$ affords a Brauer character η_s with $s>t$. Then U_j is a two sided ideal of $k[G]$. In fact, it is the inverse image of $\sum_{i=t+1}^{r} M(n_i, k)$ by the composite map $k[G]\longrightarrow k[G]/J\simeq\sum_{i=1}^{r} M(n_i, k)$. We identify $k[G]/U_j$ with $\sum_{i=1}^{t} M(n_i, k)$ and denote by ρ_i the projection of $k[G]/U_j$ onto $M(n_i, k)$. If we put $\mu=\sum_{i=1}^{t}\overline{\eta_i'(\sigma)}\operatorname{tr}\rho_i$, then μ is a (symmetric) non-singular linear function on $k[G]/U_j$. Hence by Theorem 9 of Nakayama [3] (or see [1], (55.11)), there exists an element c of $k[G]$ such that $(0:U_j)=k[G]c$ and $\eta\psi(x)=\lambda(cx)$ for all $x\in k[G]$, where ψ is the natural map $k[G]\longrightarrow k[G]/U_j$. From this, by making use of (*), we get easily that $c=\bar{h}\hat{S}_j$ and hence $(0:\hat{S}_j)=U_j$, as $k[G]$ is Frobeniusean. It is clear that $\bigcap_{j=1}^{r} U_j=J$ (namely, for any i, there exists a p-regular element σ such that $\nu(\eta_i(\sigma))=\nu(|C_G(\sigma)|)$. This follows easily, for instance, from the relation $(\eta_i, \phi_i)=1$). Thus we conclude that $\bigcap_{j=1}^{r}(0:\hat{S}_j)=J$ and the proof is complete.

From the above argument, we get also

Corollary A. *Let e be a primitive idempotent of $k[G]$, and let η*

be the Brauer character afforded by $k[G]e$. *If* S_j *is a* p'*-section of* G, *then the following are equivalent*:

(1) $\hat{S}_j e \neq 0$.

(2) $\nu(\eta(\sigma)) = \nu(|C_G(\sigma)|)$, *where* σ *is a* p*-regular element in* S_j.

We continue our argument to give an alternative proof to the following result of Okuyama.

Corollary B (Okuyama [5]). *Under the same notation as in Corollary A, let* $e = \sum_{\tau \in G} a_\tau \tau$. *Then the following are equivalent*:

(1) $\hat{S}_j e \neq 0$.

(2) $\sum a_\tau \neq 0$, *where the summation is taken over all* $\tau \in S_j^{-1} = \{\sigma^{-1} | \sigma \in S_j\}$.

Proof. Recall that if ξ is a symmetric linear function on $M(n, k)$, then there exists some $a \in k$ such that $\xi(x) = a \cdot \mathrm{tr}(x)$ for all $x \in M(n, k)$ (since the set $\{xy - yx | x, y \in M(n, k)\}$ spans the subspace consisting of the elements of trace zero). In particular, we have $\xi(e) \neq 0$ for any primitive idempotent e. Keeping the notation used in the proof of the Theorem, we know that μ is a symmetric, non-singular linear function on $k[G]/U_j = \sum_{i=1}^{t} M(n_i, k)$ and hence $\mu\psi(e) \neq 0$ for any primitive idempotent e of $k[G]$ not contained in U_j. And if $e = \sum a_\tau \tau$, then $\mu\psi(e) = \sum_{i,\tau} a_\tau \overline{\eta_i'(\sigma)}\, \overline{\phi_i(\tau)} = (\sum a_\tau)\bar{h}$, where the second summation is taken over all $\tau \in S_j^{-1}$. From these observations, we get the above assertion.

Remark. According to a result of Okuyama [5], there holds that the summation $\sum a_\tau$ in the above (2) is equal to the restricted summation $\sum' a_\tau$, where τ runs over all $\tau \in C_j^{-1}$.

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