RESTRICTED SEMIPRIMARY GROUP RINGS

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Throughout this paper a ring will mean an associative ring with identity $1 \neq 0$. A ring R is *semilocal* if $R/J(R)$ is Artinian, where $J(R)$ denotes the Jacobson radical of R. R is said to be *semiprimary*, or SP for brevity, if R is semilocal and $J(R)$ is nilpotent. If the ring R has the property that every proper homomorphic image of R is semiprimary (resp. Artinian), we call R a *restricted semiprimary* (resp. *restricted Artinian*) ring, or RSP ring (resp. RA ring) for brevity. E. P. Armendariz and K. E. Hummel [1] have studied these rings in details.

Let AG denote the group ring of the group G over the ring A. Given a subgroup H of G, ωH will denote the right ideal of AG generated by $\{1-h \mid h \in H\}$; if H is normal (written $H \lhd G$) then ωH is an ideal and $AG/\omega H \simeq A(G/H)$. If AG is SP, then so is A by $A \simeq AG/\omega G$. Further, AG being perfect, G is finite by [8, Theorem]. Conversely, suppose that A is SP and G is finite. Since $AG/J(A)G \simeq (A/J(A))G$ is Artinian and $J(AG)$ contains the nilpotent ideal $J(A)G$, AG is SP. Thus we have seen that *AG is SP if and only if A is SP and G is finite.*

In this paper, we are exclusively concerned with RSP group rings which are not SP. We shall prove that if AG with $G \neq 1$ is RSP but not SP then G is an infinite group in which every non-trivial subnormal subgroup is of finite index (Theorem 1), and that AG is RSP but not SP and G contains a non-trivial solvable subnormal subgroup if and only if A is (Artinian) simple and G is either an infinite cyclic group or an infinite dihedral group (Theorem 2).

1. In this section we shall prove the following:

Theorem 1. *Let A be a ring, and G a non-trivial group. If AG is RSP but not SP then A is (Artinian) simple and G is an infinite group in which every non-trivial subnormal subgroup is of finite index.*

In advance of proving the theorem, we state several preliminary lemmas. An ideal I of AH $(H \lhd G)$ is said to be *G-invariant* if $g^{-1}Ig \subseteq I$ for all $g \in G$.

Lemma 1. *Let H be a normal subgroup of G such that $|G:H| < \infty$*

and AH is a prime ring. Then every non-zero ideal of AH contains a non-zero G-invariant ideal.

Proof. Let $\{g_1=1, g_2, \cdots, g_n\}$ be a right transversal of H in G. If I is a non-zero ideal of AH, then $I_0=\bigcap_{i=1}^{n} g_i^{-1} I g_i$ is a G-invariant ideal of AH contained in I. Since AH is prime, I_0 is obviously non-zero.

Let $\Delta(G)=\{x\in G \mid |G:C_G(x)|<\infty\}$ be the f. c. subgroup of G. It is known [4, Theorem 8] that AG is prime if and only if A is prime and G contains no non-trivial finite normal subgroup, or equivalently, A is prime and $\Delta(G)$ is torsion free abelian. If H is a subgroup of finite index then it is easy to see that $\Delta(H)\subseteq\Delta(G)$. Hence the next lemma is immediate.

Lemma 2. *If AG is prime and H is a subgroup of G with* $|G:H|<\infty$, *then AH is prime.*

We require also the following which is included in a more general theorem [1, Theorem 2. 7 (b)] :

Lemma 3. *If AG is RSP but not SP then AG is a prime ring.*

It is known (see [2]) that a ring R is SP if and only if there exists an integer N such that R contains no strictly decreasing sequence of N principal left (right) ideals. This characterization of SP rings will be used in the proof of the next lemma.

Lemma 4. *If a ring R is a direct summand of an SP ring S as a right R-module, then R is also SP.*

Proof. Since $SI\cap R=I$ for each left ideal I of R, the assertion is immediate by the above.

Lemma 5. *If AG is RSP but not SP and H is a non-trivial normal subgroup of G, then AH is also RSP but not SP.*

Proof. Since $A(G/H)\simeq AG/\omega H$ is SP, we have $|G:H|<\infty$ (see the introduction). Since AG is prime (Lemma 3), AH is also prime (Lemma 2) and every non-zero ideal of AH contains a non-zero G-invarinant ideal (Lemma 1). Thus in order to show that AH is RSP, it is enough to prove that AH/I is SP for every non-zero G-invariant ideal I of AH. Obviously, AGI is an ideal of AG and AG/AGI is SP. To be easily seen, $(AH+AGI)/AGI$ is a direct summand of AG/AGI as a right $(AH+AGI)/AGI$-module. Noting that AH is a direct summand of AG as a right AH-module, it follows that $AH/I=AH/(AGI\cap AH)\simeq (AH+AGI)/AGI$ is SP (Lemma 4). Finally, since $|G:H|<\infty$ and G is infinite, H must be infinite and AG is not SP.

Proof of Theorem 1. Obviously A is SP and G is infinite. Suppose A contains a non-zero proper ideal I. Then $(A/I)G\simeq AG/IG$ is SP and G is finite, a contradiction. Hence, A is simple. Now, let H be a non-trivial subnormal subgroup of $G: H=H_n \lhd H_{n-1} \lhd \cdots \lhd H_1 \lhd H_0 = G$. By repeated use of Lemma 5, we see that each AH_i is RSP but not SP. Since $A(H_{i-1}/H_i)\simeq AH_{i-1}/\omega H_i$ is SP, it follows $|H_{i-1}:H_i|<\infty$. Hence $|G:H|<\infty$.

Remark 1. In view of Theorem 1, it seems worthwhile to note that if AG is RSP but not SP and G is non-trivial then G contains no non-trivial finite subnormal subgroup and that AH is RSP but not SP for every non-trivial subnormal subgroup H of G.

Corollary 1. *Let H be a non-trivial subnormal subgroup of G. If AG is RSP but not SP then either $C_G(H)=1$ or $\Delta(G)\neq 1$.*

Proof. If there exists $x\neq 1$ in $C_G(H)$ then $|G:C_G(x)|\leq |G:H|<\infty$ by Theorem 1, and therefore $x\in \Delta(G)$.

2. In this section, we shall prove our principal theorem that is stated as folows:

Theorem 2. *Let A be a ring, and G a non-trivial group. Then the following statements are equivalent*:

1) *AG is RA but not SP and G contains a non-trivial solvable subnormal subgroup.*

2) *AG is RSP but not SP and G contains a non-trivial solvable subnormal subgroup.*

3) *AG is RSP but not SP and the f. c. subgroup of G is non-trivial.*

4) *A is (Artinian) simple and G is either an infinite cyclic group or an infinite dihedral group.*

Proof. 1)$\Longrightarrow$2). This is trivial.

2)$\Longrightarrow$3). If H is a non-trivial solvable subnormal subgroup of G then H contains a non-trivial subnormal abelian subgroup L, which is obviously a non-trivial subnormal subgroup of G. Hence, for each $x \in L$, $|G: C_G(x)| \leq |G: L| < \infty$ by Theorem 1.

3)$\Longrightarrow$4). First, A is simple by Theorem 1. Now, by Lemma 3, AG is prime, so that $\Delta(G) \neq 1$ is torsion free abelian. Since $A\Delta(G)$ is RSP but not SP (Lemma 5), every non-trivial subgroup of $\Delta(G)$ is of finite index (Theorem 1). Thus, $\Delta(G)$ is finitely generated, and hence infinite cyclic: $\Delta(G) = \langle a \rangle$. For each $x \in C_G(a)$, $|G: C_G(x)| \leq |G: \langle a \rangle| < \infty$, namely, $C_G(a) = \langle a \rangle$. Here, we assume that $G \neq \langle a \rangle$ and $b \notin \langle a \rangle$. Then $bC_G(a) = b\langle a \rangle$ is the set of elements inducing the only one non-trivial automorphism of $\langle a \rangle$ $(a \mapsto a^{-1})$. Thus, $G = \langle a \rangle \cup b\langle a \rangle$ and $b^{-1}ab = a^{-1}$. Noting that $b^2 \in \langle a \rangle$, we can see $b^2 = b^{-1}b^2b = b^{-2}$, whence it follows $b^2 = 1$. We get therefore $G = \langle a, b \mid b^2 = 1,\ b^{-1}ab = a^{-1} \rangle$.

4)$\Longrightarrow$1). Since G is either an infinite cyclic group or an infinite dihedral group, G contains a non-trivial normal abelian subgroup and AG is not SP. It remains therefore to prove that AG is RA. Now, let $A \simeq D_n$ with some division ring D. Since $AG \simeq (DG)_n$, AG is RA if and only if so is DG. Thus, from the beginning, we may assume that A is a division ring.

Case 1: G is an infinite cyclic group $\langle x \rangle$. Obviously, $AG = A[x] + A[x^{-1}]$ where $A[x]$ and $A[x^{-1}]$ are polynomial rings over A in the indeterminates x and x^{-1} respectively. Given a non-zero ideal I of AG, $I_1 = I \cap A[x]$ and $I_2 = I \cap A[x^{-1}]$ are non-zero ideals of $A[x]$ and $A[x^{-1}]$ respectively. Since every non-zero ideal of $A[x]$ is a principal left (or right) ideal with a monic generator, $A[x]/I_1$ and $A[x^{-1}]/I_2$ are finite dimensional over A. Since $(r_1 + I_1,\ r_2 + I_2) \mapsto r_1 + r_2 + I$ defines an A-homomorphism of $A[x]/I_1 \oplus A[x^{-1}]/I_2$ onto AG/I, AG/I is also finite dimensional over A, and hence Artinian.

Case 2: G is an infinite dihedral group $\langle a, b \mid b^2 = 1,\ b^{-1}ab = a^{-1} \rangle$. Setting $H = \langle a \rangle$, we have $AG = AH + AHb$. Let I be a non-zero ideal of AG, and $r = r_1 + r_2b$ a non-zero element of I. If $r_1 = 0$ then $0 \neq r_2 = rb^{-1} \in AH \cap I$, and if $r_1 \neq 0$ then $0 \neq ra^2 - a^{-1}ra = r_1a^2 - r_1 \in AH \cap I$. Hence $J = AH \cap I$ is a non-zero ideal of AH. Since H is infinite cyclic, AH/J is finite dimensional over A (see Case 1). Now, $(r_1 + J,\ r_2 + J) \mapsto r_1 + r_2b + I$ defines an A-homomorphism of $AH/J \oplus AH/J$ onto AG/I. Consequently, AG/I is finite dimensional over A, and Artinian.

3. It was communicated to us by D. S. Passman that the group algebra KS_X is not RSP where K is a field and S_X is the group of restricted permutations on an infinite set X. Thus the converse of Theorem 1 is false. However we have the following theorem in this direction.

Theorem 3. *Let A be an SP-ring, and G an infiinite group in which every non-trivial normal subgroup is of finite index. If $\Omega I=\{g\in G \mid 1-g \in I\}$ is non-trivial for every non-zero ideal I of AG, then AG is RSP but not SP.*

Proof. Obviously, AG is not SP. Given a non-zero ideal I of AG, we set $H=\Omega I$. Since H is a non-trivial normal subgroup of G, G/H is a finite group. Hence, $AG/\omega H \simeq A(G/H)$ is SP. Recalling that $\omega H \subseteq I$, it follows that AG/I is SP.

Remark 2. Needless to say, the condition that ΩI is non-trivial for every non-zero ideal I of AG is not very satisfactory. Nevertheless we have some instances where this holds. Let K be a field, q a prime number different from the characteristic of K, and G an infinite group in which every non-trivial normal subgroup is of finite index. Suppose for any finite number of distinct elements $x_0=1, x_1, \cdots, x_n \in G$ there exist elements $y_0, y_1, \cdots, y_n \in G$ such that $\langle x_i^{-1} y_j x_i \mid i, j=0, 1, \cdots, n \rangle$ is an elementary abelian group of order precisely $q^{(n+1)^2}$. Then Steps I and II in the proof of Bonvallet-Hartley-Passman-Smith Theorem [7, Chap. 9, Sec. 4] (cf. also [3]) exactly prove that ΩI is non-trivial for every non-zero ideal I of KG. Moreover, in [7, Chap. 9, Sec. 4], it has been proved that algebraically closed groups and universal groups possess the property cited above.

Remark 3. Suppose the group algebra KG of the group G over the field K is RSP but not SP. We discuss the semisimplicity problem for KG.

Case 1: K is of characteristic 0. If $J(KG)\neq 0$ then KG is semilocal, so that G is finite by [5], a contradiction. This means $J(KG)=0$.

Case 2: K is of characteristic $p>0$. By the proof of [6, Theorem 9. 3], it is not difficult to prove that KG/N^*KG is SP if and only if G is locally finite and $|G:O_p(G)|<\infty$. (For more details, see [6].) Thus, either $N^*KG=0$ or G is locally finite with $|G:O_p(G)|<\infty$.

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