NUMERICAL INVESTIGATION OF SOME EQUATIONS INVOLVING EULER'S ϕ-FUNCTION

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Let $\phi(n)$ denote Euler's ϕ-function, i. e. the number of positive integers not exceeding n which are relatively prime to n. In his paper [1], P. Erdös stated that: "It seems likely that there exists, for every k, k consecutive integers n, $n+1, \cdots, n+k-1$ such that $\phi(n) = \phi(n+1) = \cdots = \phi(n+k-1)$."

The present note is a report on some results of numerical experiments relevant to the above conjecture of Erdös.

In performance of computation we have used a computer HITAC 20 in the Department of Mathematics, Okayama University.

1. The Equation $\phi(x) = \phi(x+1)$. Our problem is essentially to solve the equation involving Euler's ϕ-function

$$(E_1) \qquad \phi(x) = \phi(x+1).$$

Therefore, we have successively sought for solutions of this equation (E_1) on a computer.

In evaluation of values of the function $\phi(n)$, we make use of the well-known formula:

$$(1) \qquad \phi(n) = (p_1^{e_1} - p_1^{e_1-1})(p_2^{e_2} - p_2^{e_2-1}) \cdots (p_t^{e_t} - p_t^{e_t-1})$$

where $n = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ $(e_i \geqq 1,\ i = 1, 2, \cdots, t)$ is the canonical prime factorization of the integer n.

In a stage of test computation, we have tried a few of algorithms of evaluating values of $\phi(n)$. But, in the magnitude of n which we are now concerned with, it seems likely that using the formula (1) is more economical than others.

We have examined up to the limit $x \leqq 10928925$ and we have obtained 146 solutions of the equation (E_1).

As result, we have observed some interesting facts. Firstly, in the interval we examined, $x = 5186$ is the only one solution of the equation $\phi(x) = \phi(x+1) = \phi(x+2)$, on which P. Erdös [1] have already pointed out its existence in the interval $x \leqq 10^4$. Secondly, for $k \geqq 4$, there does not exist any solution of the equation $\phi(x) = \phi(x+1) \cdots = \phi(x+k-1)$ in our interval.

Strange enough, each of the solutions we obtained is congruent to 2 or 3 in the modulus 6. This fact seems to give some hints on property or structure of these solutions, but, to our regret, we have at present no insight into them.

Our results are listed in the following Table I.

Table I. List of Solutions of the Equation (E_1)

x	$\phi(x)$	x	$\phi(x)$	x	$\phi(x)$
1	1	57584	27840	686985	336960
3	2	57645	25920	840255	397440
15	8	64004	32000	914175	456960
104	48	65535	32768	936494	427680
164	80	73124	36000	952575	468480
194	96	105524	47520	983775	483840
255	128	107864	52992	1025504	504576
495	240	123824	60480	1091684	483840
584	288	131144	59904	1231424	604800
975	480	164175	79200	1259642	604800
2204	1008	184635	89280	1276904	583680
2625	1200	198315	96768	1390724	635040
2834	1296	214334	103680	1405845	642528
3255	1440	215775	97920	1574727	768000
3705	1728	256274	126720	1659585	826560
5186	2592	286995	142272	1759874	820800
5187	2592	307395	142560	1788254	816480
10604	4800	319275	151200	1925564	960000
11715	5600	347324	168000	2123583	1061760
13365	6480	388245	172800	2200694	1005984
18315	8640	397485	190080	2388044	1126400
22935	11040	407924	181440	2521694	1075200
25545	12480	415275	188160	2539004	1160640
32864	14976	454124	206400	2619705	1270080
38804	19008	491535	237600	2648204	1209600
39524	19200	524432	258048	2759925	1260000
46215	22464	525986	261360	2792144	1382400
48704	24320	546272	266112	2822715	1233792
49215	24576	568815	279936	2847584	1317888
49335	21120	589407	290304	3104744	1419264
56864	28416	679496	336960	3137355	1486080

x	$\phi(x)$	x	$\phi(x)$	x	$\phi(x)$
3170936	1572480	5198024	2376000	7303334	3459456
3240614	1481088	5295884	2341440	7378371	3538080
3289934	1572480	5466824	2280960	7823205	3749760
3653564	1822464	5577825	2704000	7939244	3953664
3693525	1742400	5683184	2585088	8018144	3991680
3794834	1814400	5710088	2808000	8111024	3841920
3877184	1870848	5781434	2724480	8338394	4032000
3988424	1780800	5861583	2903040	8380448	4173120
4002405	1915200	6235215	2782080	8385975	3974400
4034744	1710720	6245546	3071520	8448255	4147200
4163355	1903104	6312915	2878848	8698095	3893760
4328804	2096640	6315308	3024000	9512144	4741632
4447064	1981440	6372794	2851200	9718904	4440576
4498935	2056320	6444615	3193344	10282515	5132160
4626195	2114784	6475455	2903040	10601535	4688640
4695704	2146176	6986888	3483648	10798725	4847040
5003744	2848320	7033256	3421440	10928925	4976640
5110664	2336256	7098104	3244800		

2. The Equation $\phi(x)=\phi(x+2)$. Next, we investigate the equation

$(E_2)\qquad \phi(x) = \phi(x+2).$

If two positeve integers p and $2p-1$ are both prime numbers, then the integer $x = 4p - 2$ is a solution of this equation (E_2). Indeed, it holds that $\phi(2(2p-1))=\phi(4p)=2(p-1)$. Now, we call a solution of this type a solution of the first kind and we call a solution which is not of the first kind, a solution of the second kind.

If x is a solution of the first kind, then, evidently, we always have $x \equiv 2 \pmod 4$. On the other hand, when x is a solution of the second kind, x can take each of four possible values in the modulus 4.

For example,

$$\phi(4) = \phi(6) = 2 \text{ and } 4 \equiv 0 \pmod 4,$$
$$\phi(6497) = \phi(6499) = 6336 \text{ and } 6497 \equiv 1 \pmod 4,$$
$$\phi(70) = \phi(72) = 24 \text{ and } 70 \equiv 2 \pmod 4,$$
$$\phi(7) = \phi(9) = 6 \text{ and } 7 \equiv 3 \pmod 4.$$

In this sense, solutions of the second kind have some irregularity. And as to the existence of a squence of alternately consecutive integers n,

$n+2,\ n+4, \cdots$ such that $\phi(n)=\phi(n+2)=\phi(n+4)=\cdots$, the example $\phi(8)=\phi(10)=\phi(12)=4$ is the unique one in the interval we examined.

By a procedure similar to that in treating the equation (E_1), we have examined up to the limit $x \leqq 4\cdot 10^6$. Most of the solutions we obtained are of the first kind and solutions of the second kind are relatively not so many as those of the first kind.

According to a conjecture of G. H. Hardy and J. E. Littlewood [2], the distribution of prime pairs $(p,\ 2p-1)$ is asymptotically of the order of $n/\log^2 n$ for $p \leqq n$.

More precisely, if $P(n)$ is the number of prime pairs $(p,\ 2p-1)$ such that $p \leqq n$, then

$$P(n) \sim 2C_2 \frac{n}{\log^2 n}, \tag{2}$$

where $C_2 = \prod_{p>2}(1-1/(p-1)^2) = 0.6601618\cdots$ is the twin-prime constant [3]. Here, the coefficient in the expression (2) is slightly different from the one in the original paper of G. H. Hardy and J. E. Littlewood. As to this matter, we shall state some conclusion with numerical experiments in the next section. Therefore, we may expect that the distribution of solutions of the equation (E_2) is nearly of the same order as the number of prime pairs $(p,\ 2p-1)$.

Namely, if $Q(n)$ is the number of solutions x of the equation (E_2) such that $x \leqq n$, then

$$Q(n) \sim 2C_2 \frac{n/4}{\log^2(n/4)}$$
$$\sim \frac{C_2}{2} \frac{n}{\log^2 n}.$$

Table II. Distribution of Solutions of the Equation (E_2)

Limit n	Number of Solutions 1st kind	2nd kind	total	$2C_2 \frac{n/4}{\log^2(n/4)}$
10^3	12	11	23	10.8
10^4	59	25	84	53.9
10^5	376	58	434	321.9
10^6	2449	139	2588	2136.6
$2\cdot10^6$	4299	182	4481	3833.8
$3\cdot10^6$	6050	220	6270	5411·1
$4\cdot10^6$	7750	248	7998	6917.5

We may say that the above Table II by our experiments supports reasonably our expectation.

3. Conjecture of Hardy-Littlewood. In their paper [2], G. H. Hardy and J. E. Littlewood stated the

Conjecture D. *If* $(a, b) = 1$ *and* $P(n)$ *is the number of pairs of solution of*

$$ap - bq = k$$

such that $p < n$, *then*

$$P(n) = o\left(\frac{n}{\log^2 n}\right)$$

unless $(k, a) = 1$, $(k, b) = 1$ *and just one of* k, a, b *is even. But if these conditions are satisfied then*

$$P(n) \sim \frac{2C_2}{a} \frac{n}{\log^2 n} \Pi \left(\frac{r-1}{r-2}\right), \tag{3}$$

where r *is an odd prime factor of* k, a *and* b.

In a stage of our experments, we have become aware of the facts that the predictive values by use of the expression (3) discord with our computational results. So, we have examined the distribution of prime pairs (p, q) such that $ap - bq = 1$, $p \leqq n$ for values of a and b up to the limit $n \leqq 10^7$.

Based on our experiments, we wish to propose a correction of the expression (3) as follows:

$$P(n) \sim \frac{2C_2}{b} \frac{n}{\log^2 n} \Pi \left(\frac{r-1}{r-2}\right) \equiv M(n). \tag{4}$$

The results of our experiments are tabulated in the following Table III. In the case $a < b$, the predictive values $M(n)$ by (4) do not agree with actual values $P(n)$, yet giving approximations sufficiently good enough. This discrepancy is, probably, due to the magnitude of the limit n which is not quite large. However, we believe that the Table III gives a numerical verification for the expression (4).

Table III. Distribution of Prime Pairs (p, q) such that $ap - bq = 1$ in the Limit $n = 10^7$

a	b	$P(n)$	$M(n)$	$M(n)/P(n)$
2	1	56157	50822.2	0.90500
4	1	53917	50822.2	0.94260
6	1	105068	101644.3	0.96741
8	1	51403	50822.2	0.98870
10	1	67731	67763.0	1.00047
12	1	100620	101644.3	1.01018
14	1	59814	60986.6	1.01960
16	1	49635	50822.2	1.02392
18	1	98344	101644.3	1.03356
3	2	57043	50822.2	0.89094
5	2	36687	33881.4	0.92352
7	2	32410	30493.3	0.94086
9	2	53230	50822.2	0.95477
11	2	29318	28234.5	0.96304
13	2	28448	27721.2	0.97445
15	2	68776	67763.0	0.98527
17	2	27309	27105.2	0.99254
4	3	38438	33881.4	0.88145
8	3	36798	33881.4	0.92074
10	3	48236	45175.3	0.93655
14	3	42697	40657.7	0.95224
16	3	35019	33881.4	0.96751
5	4	19120	16940.7	0.88602
7	4	17053	15246.6	0.89407
9	4	27749	25411.1	0.91575
11	4	15212	14117.3	0.92804
13	4	14762	13860.6	0.93894
15	4	35777	33881.4	0.94701

a	b	$P(n)$	$M(n)$	$M(n)/P(n)$
1	2	30657	25411.1	0.82888
1	4	16197	12705.5	0.78444
1	6	22279	16940.7	0.76039
1	8	8562	6352.8	0.74197
1	10	9268	6776.3	0.73115
1	12	11791	8470.4	0.71837
1	14	6109	4356.2	0.71308
1	16	4523	3176.4	0.70227
1	18	8117	5646.9	0.69569
2	3	40296	33881.4	0.84081
2	5	16691	13552.6	0.81197
2	7	10969	8712.4	0.79427
2	9	14590	11293.8	0.77408
2	11	6593	5133.5	0.77864
2	13	5613	4264.8	0.75981
2	15	12059	9035.1	0.74924
2	17	4404	3188.8	0.72408
3	4	29887	25411.1	0.85024
3	8	15677	12705.5	0.81046
3	10	17111	13552.6	0.79204
3	14	11182	8712.4	0.77914
3	16	8412	6352.8	0.75520
4	5	15901	13552.6	0.85231
4	7	10436	8712.4	0.83484
4	9	13879	11293.8	0.81373
4	11	6319	5133.5	0.81240
4	13	5385	4264.8	0.79198
4	15	11475	9035.1	0.78737

REFFERENCES

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