NOTES ON A CONJECTURE OF P. ERDÖS. II

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This is a continuation of the previous paper [2] in which some numerical studies have been made in connexion with a conjecture of P. Erdös stated in [1]. For convenience' sake we repeat our basic definitions in [2]. Let a and b be integers with $1 \leqq a < b$ and G. C. D. $(a, b) = 1$. For any natural number n we denote by $f(n) = f(n; a, b)$ the number of those integers k satisfying $1 \leqq k < (\log n)/\log(b/a)$, for which $a^k n - b^k$ is a prime number. A positive integer n is said to have the property $\mathrm{P}(a, b)$, if all of the integers

$$a^k n - b^k \qquad (1 \leqq k < (\log n)/\log(b/a))$$

are prime numbers. We conjecture with Erdös [1] that for any fixed pair of integers a, b with $1 \leqq a < b$ there are at most finitely many natural numbers having the property $\mathrm{P}(a, b)$.

1. Further numerical results. We shall present below for $2 \leqq a \leqq 5$, $a < b \leqq 21$, G. C. D. $(a, b) = 1$, tables of the number of natural numbers $n \leqq x$ with the property $\mathrm{P}(a, b)$ for $x = 2^m \mathrm{E}\,4$, $m = 0(1)M$ $(M = 9$ or $8)$.

In order to carry out computations relative to tabulating the number of numbers n with a property $\mathrm{P}(a, b)$, one needs to factorize a great many of so-called double length integers. We have disposed of this task in the following manner. *)

Step 1° We represent a natural number n as $A \cdot 10^9 + B$, where A, B are integers in simple precision.

Step 2° We test the divisibility of n by the numbers 2, 3, 5 and 7.

Step 3° By making use of the increment table with respect to the number $2 \cdot 3 \cdot 5 \cdot 7 = 210$, we produce a sequence of positive integers which are relatively prime to 210.

Step 4° We test the divisibility of n by each member p of this sequence, by evaluating $n = A \cdot 10^9 + B$ to the modulus p.

*) This applies also in the computation of the 'average' of $f(n)$ for the case of $a = 2$, $b = 3$, in particular (cf. § 2 below).

Number of natural numbers $n \leqq x$ with $P(2, b)$, $2 < b \leqq 21$, $(2, b) = 1$, for $x = 2^m E4$, $m = 0(1)9$

b \ x	1E4	2E4	4E4	8E4	16E4	32E4	64E4	128E4	256E4	512E4	largest n found
3	5	5	5	5	5	5	5	5	5	5	8
5	34	34	34	34	34	34	34	34	34	34	507
7	44	46	46	46	46	47	47	47	47	48	3305430
9	84	87	94	94	95	97	98	98	99	99	1594135
11	99	113	123	128	142	147	153	157	158	163	4373430
13	122	135	147	169	172	179	182	185	193	196	4534467
15	321	404	475	551	671	737	819	942	987	1039	5104333
17	337	393	492	529	576	627	661	689	723	751	4946795
19	163	187	208	240	260	297	334	377	416	476	5119215
21	393	460	496	560	657	703	770	898	950	1026	5074165

Number of natural numbers $n \leqq x$ with $P(3, b)$, $3 < b \leqq 21$, $(3, b) = 1$, for $x = 2^m E4$, $m = 0(1)9$

b \ x	1 E 4	2 E 4	4 E 4	8 E 4	16 E 4	32 E 4	64 E 4	128 E 4	256 E 4	512 E 4	largest n found
4	3	3	3	3	3	3	3	3	3	3	5
5	4	4	4	4	4	4	4	4	4	4	24
7	15	15	15	15	15	15	15	15	15	15	352
8	12	12	12	12	12	12	12	12	12	12	497
10	42	42	42	42	42	42	42	42	42	42	4409
11	25	25	25	25	25	25	25	25	25	25	2160
13	73	76	78	80	81	81	83	83	83	83	537808
14	57	63	67	70	77	81	83	85	86	87	4696235
16	45	45	45	45	46	47	47	47	47	47	202545
17	104	114	126	134	140	147	153	157	161	169	4812192
19	68	71	75	82	84	92	96	99	103	106	4502550
20	151	175	190	221	233	259	278	292	314	324	4893031

Number of natural numbers $n \leqq x$ with $P(4, b)$, $4 < b \leqq 21$, $(4, b) = 1$, for $x = 2^m E\,4$, $m = 0(1)8$

$b \backslash x$	1 E 4	2 E 4	4 E 4	8 E 4	16 E 4	32 E 4	64 E 4	128 E 4	256 E 4	largest n found
5	1	1	1	1	1	1	1	1	1	2
7	5	5	5	5	5	5	5	5	5	20
9	13	13	13	13	13	13	13	13	13	143
11	12	12	12	12	12	12	12	12	12	417
13	14	14	14	14	14	14	14	14	14	155
15	49	51	52	53	53	54	54	54	54	176879
17	29	29	30	30	31	31	31	31	31	81870
19	71	74	75	76	76	76	76	76	76	46410
21	98	113	113	119	120	122	125	125	126	2448095

Number of natural numbers $n \leqq x$ with $P(5, b)$, $5 < b \leqq 21$, $(5, b) = 1$, for $x = 2^m E4$, $m = 0(1)8$

b \ x	1 E 4	2 E 4	4 E 4	8 E 4	16 E 4	32 E 4	64 E 4	128 E 4	256 E 4	largest n found
6	0	0	0	0	0	0	0	0	0	—
7	0	0	0	0	0	0	0	0	0	—
8	3	3	3	3	3	3	3	3	3	5
9	2	2	2	2	2	2	2	2	2	40
11	4	4	4	4	4	4	4	4	4	24
12	12	12	12	12	12	12	12	12	12	721
13	6	6	6	6	6	6	6	6	6	66
14	9	9	9	9	9	9	9	9	9	165
16	9	9	9	9	9	9	9	9	9	663
17	23	24	24	24	24	24	24	24	24	17540
18	25	25	25	25	25	25	25	25	25	3731
19	15	15	15	15	16	16	16	16	16	89190
21	47	50	51	52	52	52	52	52	52	67958

2. The mean value of $f(n)$. P. Erdös has proved in [1] among other things that in the case of $a=1$, $b=2$ the upper limit

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n\leq N} (f(n;\ 1,2))^k$$

is finite for every natural k. For $k=1$ it is not difficult to show that one has

$$(1) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} f(n;\ 1,\ b) = \frac{1}{\log b}$$

for every fixed integral $b\geqq 2$. Indeed, (1) is a simple consequence of the prime number theorem in the form

$$(2) \qquad \pi(x) = \int_2^x \frac{du}{\log u} + R(x) \qquad (x\geqq 2),$$

where $\pi(x)$ denotes as usual the number of prime numbers not exceeding x and where $R(x)$ is the remainder term,

$$R(x) = O\left(\frac{x}{(\log x)^A}\right)$$

for any fixed $A>1$. Thus, after a straightforward computation with (2), we have for all sufficiently large values of N

$$\begin{aligned}\frac{1}{N} \sum_{n\leq N} f(n;\ 1,\ b) &= \frac{1}{N} \sum_k{}' \pi(N-b^k) \\ &= \frac{1}{\log b} \frac{1}{N} \int_b^{N-b} \frac{\log v}{\log(N-v)} dv + O\left(\frac{1}{\log N}\right) \\ &= \frac{1}{\log b} + O\left(\frac{1}{\log N}\right),\end{aligned}$$

where in the summation $\sum_k'$ k ranges over the integers lying in the interval

$$1\leqq k<(\log N)/\log b.$$

For the general case of $1\leqq a<b$, G. C. D. $(a,\ b)=1$ we may expect that the mean value

$$(3) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} f(n;\ a,\ b) = \frac{a}{\phi(a)\log a} \log\left(\frac{\log b}{\log(b/a)}\right)$$

exists, where $\phi(a)$ is the Euler totient function. As a matter of fact, if we denote by $\pi(x;\ q,\ h)$ the number of primes $p\leqq x$ satisfying the congruence $p \equiv h \pmod q$, then we have

$$(4) \qquad \frac{1}{N} \sum_{n \leq N} f(n; a, b) = \frac{1}{N} \sum_{k} \pi(a^k N - b^k; a^k, -b^k),$$

where in the summation $\sum_k$ k runs through the integers in the interval

$$(5) \qquad 1 \leq k < (\log N) / \log (b/a).$$

However, we have not succeeded in rigorously deducing (3) from (4) if $a > 1$; the reason why we failed to prove (3) is due to the fact that the maximal value of $a^k N - b^k$ when k varies over the interval (5) is approximately equal to

$$(a^{-\kappa} - b^{-\kappa}) N^{\lambda}$$

with

$$\kappa = \frac{\log ((\log b)/\log a)}{\log (b/a)}, \qquad \lambda = \frac{\log b}{\log (b/a)},$$

the order of magnitude of which is strictly higher than N if $a > 1$.

Here we shall give a table of values of the average

$$\bar{f}(N; a, b) = \frac{1}{N} \sum_{n \leq N} f(; a, b)$$

for several pairs of integers a and b with $1 \leq a < b$, G. C. D. $(a, b) = 1$. In the table, the figures corresponding to $N = \infty$ are the mean values of $f(n; a, b)$ expected from the formulae (1) and (3). We note that (1) is the limiting case of (3) when $a = 1$.

Values of $\bar{f}(N; a, b)$ for some pairs of integers a, b

N	$\bar{f}(N; 1, 2)$	$\bar{f}(N; 1, 3)$	$\bar{f}(N; 1, 4)$
10E4	1.418090	0.880360	0.689240
20E4	1.418795	0.875425	0.683175
30E4	1.416786	0.881757	0.683590
40E4	1.419785	0.878857	0.692210
50E4	1.414068	0.872848	0.693326
60E4	1.418842	0.877855	0.692038
70E4	1.420786	0.882207	0.689506
80E4	1.420425	0.884135	0.686960
90E4	1.418361	0.884438	0.684220
100E4	1.414701	0.883383	0.681085
110E4	1.416012	0.882210	0.683137
120E4	1.419858	0.881127	0.687672
130E4	1.421067	0.879177	0.690288
140E4	1.421782	0.877436	0.692331
150E4	1.421661	0.875503	0.693688
160E4	1.420724	0.873776	0.694449
170E4	1.419985	0.877481	0.695131
180E4	1.418607	0.879735	0.695365
190E4	1.417529	0.881686	0.695650
200E4	1.416008	0.883032	0.695592
210E4	1.414594	0.884083	0.695436
220E4	1.417205	0.884843	0.695197
230E4	1.418934	0.885341	0.694883
240E4	1.419980	0.885574	0.694378
250E4	1.420706	0.885671	0.693840
260E4	1.421500	0.885842	0.693433
270E4	1.421911	0.885810	0.692891
∞	1.442695	0.910239	0.721348

N	$\bar{f}(N; 2, 3)$	$\bar{f}(N; 4, 9)$
1E4	2.778100	1.334000
2E4	2.785550	1.341300
3E4	2.790000	1.346833
4E4	2.785500	1.342700
∞	2.876065	1.438033

REFERENCES

[1] P. ERDÖS: On integers of the form 2^k+p and some related problems. Summa Brasil. Math. **2** (1950), 113—123.
[2] S. UCHIYAMA and M. YORINAGA: Notes on a conjecture of P. Erdös. I. Math. J. Okayama Univ. **19** (1977), 129—140.

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(Received September 19, 1977)