NOTE ON THE n-CENTER OF AN ALGEBRA

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Let R be a ring with 1, and $S_n[x_1, x_2, \cdots, x_n]$ the standard polynomial of degree n. In [6], A. Kovacs defined the n-center $C_n(R)$ of R as the set $\{a \in R \mid S_n[a, r_1, \cdots, r_{n-1}] = 0$ for all $r_i \in R\}$, and denoted by $R^{(n)}$ the additive subgroup of R generated by all the substitutions of R in $S_n[x_1, x_2, \cdots, x_n]$. He deduced there general properties of $C_n(R)$, and characterized $C_n(R)$ for prime rings and semiprime rings. In general $C_n(R)$ is a module over the center $C(R)$ of R, and if n is even then $C(R)$ is contained in $C_n(R)$ ([6, Lemma 2]).

In this note, we shall prove the following two theorems.

Theorem 1. *Let C be a commutative algebra over a field K of characteristic 0, and R an Azumaya algebra of rank m^2 over C. Then there hold the following:*

(a) $$C_n(R) = \begin{cases} R & (n \geq 2m), \\ C & (n \text{ even}, \ n < 2m), \\ 0 & (n \text{ odd}, \ n < 2m). \end{cases}$$

(b) $$R^{(n)} = \begin{cases} 0 & (n \geq 2m), \\ R^{(2)} & (n \text{ even}, \ n < 2m), \\ R & (n \text{ odd}, \ n < 2m). \end{cases}$$

(c) $R = C_n(R) \oplus R^{(n)}$ *(as $C(R)$-modules) for each n.*

Theorem 2. *If R is a semiprime PI-algebra over a field K of characteristic 0, then $C_n(R) \cap R^{(n)} = 0$ for each n.*

Throughout the subsequent study, K will represent a field of characteristic 0, and R a K-algebra with 1. First we prove the next

Lemma 1. *If $R = M_m(C)$ with a commutative subalgebra C then*

$$R^{(n)} = \begin{cases} 0 & (n \geq 2m), \\ R^{(2)} & (n \text{ even}, \ n < 2m), \\ R & (n \text{ odd}, \ n < 2m). \end{cases}$$

Proof. As is well known, if $n \geq 2m$ then $R^{(n)} = 0$ by a theorem of Amitsur and Levitzki (see for instance [4, p. 21]). Henceforth, we limit ourselves to the case $n < 2m$, and distinguish between two cases.

(1) $n = 2k$. Evidently,

$$e_{1,k+1} = S_n [e_{12}, e_{22}, e_{23}, e_{33}, \cdots, e_{kk}, e_{k,k+1}, e_{k+1,k+1}] \in R^{(n)}$$

and

$$e_{11} - e_{k+1,k+1} = S_n [e_{12}, e_{22}, e_{23}, e_{33}, \cdots, e_{kk}, e_{k,k+1}, e_{k+1,1}] \in R^{(n)}.$$

Hence $\{e_{ij} \mid i \neq j\} \cup \{e_{11} - e_{ii} \mid i \neq 1\}$ is a subset of $R^{(n)}$. Since $me_{11} = 1 + \sum_{i=2}^{m} (e_{11} - e_{ii}) \in C + R^{(n)}$ and K is of characteristic 0, it follows $e_{11} \in C + R^{(n)}$. Thus, $e_{ij} \in C + R^{(n)}$ for all i, j, so $R = C + R^{(n)}$. Considering the trace, it is obvious that $R^{(2)} \cap C = 0$. Since $R^{(n)} \subseteq R^{(2)}$ by [6, Corollary 10 (iii)], we have then $R^{(n)} = R^{(2)}$.

(2) $n = 2k + 1$. Evidently,

$$e_{1,k+1} = S_n [e_{11}, e_{12}, e_{22}, e_{23}, e_{33}, \cdots, e_{kk}, e_{k,k+1}, e_{k+1,k+1}] \in R^{(n)},$$

so that $e_{ij} \in R^{(n)}$ for all $i \neq j$. In order to prove $R^{(n)} = R$, it suffices therefore to show that $e_{11} \in R^{(n)}$. Since

$$2(e_{11} + e_{22} + \cdots + e_{kk}) + e_{k+1,k+1} = S_n [e_{11}, e_{12}, e_{22}, e_{23}, e_{33}, \cdots, e_{kk}, e_{k,k+1}, e_{k+1,1}] \in R^{(n)},$$

there hold the following:

$$2(e_{22} + e_{33} + \cdots + e_{k+1,k+1}) + e_{11} \in R^{(n)}.$$

$$2(e_{33} + e_{44} + \cdots + e_{11}) + e_{22} \in R^{(n)}.$$

$$\cdots\cdots$$

$$2(e_{k+1,k+1} + e_{11} + \cdots + e_{k-1,k-1}) + e_{kk} \in R^{(n)}.$$

Now, summing up those above, we get

$$(2k + 1)(e_{11} + e_{22} + \cdots + e_{k+1,k+1}) \in R^{(n)}.$$

Since K is of characteristic 0, it follows then $e_{11} + e_{22} + \cdots + e_{k+1,k+1} \in R^{(n)}$. Hence, $e_{11} = 2(e_{11} + e_{22} + e_{33} + \cdots + e_{k+1,k+1}) - \{2(e_{22} + e_{33} + \cdots + e_{k+1,k+1}) + e_{11}\} \in R^{(n)}$

Proof of Theorem 1. Since R is an Azumaya algebra of rank m^2 over C, we know that there is a commutative faithfully flat C-algebra B such that $R \otimes_C B = M_m(B) = M_m(K) \otimes_K B$ (see e. g. [7, Corollaire III. 6. 7]). So, if $n \geq 2m$ then it is easy to see that $R^{(n)} = 0$ and $C_n(R) = R$. In what follows, we may therefore restrict our attention to the case

$n<2m$. It is evident that $C_n(R)\otimes_C B\subseteq C_n(R\otimes_C B)=C_n(M_m(K))\otimes_K B$. By [6, Theorems 13 and 16],

$$C_n(M_m(K))=\begin{cases} K & (n \text{ even}),\\ 0 & (n \text{ odd}).\end{cases}$$

Hence,

$$C_n(R)\otimes_C B=\begin{cases} B & (n \text{ even}),\\ 0 & (n \text{ odd}).\end{cases}$$

Recalling that ${}_C B$ is faithfully flat, it follows at once

$$C_n(R)=\begin{cases} C & (n \text{ even}),\\ 0 & (n \text{ odd}).\end{cases}$$

Next, it is easy to see that $R^{(n)}\otimes_C B=(R\otimes_C B)^{(n)}=M_m(K)^{(n)}\otimes_K B$. By Lemma 1,

$$M_m(K)^{(n)}=\begin{cases} M_m(K)^{(2)} & (n \text{ even}),\\ M_m(K) & (n \text{ odd}).\end{cases}$$

Hence,

$$R^{(n)}\otimes_C B=\begin{cases} R^{(2)}\otimes_C B & (n \text{ even}),\\ R\otimes_C B & (n \text{ odd}).\end{cases}$$

Finally, noting that K is of characteristic 0, one can easily see that R is a strongly separable algebra in the sense of Kanzaki [5] (cf. also [3, Theorem 1]). Then, $R=C\oplus R^{(2)}$, and (c) is immediate from (a) and (b).

Corollary. *Let R be a regular self-injective PI-algebra over a field K of characteristic 0, then $R=C_n(R)\oplus R^{(n)}$ (as $C(R)$-modules) for each n.*

Proof. By [1, Theorem 3. 5], $R=R_1\oplus\cdots\oplus R_m$, where each R_i is an Azumaya algebra of constant rank. Then it is easy to see that $C_n(R)=C_n(R_1)\oplus\cdots\oplus C_n(R_m)$ and $R^{(n)}=R_1^{(n)}\oplus\cdots\oplus R_m^{(n)}$. Since $R_i=C_n(R_i)\oplus R_i^{(n)}$ by Theorem 1 (c), we readily obtain $R=C_n(R)\oplus R^{(n)}$.

In advance of proving Theorem 2, we state the next

Lemma 2. *Let S be a semiprime PI-ring, and Q the maximal quotient ring of S. Then $C_n(S)\subseteq C_n(Q)$ for each n.*

Proof. Our proof is quite similar to that of [8, Theorem 2]. Let $a \in C_n(S)$. Given $q_1, q_2, \cdots, q_{n-1} \in Q$, we set $q = S_n[a, q_1, q_2, \cdots, q_{n-1}]$. Then there exists an essential left ideal J of S such that $Jq_1, Jq_2, \cdots, Jq_{n-1}$ and Jq are contained in S. Now, let $x \in J$, and $y = xq$ $(\in S)$. If y is non-zero, then $U = SyS \cap J \neq 0$. Since J is a semiprime *PI*-ring by [8, Lemma 2], U contains a non-zero element c in the center of J ([9, Theorem 2]). Noting that the center of J coincides with $J \cap C(S)$ (see [8. Lemma 1]) and each q_ic is in S, we see that $c^{n-1}y = c^{n-1}xq = c^{n-1}xS_n[a, q_1, q_2, \cdots, q_{n-1}] = S_n[a, q_1c, q_2c, \cdots, q_{n-1}c] = 0$. This implies $c^n \in c^{n-1}SyS = S(c^{n-1}y)S = 0$, which is a contradiction. Hence, $Jq = 0$, and so $q = 0$, proving $a \in C_n(Q)$.

Proof of Theorem 2. Since the maximal quotient ring Q of R is a regular self-injective *PI*-ring (see [2, Theorem 3] and [8, Theorem 2]), $Q = C_n(Q) \oplus Q^{(n)}$ by Corollary to Theorem 1. Since $C_n(R) \subseteq C_n(Q)$ by Lemma 2, we obtain $C_n(R) \cap R^{(n)} \subseteq C_n(Q) \cap Q^{(n)} = 0$.

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