TWO COMMUTATIVITY THEOREMS FOR RINGS

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Throughout R will represent a ring (with or without 1), N_0 the set of all nilpotent elements of R, N the prime radical of R, and J the Jacobson radical of R. Given subsets S, T of R, we set $V_S(T) = \{s \in S \mid st = ts \text{ for all } t \in T\}$ and $V_S^-(T) = \{s \in S \mid st = -ts \text{ for all } t \in T\}$. We denote by C the center of R, and by C' the set of all $x \in R$ such that for each $y \in R$ there holds $[x, y - yy'] = 0$ with some y' in the subring $[y]$ generated by y. In [1], C' is called the *cohypercenter* of R.

In this paper, we consider the following conditions:

A) For each $x \in R$ there exists a positive integer n such that $x - x^{n+1} \in N_0$.

A′) For each $x \in R$ there exist positive integers m and n such that $x^m - x^{m+n} \in N_0$.

A″) For each $x \in R$ there exist a positive integer n and an element $x' \in [x]$ such that $x^n = x^n x'$.

A‴) For each $x \in R$ there exists an element $x' \in [x]$ such that $x - xx' \in N_0$.

B) $x - y \in N_0$ and $y - z \in N_0$ $(x, y, z \in R)$ imply $x^2 = z^2$ or $xy = yx$.

B′) $x - y \in N_0$ $(x, y \in R)$ implies that $x^2 = y^2$ or both x and y are contained in $V_R(N_0)$.

B″) Either R is commutative or $R = V_R^-(N_0)$ and $u^2 = 0$ for all $u \in N_0$.

C) For each $x, y \in R$ there exist $x' \in [x]$ and $y' \in [y]$ such that $[x - xx', y - yy'] = 0$.

C′) For each $x, y \in R$ there exists some $x' \in [x]$ such that $x - xx'$ is in $V_{N_0}(y)$.

Recently, in [1, Theorem 3], M. Chacron proved that if R satisfies the condition C) then both R/N and N are commutative. The proof depends heavily on another (perhaps more difficult) result in [1] that the cohypercenter of a semi-prime ring coincides with its center. In §1 of this paper, we shall give a somewhat direct and economical proof to the above theorem. And in §2, we shall deal with the commutativity theorem of S. Ligh and J. Luh (cf. [3]) and D. L. Outcalt and A. Yaqub [4] without assuming the existence of 1.

1. The next has been shown in [1, Remarks 12 and 14]. However, for the sake of completeness, we shall give the proof.

Lemma 1. *If* C) *is satisfied then* C' *is a commutative subring of* R *containing* N_0.

Proof. Let $x\in N_0$, and $y\in R$. There exist $x_1'\in[x]$ and $y_1'\in[y]$ such that $[x-xx_1', y-yy_1']=0$. To be easily seen, there exist $x_2'\in[xx_1']$ and $y_2'\in[y]$ such that $[x-xx_1'x_2', y-yy_2']=0$. Repeating the same procedure, we obtain eventually $0=[x-xx_1'x_2'\cdots x_k', y-yy_k']=[x, y-yy_k']$ for some k, which proves $N_0\subseteq C'$. Next, for any $a, b\in C'$ we consider the subring S generated by $\{a, b\}$. Given $x\in S$, one will easily see that there exist some $x'\in[x]$ such that $[a, x-xx']=0=[b, x-xx']$, namely, $x-xx'\in V_S(S)$. Then, by [2, Theorem 19], S is a commutative ring contained in C'. This proves that C' is a commutative subring of R.

Corollary 1. *Assume that* R *satisfies the condition* C). *If* R *is either a division ring or a radical ring without non-zero zero-divisors, then* R *is commutative.*

Proof. If z is a quasi-regular element of R with the quasi-inverse z^* then $y-yz$ and $y-yz^*$ will be written formally as $y(1-z)$ and $y(1-z)^{-1}$ respectively. Obviously, the map defined by $y\mapsto(1-z)y(1-z)^{-1}$ $(y\in R)$ is an automorphism of R. Now, let a and x be elements of R such that $a\notin C[x]$ and $x\notin C[a]$. First we shall show that there exist a', $a_0'\in[a]$ such that

$$(1)\qquad \begin{aligned}&[(1-x)(a-aa')(1-x)^{-1}, a-aa_0']=0,\\ &[(1-ax)(a-aa')(1-ax)^{-1}, a-aa_0']=0.\end{aligned}$$

In any rate, $[(1-x)(a-aa'')(1-x)^{-1}, a-aa_0'']=0$ for some $a'', a_0''\in[a]$. Then $[(1-ax)\{(a-aa'')(1-b')\}(1-ax)^{-1}, (a-aa_0'')(1-b_0')]=0$ for some $b'\in[a-aa'']$ and $b_0'\in[a-aa_0'']$. Evidently, setting $a'=a''+b'-a''b'$ and $a_0'=a_0''+b_0'-a_0''b_0'$, we obtain (1). For the brevity, we set $\alpha=a-aa'$, $\alpha_0=a-aa_0'$, $\beta=(1-x)\alpha(1-x)^{-1}$, and $\beta'=(1-ax)\alpha(1-ax)^{-1}$. Then (1) becomes $[\beta, \alpha_0]=0=[\beta', \alpha_0]$, and we have

$$(2)\qquad (1-x)\alpha=\beta(1-x),$$

$$(3)\qquad (1-ax)\alpha=\beta'(1-ax).$$

Now, subtracting (3) from (2) multiplied by a, we get

$$(4)\qquad (a-1)\alpha=(\beta'a-a\beta)x+a\beta-\beta'.$$

This deduces $(\beta'a-a\beta)x=(a-1)\alpha-a\beta+\beta'\in V_R(\alpha_0)$, so that $(\beta'a-a\beta)$

$[x, \alpha_0]=0$. If $[x, \alpha_0]\neq 0$, then $a\beta=\beta' a$, and by (4) it follows $\alpha(a-1)= a\beta-\beta'=\beta'(a-1)$, that is, $\alpha=\beta'$. Going back to (3), we get $(1-ax)\alpha =\alpha(1-ax)$, namely, $a[x, \alpha]=0$. Hence, $[x, \alpha]=0$. We have therefore seen that every element of R is in C', whence it follows the commutativity of R by Lemma 1.

Lemma 2. *If R is a semi-prime ring satisfying* C) *then R is commutative.*

Proof. Without loss of generality, we may assume R is prime.

Case I: R is semi-primitive. We may assume further R is primitive. Every homomorphic image of a subring of R inherits the condition C). Since matrix rings over division rings of degree >1 contains non-commutative nilpotent elements, by Lemma 1 a routine argument enables us to see that R is a division ring, so R is commutative by Corollary 1.

Case II: R is not semi-primitive. If $J\neq 0$ is shown to be commutative, then one will easily see that $V_R(J)\subseteq C$, so that $R=C$. Therefore, we assume henceforth $R=J$. Suppose R contains a non-zero element a with $a^2=0$. Then for each $y\in aR$ there exists some $y'\in[y]$ such that $0=[a, y-yy']=yy'a-ya$. Hence, we have $y^2=y^2y'$. This implies evidently $y^2=0$. Combining this with $0=yy'a-ya$, we readily obtain $ya=0$, namely, $aRa=0$. This contradiction shows that R is a reduced ring. Thus, R is a ring without non-zero zero-divisors, and so commutative by Corollary 1.

Now, as a combination of Lemmas 1 and 2, we readily obtain

Theorem 1 ([1, Theorem 3]). *If R satisfies the condition* C) *then both R/N and N are commutative.*

2. Evidently, in A''), $x^n=x^nx'$ may be replaced by $x^n=x^{n+1}x'$. Similarly, in A''') (resp. C')), $x-xx'\in N_0$ (resp. $x-xx'\in V_{N_0}(y)$) may be replaced by $x-x^2x'\in N_0$ (resp. $x-x^2x'\in V_{N_0}(y)$).

Lemma 3. (1) *If* B) *is satisfied then N_0 is an ideal and $x^2\in V_R(N_0)$ for all $x\in R$, especially, every idempotent of R is central.*

(2) *If N_0 is an ideal then the conditions* A)$-$A''') *are equivalent.*

(3) B'') *implies* B'), *and* B') *implies* B).

(4) *If for each $x, y\in R$ there exists some $z\in R$ such that $[x-x^2z, y] =0$ then $N_0\subseteq C$.*

Proof. (1) is contained in [3, Lemma 1], and (3) is easy.

(2) Obviously, A)$\Longrightarrow$A')$\Longrightarrow$A''). For any $x'\in[x]$ we have $(x-xx')^n=(x^n-x^nx')-(x^n-x^nx')x''$ with some $x''\in[x]$, which proves A'')$\Longrightarrow$A'''). Finally, A''')$\Longrightarrow$A) by [5, Corollary 3. 5].

(4) Let $x^n=0$ $(n>1)$. We proceed by the induction with respect to n. Given $y\in R$, there exists z with $[x-x^2z, y]=0$. Since x^2 is central by $(x^2)^{n-1}=0$, it follows $(x^2z)^{n-1}=0$, so that x^2z is central. Hence, $[x, y]=[x^2z, y]=0$.

Lemma 4. (1) *If* B) *is satisfied then* N_0 *is either commutative or anti-commutative with* $u^2=0$ *for all* $u\in N_0$.

(2) *If* A) *and* B) *are satisfied and* R *is left* (*or right*) *s-unital then* N_0 *is commutative.*

(3) *If* A) *and* B) *are satisfied and* N_0 *is commutative then* R *is commutative.*

Proof. (1) By Lemma 3 (1), N_0 is an ideal of R and $z^2\in V_R(N_0)$ for all $z\in R$. Suppose there exist x, $y\in N_0$ such that $xy\neq yx$. Since $x+y\equiv x\equiv 0$ (mod N_0) and $(x+y)x\neq x(x+y)$, B) implies $0=(x+y)^2=x^2$, and similarly $0=(x+y)^2=y^2$. From these it follows $xy=-yx$. Now, by making use of Brauer's trick, one will easily see that N_0 is anti-commutative. If v is an arbitrary element of the center of N_0, then $xv=vx=-xv$. Hence, we obtain $v^2=(x+v)^2-2xv-x^2=0$.

(2) Suppose there exist $x, y\in N_0$ such that $xy\neq yx$. Then, by (1), N_0 is an anti-commutative ideal and $u^2=0$ for all $u\in N_0$. By [6, Theorem 1], there exists an element c such that $cx=x$ and $cy=y$. Choose an element $d\in[c]$ such that $c^n=c^{n+1}d$ for some positive integer n (Lemma 3 (2)). Then $e=c^nd^n$ is a central idempotent with $ec^n=c^n$ (Lemma 3 (1)). Hence, $ex=ec^nx=x$ and $ey=y$. Noting that $e+x+y\equiv e+x$ (mod N_0), B) implies then $(e+x+y)^2=(e+x)^2$, whence it follows $2(x+y)=2x$, namely, $2y=0$. This forces a contradiction $xy=-yx=yx$.

(3) Let x, y be arbitrary elements of R, and $S=[x, y, N_0]$. Then, by [5, Theorem 3. 4], $\bar{S}=S/N_0=S_1/N_0\oplus\cdots\oplus S_m/N_0$, where each S_i/N_0 is a finite field. As is well-known, the identity of $\bar{S}$ can be lifted to an idempotent e of S, which is central by Lemma 3 (1). Suppose there exist $s\in S$ and $t\in N_0$ such that $st\neq ts$. Then there exists some $s_j\in S_j$ such that $s_jt\neq ts_j$. We set $es_j=s_j+u$ with some $u\in N_0$. By Lemma 3 (1) and the assumption, one will easily see that $2s_j=(e+s_j)^2-e-s_j^2-2u\in V_R(N_0)$. Hence, there holds $2(s_jt-ts_j)=0$. If the characteristic p of S_j/N_0 is different from 2 then the last together with $p(s_jt-ts_j)=(ps_j)t-$

$t(ps_j)=0$ deduces a contradiction $s_jt-ts_j=0$. While, if $S_j/N_0=\mathrm{GF}(2^k)$ then $0=(s_j^{2^k}-s_j)t-t(s_j^{2^k}-s_j)=ts_j-s_jt$ by $s_j^2t=ts_j^2$, which is a contradiction. Thus, we have seen that N_0 is contained in the center of S. Consequently, S is commutative by [2, Theorem 19], which means that R is commutative.

Now, we are ready to prove the principal theorem of this section.

Theorem 2. *The following statements are equivalent:*

1) A) *and* B) *are satisfied.*
1′) A′) *and* B) *are satisfied.*
1″) A″) *and* B) *are satisfied.*
1‴) A‴) *and* B) *are satisfied.*
2) A) *and* B′) *are satisfied.*
2′) A′) *and* B′) *are satisfied.*
2″) A″) *and* B′) *are satisfied.*
2‴) A‴) *and* B′) *are satisfied.*
3) A) *and* B″) *are satisfied.*
3′) A′) *and* B″) *are satisfied.*
3″) A″) *and* B″) *are satisfied.*
3‴) A‴) *and* B″) *are satisfied.*

Proof. By Lemma 1 (3), B″) $\Longrightarrow$ B′) $\Longrightarrow$ B). Hence, by Lemma 3 (1) and (2), 1)−1‴), 2)−2‴) and 3)−3‴) are respectively equivalent and 3) $\Longrightarrow$ 2) $\Longrightarrow$ 1). It remains therefore to prove 1) $\Longrightarrow$ 3). Suppose R is not commutative. By Lemma 4 (1) and (3), N_0 is anti-commutative and $u^2=0$ for all $u\in N_0$. If $xv\neq vx$ for some $x\in R$ and $v\in N_0$, then B) implies $(x+v)^2=x^2$, whence it follows $xv=-vx$. Now, by making use of Brauer's trick, one can easily see that $R=V_R(N_0)$ or $V_R^-(N_0)$. Since $R=V_R(N_0)$ yields the commutativity of R by [2, Theorem 19], R must be $V_R^-(N_0)$.

The next includes [3, Theorem 2] as well as [4, Theorem 2].

Corollary 2. *If* R *is left* (*or right*) *s-unital, then the follwing statements are equivalent:*

1) A) *and* B) *are satisfied.*
1′) A′) *and* B) *are satisfied.*
1″) A″) *and* B) *are satisfied.*
1‴) A‴) *and* B) *are satisfied.*
2) A) *and* B′) *are satisfied.*

2$'$) A$'$) *and* B$'$) *are satisfied.*
2$''$) A$''$) *and* B$'$) *are satisfied.*
2$'''$) A$'''$) *and* B$'$) *are satisfied.*
3) *R is a commutative ring satisfying* A).
3$'$) *R is a commutative ring satisfying* A$'$).
3$''$) *R is a commutative ring satisfying* A$''$).
3$'''$) *R is a commutative ring satisfying* A$'''$).
4) C$'$) *is satisfied.*

Proof. 4) implies 3$'''$) by Lemma 3 (4) and [2, Theorem 19], and 1) implies 3) by Lemma 4 (2). Hence, the corollary is evident by Theorem 2.

Remark. Let R be the module $Z \oplus Z \oplus Z$. If we define the multiplication by $(a_1, a_2, a_3)(b_1, b_2, b_3) = (0, 0, a_1b_2 - a_2b_1)$, then R is an anti-commutative, non-commutative ring and the square of each element is 0.

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