SIMPLY CONNECTED SMOOTH 4-MANIFOLDS WHICH ADMIT NONTRIVIAL SMOOTH S^1 ACTIONS

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1. Introduction

In this paper, we study the diffeomorphism types of the closed simply connected 4-dimensional smooth manifolds which admit non-trivial smooth circle group actions.

Let S^1 denote the circle group, the multiplicative group consisting of the complex numbers with absolute value 1. Let CP^2 be the complex projective plane with the usual orientation which gives $[+1]$ as the intersection form on $H_2(CP^2: Z)$, and let $-CP^2$ be CP^2 with the opposite orientation.

Our main result is the following,

Theorem. *Let M be an oriented simply connected closed smooth 4-manifold. If M has a non-trivial smooth S^1 action, then M is orientation preservingly diffeomorphic to the connected sum*

$$\Sigma^4 \sharp k(CP^2) \sharp m(-CP^2) \sharp n(S^2 \times S^2)$$

for some integers k, m and $n \geqq 0$, where Σ^4 denotes a homotopy 4-sphere and S^2 is the 2-sphere.

Remark: In [6], P. Orlik and F. Raymond proved that if the 2-torus $T^2 = S^1 \times S^1$ acts effectively on a closed simply connected 4-manifold M, then M is an equivariant connected sum of the copies of CP^2, $-CP^2$, $S^2 \times S^2$ and $CP^2 \sharp (-CP^2)$ with some effective T^2 actions. The decomposition in the above Theorem is not equivariant in general.

§ 2 ~ § 5 will be devoted to the proof of the above Theorem. In § 6, some S^1 actions on the 4-sphere S^4 and the connected sum $k\,CP^2 \sharp m(-CP^2) \sharp n(S^2 \times S^2)$ will be given together with some other comments.

Throughout this paper, S^1 manifolds mean smooth S^1 manifolds, and two S^1 manifolds M and M' are called equivalent if they are equivariantly diffeomorphic to each other.

2. Preliminary lemmas and definitions

Let M be a simply connected 4-dimensional closed S^1 manifold. We always assume that the S^1 action is effective and M is endowed with some invariant Riemannian metric. For a point $x \in M$, the subgroup of S^1 defined by $G_x = \{g \in S^1 \mid gx = x\}$ is called the isotropy group at x, where $g\,x$ denotes the point in M obtained by transforming x by g. Let Z_m denote the subgroup of S^1 of order $m \geqq 2$. Put

$$F = \{x \in M \mid G_x = S^1\}$$
$$F(Z_m) = \{x \in M \mid G_x \supset Z_m\} \qquad \text{and}$$
$$M_0 = M - (\cup_m F(Z_m)).$$

F is the fixed point set and $F \subset F(Z_m)$ for each $m \geqq 2$.

Let M^* be the orbit space of M and let $\pi: M \longrightarrow M^*$ be the projection. Put $M_0^* = \pi(M_0)$. Then $\pi \mid M_0: M_0 \longrightarrow M_0^*$ is a principal S^1 bundle and M_0^* has a natural smooth structure.

Lemma 2.1. *M^* is a simply connected topological 3-manifold and the connected components of its boundary ∂M^* (possibly empty) correspond in one-one way with the 2-dimensional connected components of F by π.*

Proof. Since M is simply connected and $\pi_*: \pi_1(M) \longrightarrow \pi_1(M^*)$ is onto ([2] p. 91), M^* is simply connected. By the differentiable slice theorem ([2] p. 171), at each $x \in M$, there is a slice S_x which is diffeomorphic to the 3-disc D^3 or the 4-disc D^4 according as G_x is finite or not. This means that G_x acts on S_x linearly and the twisted product space $S^1 \times_{G_x} S_x$ with the S^1 action defined by $g[g_0, u] = [gg_0, u]$ ($g, g_0 \in S^1$ and $u \in S_x$) is equivariantly diffeomorphic to an invariant closed neighborhood of the orbit S^1x. Then the orbit space S_x/G_x is homeomorphic to a closed neighborhood of $\pi(x)$. Now it is easy to see that the orbit space S_x/G_x is homeomorphic to the 3-disc D^3 and the 2-dimensional fixed point set projects to the boundary by π. q. e. d.

Each connected component of F and $F(Z_m)$ is an orientable surface or a point.

Lemma 2.2. (1) *Each 2-dimensional connected component of F is S^2.* (2) *Each 2-dimensional connected component of $F(Z_m)$ is S^2 for each $m \geqq 2$.*

Proof. (1) By Lemma 2.1 and the Poincare duality, $H_1(M^*: Z) = 0$. and $H_2(M^*, \partial M^*: Z) = H^1(M^*: Z) = 0$. Hence $H_1(\partial M^*: Z) = 0$ by the homology exact sequence of the pair $(M^*, \partial M^*)$. Again by Lemma 2.1, each connected 2-dimensional component of F is identified with one of the 2-dimensional connected components of ∂M^*, and this proves (1). (2) Let p be a prime number $\geqq 2$ dividing m. Since Z_p acts trivially on $H^*(M: Z_p)$ and $H^{odd}(M: Z)=0$ (here $Z_p=Z/pZ$), by a theorem of A. Borel ([1] XII), the following equalities hold

$$\dim_{Z_p} H^*(M: Z_p) = \dim_{Z_p} H^*(F(Z_p): Z_p) \quad \text{and}$$
$$\dim_{Q} H^*(M: Q) = \dim_{Q} H^*(F: Q),$$

where Q is the field of the rational numbers and $\dim_k$ means the dimension of the vector space over k, k $= Z_p$ or Q. Now let E be a 2-dimensional connected component of $F(Z_m)$ ($\subset F(Z_p)$) which is not contained in F. Then E is an orientable surface on which S^1 acts non-trivially, hence it is S^2 or T^2 (the 2-torus). Since $H^*(M: Z)$ and $H^*(F(Z_p): Z)$ are torsion free, the above equalities imply that $\dim_Q H(F(Z_p): Q) = \dim_Q H(F: Q)$. If E is T^2, then this equality does not hold since any non-trivial S^1 action on T^2 has no fixed point. This proves (2) by (1).

q. e. d.

Remark : The equality of A. Borel in the above proof implies that F is not empty in our situation.

Now put for each integer $m \geqq 2$,

$F(m)$ = the disjoint union of the connected components $\{E\}$ of $F(Z_m)$ on which S^1 acts non-trivially with $G_x = Z_m$ or S^1 for each $x \in E$.

Then by Lemma 2. 2, $F(m)$ is a disjoint union of 2-spheres and $F(Z_m) = F \cup (\bigcup_{m|m'} F(m'))$. By Lemma 2. 1, M is a punctured homotopy 3-sphere or a homotopy 3-sphere and $\pi(\bigcup_m F(m))$ is a disjoint union of finite circles and arcs in the interior of M.

3. Two processes of equivariant decompositions

Let M_1 and M_2 be oriented closed S^1 manifolds. Let p_1 and p_2 be fixed points in M_1 and M_2 respectively. Suppose that there is a closed disc D_i in M_i which is a closed invariant neighborhood of $p_i (i=1, 2)$ and there is an orientation reversing equivariant diffeomorphism $f: D_1 \longrightarrow D_2$. Then the equivariant connected sum of M_1 and M_2 at p_1 and p_2, $M_1 \# M_2$, is defined as the S^1 manifold obtained from the disioint union $(M_1 - \mathring{D}_1) \cup (M_2 - \mathring{D}_2)$ by identifying each point $x \in \partial(M_1 - \mathring{D}_1) = \partial D_1$ with $f(x) \in \partial(M_2 - \mathring{D}_2) = \partial D_2$ (here $\mathring{D}$ denotes the interiorof D_i, $i = 1, 2$). The orientation of $M_1 \# M_2$ is the one compatible with the orientations of M_1 and M_2.

Now let M be a closed oriented 4-dimensional S^1 manifold. Under suitable conditions, M can be be decomposed as an equivariant connected sum such as $M' \# \pm CP^2$ or $M' \# (S^2 \times S^2)$ or $M' \# CP^2 \# (-CP^2)$ as follows.

Case (I) $M \approx M' \# \pm CP^2$

Let S be an invariant 2-sphere smoothly embedded in M. Let ν be the equivariant normal bundle of S in M. We identify the disc bundle of ν, $D(\nu)$, with a closed invariant neighborhood of S in M. Suppose that the self-intersection number of S, $S \cdot S$, is ± 1. Then the sphere bundle of ν, $S(\nu) = \partial D(\nu)$ is diffeomorphic to the 3-sphere S^3. Now any smooth S^1 action on S^3 is equivalent to some linear S^1 action on S^3 ([4][5]), hence it can be extended to the action on D^4 linearly. Let M' be the S^1 manifold obtained from the disjoint union $\overline{M - D(\nu)} \cup D^4$ by identifying $\partial\,\overline{(M - D(\nu))} = \partial(D(\nu))$ and ∂D^4 by an equivariant diffeomorphism of S^3, and let J be the S^1 manifold obtained from the disjoint union $D(\nu) \cup D^4$ by identifying $\partial D(\nu)$ and ∂D^4 by an equivariant diffeomorphism of S^3. Since $\pi_0\,(\mathrm{Diff}\ S^3) = 0$ ([3]), the diffeomorphism classes of M and J are unique. We orient M' (resp. J) by extending the orientation of $\overline{M - D(\nu)}$ (resp. $D(\nu)$). Then M' and J have the structures of oriented S^1 manifolds naturally. If $S \cdot S = 1$ (resp. -1), then $D(\nu)$ is the disc bundle of the Hopf bundle (resp. the conjugate bundle of the Hopf bundle) over S^2, and J is orientation preservingly diffeomorphic to CP^2 (resp. $-CP^2$). Now it is clear that M is equivariantly dffeomorphic to the equivariant connected sum $M \# J = M \# \pm CP^2$.

Case (II) $M \approx M' \# (S^2 \times S^2)$ or $M' \# CP^2 \# (-CP^2)$

Let S_1 and S_2 be two invariant 2-spheres smoothly embedded in M. Let ν_i be the equivariant normal bundle of S_i in M, and as before we identify the disc bundle $D(\nu_i)$ with a closed invariant tubular neighborhood of S_i in $M (i=1, 2)$. Suppose that S_1 and S_2 intersect mutually and transversally at a fixed point $p = S_1 \cap S_2$, and suppose that the self intersection number of S_1, $S_1 \cdot S_1$, is equal to 0. $K = D(\nu_1) \cup D(\nu_2)$ is considered an invariant submanifold in M which is a closed invariant neighborhood of $S_1 \cup S_2$. K is a so called equivariant plumbing of $D(\nu_1)$ and $D(\nu_2)$ at p, and it has a natural smooth structure (by smoothing the corner). $H_2(K : \boldsymbol{Z})$ is generated by the classes of S_1 and S_2 with suitable orientations, and the intersection matrix of $H_2(K : \boldsymbol{Z})$ has the form $\begin{bmatrix} 0 & \varepsilon \\ \varepsilon & \alpha \end{bmatrix}$ ($\varepsilon = \pm 1$), where $\alpha = S_2 \cdot S_2$, the self intersection number of S_2. Hence the boundary ∂K is diffeomorpfic to S^3. Therefore the same process as in Case (I) can be applied. That is, the S^1 action on S^3 can be extended to D^4 linearly and we obtain an S^1 manifold M' (resp. K') from the disjoint union $\overline{M - K} \cup D^4$ (resp. $K \cup D^4$) by identifying $\partial\,\overline{M - K}$ (resp. ∂K) and ∂D^4 by an equivariant diffeomorphism of S^3. As in Case (I), the diffeomorphism classes of M' and K' are unique. We orient M' and K' by extending the orientations of $\overline{M-K}$ and K respectively. Clearly M is equivariantly diffeomorphic to the equivariant connected sum, $M' \# K'$. Now K' is diffeomorphic to $S^2 \times S^2$ or $CP^2 \# (-CP^2)$ according as $\alpha = S_2 \cdot S_2$ is even or odd. To see this, let D_α be the disc bundle of the complex line bundle over S^2 with $\langle c, [S^2] \rangle = \alpha$, where c is the first Chern class of this bundle and $[S^2]$ is the fundamental class of S^2 and $\langle\ ,\ \rangle$ denotes the Kronecker pairing. Let B_α be the double of D_α. Then B_α is a S^2 bundle over S^2 such that each fibre is the double of each fibre of D_α. B_α is diffeomorphic to $S^2 \times S^2$ or $CP^2 \# (-CP^2)$ according as α is even or odd since $\pi_1(SO(3)) = Z_2$. Let E be the zero section of D_α and let F be a fibre of the S^2 bundle B_α. Then D_α is the normal disc bundle of E in B_α and the normal disc bundle of F in B_α, D_F, is a trivial D^2 bundle. Now $D_\alpha \cup D_F \subset B$ is a plumbing of these disc bundle and $\overline{B - D_\alpha \cup D_F}$ is diffeomorphic to the 4-disc D^4. Hence $D_\alpha \cup D_F$ is diffeomorphic to the above K and B_α is diffeomorphic to the above K'. Consequently M is equivariantly diffeomorphic to the S^1 manifolds of the form $M' \# (S^2 \times S^2)$ or $M \# CP^2 \# (-CP^2)$.

4. Circular system

Definition 4.1. *A circular system of length* k ($\geqq 2$) in a 4-dimensional S^1 manifold M is a set consisting of isolated fixed points $\{p_1, p_2, \cdots, p_k\}$ and invariant 2-spheres smoothly embedded in M, $\{S_1, S_2, \cdots, S_k\}$, such that

1) S_i intersects transversally at p_{i+1} with S_{i+1} $(1 \leqq i \leqq k-1)$ and S_k intersects transversally at p_1 with S_1,

2) each S_i is a connected component of $F(m_i)$ for some $m_i \geqq 2$ except at most one S_i on which the S^1 action is free outside the fixed point set, and

3) $p_i \neq p_j$, and $S_i \neq S_j$ for $1 \leqq i < j \leqq k$.

We write the circular system defined as above in the form $L = p_1 S_1 p_2 S_2 \cdots p_k S_k p_1$.

Lemma 4.2. *If* $\bigcup_{m \geqq 2} F(m)$ *is not empty, then there is a circular system.*

Proof. Suppose that $F(m)$ is not empty for some $m \geqq 2$. Let S be a connected component of $F(m)$. S has two fixed points in it, say p and p'. Denote this situation by pSp'. If there is a 2-sphers $S'(\neq S)$ $\subset F(m')$ $(m' \geqq 2)$ containing p' (resp. p) and p'' as the two fixed points, then we obtain a sequence $pSp'S'p''$ (resp. $p''S'pSp'$). Note that m and m' are mutually prime and S intersects transversally at p' (resp. p) with S'. It may be happen that $p=p''$ (resp. $p'=p''$). In this case $S\cap S'=p\cup p'$ and S intersects transversally at p and p' with S'. Repeating this process we obtain a sequence $p_1S_1p_2S_2\cdots p_{k-1}S_{k-1}p_k$ such that $p_i\neq p_j$, $S_i\neq S_j$ for each $1\leqq i<j\leqq k-1$ and $S_i\subset F(m_i)$ $(m_i\geqq 2)$ and S_{i-1} intersects transversally at p_i with S_i $(2\leqq i\leqq k-1)$. Now suppose that this sequence is maximal, that is, there is no other sequence strictly containing this sequence and having the above properties. Then the two cases may happen, $p_1=p_k$ or $p_1\neq p_k$. If $p_1=p_k$, then the above sequence is a circular system. Assume that $p_1\neq p_k$. Let D (resp. D') be an invariant 2-disc smoothly embedded in M which intersects transeversally at its center p_1 (resp. p_k) with S_1 (resp. S_{k-1}). We assume that D and D' are so small that $D\cap D'=\phi$ and $(D\cap D')\cap(\cup_m F(Z_m))=p_1\cup p_k$. S^1 acts on $D-p_1$ and $D-p_k$ freely, that is $(D-p_1)\cup(D-p_k)\subset M_0$ (§ 2). $\pi(D)$ and $\pi(D')$ in M^* are two disjoint arcs with one endpoint $\pi(p^1)$ and $\pi(p_k)$ respectively. Joining the another endpoints of their arcs by a smooth curve in M_0^*, we obtain an smoothly embedded arc γ in M^* such that the endpoints of γ are $\pi(p_1)\cup\pi(p_k)$ and $\gamma\text{-}(\pi(p_1)\cup\pi(p_k))\subset M_0^*$. Then $S_k=\pi^{-1}(\gamma)$ is an invariant 2-sphere smoothly embedded in M and it intersects transversally at p_1 and p_k with S_1 and S_{k-1} respectively. Hence we obtain a circular system $p_1S_1p_2\cdots S_{k-1}p_kS_kp_1$. This proves the Lemma. q. e. d.

Let $L=p_1S_1p_2S_2\cdots p_kS_kp_1$ be a circular system in M. We choose and fix an orientation o_1 of S_1, and choose an orientation o_i of S_i so that the intersection number of S_{i-1} and S_i at p_i may be $+1$ $(2\leqq i\leqq k)$. Then o_1 and o_k determine the intersection number $\varepsilon(=\pm 1)$ of S_1 and S_k at p_1. Note that ε does not depend on the choice of o_1. The self intersection number $S_i\cdot S_i=\alpha_i$ is independent of o_i $(1\leqq i\leqq k)$. We write these situations as $L=L(\alpha_1, \alpha_2, \cdots, \alpha_k; \varepsilon)$. The rest of this section is devoted to the proof of Lemma 4. 5 given at the last of this section.

A smooth S^1 action on a smooth manifold induces a linear S^1 action on the tangent space at a fixed point by derivation. We call this S^1 action the tangent representation at the fixed point. Denote the standard complex representation of S^1, $S^1=U(1)$ (the unitary group of degree 1), by t and denote its k-th tensor product by t^k. Choose a suitable complex structure on the tangent plane of S_i at p_i which gives the same orientation as o_i, we may write the representation of S^1 on the tangent plane at p_i in the form $t^{\delta_i m_i}$ $(\delta_i=\pm 1)$ $(1\leqq i\leqq k)$ (note that $S_i\subset F(m_i)$). Put $n_i=\delta_i m_i (1\leqq i\leqq k)$. Then the tangent representation of S^1 on TM_{p_i}

is $t^{-\varepsilon n_k} + t^{n_1}$ $(i = 1)$ or $t^{-n_{i-1}} + t^{n_i}$ $(2 \leqq i \leqq k)$ under a complex structure on TM_{p_i} which gives the same orientation with that of M on TM_{p_i} and contain the tangent plane of S_i at p_i with the above complex structure as a complex subspace.

Lemma 4. 3. *The following relations hold, for* $k \geqq 3$,

$$-\varepsilon n_k + \alpha_1 n_1 - n_2 = 0$$
$$-n_{i-1} + \alpha_i n_i - n_{i+1} = 0 \quad (2 \leqq i \leqq k-1)$$
$$-n_{k-1} + \alpha_k n_k - \varepsilon n_1 = 0$$

and for $k = 2$,

$$\alpha_1 n_1 - (\varepsilon + 1) n_2 = 0$$
$$\alpha_2 n_2 - (\varepsilon + 1) n_1 = 0.$$

Proof. Suppose that S^1 acts on the complex projective line CP^1 by the equation $g[z_0, z_1] = [z_0, g^m z_1]$ $([z_0, z_1] \in CP^1,\ m \in \boldsymbol{Z})$. Let ξ be an S^1 complex line bundle over the S^1 space CP^1 with $\langle c_1(\xi), [CP^1] \rangle = \alpha \in \boldsymbol{Z}$, where $c_1(\xi)$ is the first Chern class of ξ and $[CP^1]$ is the fundamental class. Let $E(\xi)$ be the total space of ξ and let $p: E(\xi) \longrightarrow CP^1$ be the projection. Then the fibres $p^{-1}([1, 0])$ and $p^{-1}([0, 1])$ are invariant by the S^1 action on $E(\xi)$ and S^1 acts on these fibres linearly. Assume that the representations of S^1 on $p^{-1}([1, 0])$ and $p^{-1}([0, 1])$ are t^{a_0} and t^{a_1} respectively under the complex structures on these fibres. Then the following equality holds,

$(*)\quad a_1 - a_0 = m\alpha\,.$

Proof of $(*)$. Let $ES^1 \longrightarrow BS^1$ be the universal S^1 bundle. Let $ES^1 \times_{S^1} CP^1$ be the twisted product space. We consider the equivariant cohomology $H^*(ES^1 \times_{S^1} CP^1 : \boldsymbol{Z})$([2] Chap VII). Let $\rho: ES^1 \times_{S^1} CP^1 \longrightarrow BS^1$ be the projection, then this is a fibre bundle with fibre CP^1. Since $H^{odd}(CP^1 : \boldsymbol{Z}) = H^{odd}(BS^1 : \boldsymbol{Z}) = 0$, the cohomology Serre spectral sequence of this fibre bundle collapse, and CP^1 is totally non-homologoues to zero in $ES^1 \times_{S^1} CP^1$. Let S^3 be the unit sphere in the complex plane, $S^3 = \{(z_0, z_1) \in \boldsymbol{C} + \boldsymbol{C} \mid |z_0|^2 + |z_1|^2 = 1\}$, and $q: S^3 \longrightarrow CP^1$ $((z_0, z_1) \longrightarrow [z_0, z_1])$ be the canonical projection. Let S^1 act on S^3 by the equation $g(z_0, z_1) = (z_0, g^m z_1)$. Then q is an S^1 circle bundle over the S^1 space CP^1, and $q_G: ES^1 \times_{S^1} S^3 \longrightarrow ES^1 \times_{S^1} CP^1$ is a circle bundle. Let $c(\in H^2(ES^1 \times_{S^1} CP^1 : \boldsymbol{Z}))$ be the first Chern class of this circle bundle. Let u be the first Chern class of the universal bundle $ES^1 \longrightarrow BS^1$. Then c and $\rho^*(t) = u'$ generate $H^2(ES^1 \times_{S^1} CP^1 : Z)$. Let $j_0 : BS^1 = BS^1 \times [1, 0] \subset ES^1 \times_{S^1} CP^1$ and $j_1 : BS^1 = BS^1 \times [0, 1] \subset ES^1 \times_{S^1} CP^1$ be the inclusions. Then $j_0^*(c)$ and $j_1^*(c)$ are the first Chern classes of the restrictions of q_G to $BS^1 \times [1, 0]$ and $BS^1 \times [0, 1]$ respectively. Hence $j_0^*(c) = 0$ and $j_1^*(c) = mu$ in $H^2(BS^1 : Z)$. Now consider the complex line buncle $p_G : ES^1 \times_{S^1} E(\xi) \longrightarrow ES^1 \times_{S^1} CP^1$. Let $c_1(p_G)$ be the first Chern class of p_G. Then $c_1(p_G) = rc + su'$ for some integers r and s. By the inclusion $CP^1 (= e \times CP^1) \subset ES^1 \times_{S^1} CP^1$ $(e \in ES^1)$, $c_1(p_G)$ corresponds to the first Chern class of p. This implies that $r = \alpha$, and $c_1(p_G) = \alpha c + su'$. The restrictions $j_0^*(c_1(p_G))$ and $j_1^*(c_1(p_G))$ are the first Chern classes of the restrictions of the bundle p_G to $BS^1 \times [1, 0]$ and $BS^1 \times [0, 1]$ respectively. Hence $j_0^*(c_1(p_G)) = a_0 u$ and $j_1^*(c_1(p_G)) = a_1 u$. On the other hand $j_0^*(\alpha c + su') = j_0^*(c) + sj_0^*(u') = su$ and $j_1^*(\alpha c + su') = j_1^*(c) + sj_1^*(u') = m\alpha u + su$. From these equations, we obtain $a_1 - a_0 = m\alpha$. q. e. d.

We proceed to the proof of Lemma 4. 3. First suppose that $k \geqq 3$ and $2 \leqq i \leqq k-1$. Let ν_i be the equivariant normal bundle of S_i in M. ν can be considered as an S^1 complex line bundle over S_i such that the complex struture and the orientation o_i of S_i give the same orientation as that of M. Then the representations of S^1 at the fibres over p_i and p_{i+1} are $t^{-n_{i-1}}$ and $t^{n_{i+1}}$ respectively. Now let S^1 act on CP^1 by the equation $g[z_0, z_1] = [z_0, g^{n_i} z_1]$. Let $l = \{l(s) = [1-s, s] \in CP^1 \ (0 \leqq s \leqq 1)\}$ be an arc in CP^1 joining $[1, 0]$ and $[0, 1]$. Let $h : l \longrightarrow S_i$ be a smooth map such that $h([1, 0]) = p_i$ and $h([0, 1]) = p_{i+1}$ and the image $h(l)$ meets with each orbit in S_i at exactly one point. Using the S^1 actions, we obtain an equivariant map $\widetilde{h} : CP^1 \longrightarrow S_i$ of degree 1 extending h. The induced bundle $\widetilde{h}^* \nu_i$ is an S^1 complex line bundle over CP^1, and the representations of S^1 on the fibres over $[1, 0]$ and $[0, 1]$ are $t^{-n_{i-1}}$ and $t^{n_{i+1}}$ respectively, and $\langle c_1(\widetilde{h}^* \nu_i), [CP^1] \rangle = S_i \cdot S_i = \alpha_i$, where $c_1(\widetilde{h}^* \nu_i)$ is the first Chern class of $\widetilde{h}^* \nu_i$. Hence $n_{i+1} + n_{i-1} = \alpha_i n_i$ by (*). For $i = 1, k$ $(k \geqq 3)$ or $k = 2$, all is the same. q. e. d.

Now assume $k \geqq 3$. Considering the relations in the above Lemma 4. 3 as a linear equations of $n_1, n_2, \cdots, n_k$, we have an equation

$$\det \begin{vmatrix} \alpha_1 & -1 & & & & -\varepsilon \\ -1 & \alpha_2 & -1 & & 0 & \\ & -1 & \alpha_3 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & -1 \\ -\varepsilon & & & & -1 & \alpha_k \end{vmatrix} = 0$$

here det denotes the determinant of the matrix.

Lemma 4.4 *If* $k \geqq 3$ *and* $|\alpha_i| \geqq 2$ *for all* $1 \leqq i \leqq k$, *then* $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 2$, $\varepsilon = 1$, *or* $\alpha_1 = \alpha_2 = \cdots = \alpha_k = -2$, $\varepsilon = (-1)^k$.

Proof. We denote the above determinant by $A(\alpha_1, \cdots, \alpha_k; \varepsilon)$. Put

$$B(\alpha_1, \cdots, \alpha_k) = \det \begin{vmatrix} \alpha_1 & -1 & & & \\ -1 & \alpha_2 & -1 & & \\ & -1 & \alpha_3 & -1 & 0 \\ & & \ddots & \ddots & \ddots \\ & 0 & & \ddots & -1 \\ & & & -1 & \alpha_k \end{vmatrix}$$

Then a simple calculation shows the following equation, $A(\alpha_1, \cdots, \alpha_k; \varepsilon) = B(\alpha_1, \cdots, \alpha_k) - B(\alpha_2, \cdots, \alpha_{k-1}) - 2\varepsilon$. Now for a sequence of the integers $a_1, \cdots, a_r$ ($|a_i| \geqq 2$, $i = 1, \cdots, r$), we write the finite continued fractions as $b(a_1, \cdots, a_r) = a_r + \frac{-1}{a_{r-1}} + \cdots + \frac{-1}{a_1}$, and $b'(a_1, \cdots, a_r) = a_1 + \frac{-1}{a_2} + \cdots + \frac{-1}{a_r}$. Then $|b(a_1, \cdots, a_r)| > |a_r| - 1$, $|b'(a_1, \cdots, a_r)| > |a_1| - 1$ and if $a_r a_{r-1} < 0$, then $|b(a_1, \cdots, a_r)| > 2$ and if $a_1 a_2 < 0$, then $|b'(a_1, \cdots, a_r)| > 2$. Now it is easy to see that $B(\alpha_1, \cdots, \alpha_k) = \prod_{r=1}^{k} b(\alpha_1, \cdots, \alpha_r) = \prod_{r=1}^{k} b'(\alpha_r, \cdots, \alpha_k)$.
Hence $B(\alpha_1, \cdots, \alpha_k) = b(\alpha_1, \cdots, \alpha_k)\, b'(\alpha_1, \cdots, \alpha_{k-1})\, B(\alpha_2, \cdots, \alpha_{k-1})$. Hence $A(\alpha_1, \cdots, \alpha_k; \varepsilon) = (b(\alpha_1, \cdots, \alpha_k)\, b'(\alpha_1, \cdots, \alpha_{k-1}) - 1)\, B(\alpha_2, \cdots, \alpha_{k-1}) - 2\varepsilon = (b(\alpha_1, \cdots, \alpha_k)\, b'(\alpha_1, \cdots, \alpha_{k-1}) - 1) \prod_{r=2}^{k-1} b(\alpha_2, \cdots, \alpha_r) - 2\varepsilon$. By the above observation, we see that if $|\alpha_i| \geqq 3$ for some $1 \leqq i \leqq k$, or $\alpha_i \alpha_{i+1} < 0$

for some $1 \leqq i \leqq k-1$, then $A(\alpha_1, \cdots, \alpha_k; \varepsilon)$ cannot be 0. Hence $A(\alpha_1, \cdots, \alpha_k; \varepsilon) = 0$ implies that $\alpha_1 = \cdots = \alpha_k = 2$ or $\alpha_1 = \cdots = \alpha_k = -2$. Now $B(\underbrace{2, \cdots, 2}_{r}) = r+1$ and $B(\underbrace{-2, \cdots, -2}_{r}) = (-1)^r(r+1)$, so that $A(\underbrace{2, \cdots, 2}_{k}; \varepsilon) = 2 - 2\varepsilon$ and $A(\underbrace{-2, \cdots, -2}_{k}; \varepsilon) = (-1)^k 2 - 2\varepsilon$. The Lemma follows. q. e. d.

Now if $k \geqq 3$ and $\alpha_1 = \cdots = \alpha_k = 2$, $\varepsilon = 1$, then each solution of the equations in Lemma 4. 3, $(n_1, \cdots, n_k)$, is a scalar multiple of $(1, \cdots, 1)$, and if $\alpha_1 = \cdots = \alpha_k = -2$ and $\varepsilon = (-1)^k$, then it is a scalar multiple of $(1, -1, \cdots, (-1)^{i+1}, \cdots, (-1)^{k+1})$. But by the definition of a circular system and $n_i (= \delta_i m_i,\ \delta_i = \pm 1, i = 1, \cdots, k)$, n_i and n_{i+1} are mutually prime for each $1 \leqq i \leqq k-1$, and $|n_i| > 1$ except at most one n_i. Therefore we get the following.

Lemma 4. 5. *If $L = L(\alpha_1, \cdots, \alpha_k; \varepsilon)$ $(k \geqq 3)$ is a circular system in a 4-dimensional S^1 manifold M, then some α_i is equal to* 0 *or* ± 1.

5. Proof of Theorem Let M be an oriented simply connected closed 4-dimensional S^1 manifold. First assume that there is a circular system of length $k \geqq 3$ in M. Let $L = L(\alpha_1, \cdots, \alpha_k; \varepsilon)$ be a circular system of length $k \geqq 3$ in M. By Lemma 4. 5, some $\alpha_i = \pm 1$ or 0 $(1 \leqq i \leqq k)$. If $\alpha_i = +1$ (resp. -1), then we can apply the process of Case (I) in § 3 to S_i, and M is equivalent to $M' \# CP^2$ (resp. $M \# -CP^2$) for some S^1 manifold M'. If $\alpha_i = 0$, then the process of Case (II) in § 3 can be applied to S_i and S_{i+1} (or S_i and S_{i-1}), and M is equivalent to $M' \# (S^2 \times S^2)$ or $M' \# CP^2 \# (-CP^2)$. In each case the number of the isolated fixed points in M' is strictly smaller then that in M. If there is a circular system of length $k \geqq 3$ in M', then the same process can be applied to M'. By the induction on the number of the isolated fixed points, we obtain an equivariant decomposition of M such as $M \approx N \# kCP^2 \# m(-CP^2) \# n(S^2 \times S^2)$, where N is an S^1 manifold such that length of each circular system in N is 2 or the S^1 action on N is semifree.

Asssume that N has a circular system of length 2. Let $L(\alpha_1, \alpha_2; \varepsilon) = p_1 S_1 p_2 S_2 p_1$ be a circular system in N. By Lemma 4. 3, $\alpha_1 n_1 = (\varepsilon + 1) n_2$ and $\alpha_2 n_2 = (\varepsilon + 1) n_1$, where n_1 and n_2 are as in § 4. Note that n_1 and n_2 are mutually prime integers and $|n_1||n_2| \geqq 2$. It follows that if $\varepsilon = -1$, then $\alpha_1 = \alpha_2 = 0$, and if $\varepsilon = +1$, then $\alpha_1 = \pm 4$, $\alpha_2 = \pm 1$, $|n_1| = 1$ and $|n_2| = 2$ or $\alpha_1 = \pm 1$, $\alpha_2 = \pm 4$, $|n_1| = 2$ and $|n_2| = 1$.

Remark : An example of the circular system of the latter type ($\varepsilon=+1$) is the following ; Let S^1 act on CP^2 by the equation $g[z_0, z_1, z_2]=[g^{-1}z_0, gz_1, z_2]$ ($g\in S^1$, $[z_0, z_1, z_2]\in CP^2$). Put $S_1=\{z_0z_1-z_2^2=0\}$, $S_2=\{z_2=0\}$, $p_1=[1, 0, 0]$ and $p_2=[0, 1, 0]$.

Thus if $\varepsilon=+1$, then α_1 or $\alpha_2=\pm 1$, hence the process of Case (I) in § 3 can be applied to S_1 or S_2, and N is decomposed as $N' \# \pm CP^2$. Again repeating this process, by the induction on the number of the circular systems of the forms $L=L(\pm 4, \pm 1 ; 1)$ and $L(\pm 1, \pm 4 ; 1)$, we obtain the following,

Lemma 5. 1. *M is equivarianty decomposed as $M\approx N' \# kCP^2 \# m(-CP^2) \# n(S^2\times S^2)$, where N' is a closed S^1 manifold such that every circular system in N' has length 2 and has the form $L(0, 0 ; -1)$ or the S^1 action on N' is semifree.*

Now to decompose N', we will change the S^1 action on N' into a semifree S^1 action if it is not semifree.

Lemma 5. 2. *Let N' be such an S^1 manifold as in Lemma 5. 1. Then there is a semifree S^1 action on N'.*

Proof. If the S^1 action on N' is not semifree, then there is a circular system of length 2, $L=p_1S_1p_2S_2p_1$ which has the form $L(0, 0 ; -1)$. Since $S_i \cdot S_i=0$, the equivariant normal bundle ν_i of S_i is trivial ($i=1, 2$). Let $D(\nu_i)$ and $S(\nu_i)$ be the disc bundle and the sphere bundle of ν_i respectively. Then $D(\nu_i)$ is diffeomorphic to $S^2\times D^2$ and $S(\nu_i)$ is diffeomorphic to $S^2\times S^1$. We identify $D(\nu_i)$ with an invariant closed neighborhood of S_i in N'. Suppose that $S_i\in F(m_i)$ ($i=1, 2$). Then m_1 and m_2 are mutually prime positive integers. Assume that $m_1>m_2$ for instance. The S^1 action restricted on $S(\nu_1)$ has two orbit types S^1 and S^1/Z_{m_2} with two exceptional orbits $S(\nu_1)\cap S_2$ (for the definition of the orbit types and exceptional orbit, see [2] [4]). Let $\rho : S^1 = SO(2) \longrightarrow SO(3)$ be the natural inclusion, where $SO(2)$ and $SO(3)$ are the special orthogonal groups of degree 2 and 3 respectively. Then the classification theorem of the 3-dimensional S^1 manifolds by P. Orlik and F. Raymond implies that the S^1 manifold $S(\nu_1)$ is equivalent to $S^2\times S^1$ endowed with the linear S^1 action defined by $g(x, y) = (\rho(g)^{m_1}x, g^{m_2}y)$ ($g\in S^1$ and $(x, y)\in S^2\times S^1$), where $g^{m_2}y$ is the multiplication in S^1 ([4] p. 20). Now there are integers q and r such that $m_1 = 2qm_2 + r$ and $-m_2<r<m_2$. Let $(S^2\times D^2)'$ be $S^2\times D^2$ endowed with the S^1 action defined by $g(x, y) = (\rho(g)^r x, g^{m_2}y)$. Let f be the diffeomorphism from

$S^2\times S^1$ to itself defined by $f(x, y)=(\rho(y)^{2q}x, y)$ $((x, y)\in S^2\times S^1)$. Then f is an equivariant diffeomorphism from $(S^2\times S^1)'=\partial(S^2\times D^2)'$ to $S^2\times S^1$ endowed with the above linear S^1 action which is equivalent to $S(\nu_1)$. Hence f gives an equivariant diffeomorphism $\tilde{f}$ from $(S^2\times S^1)'$ to $S(\nu_1)$. We obtain a new S^1 manifold N'' from the disjont union $\overline{(N'-D(\nu_1))}\cup(S^2\times D^2)'$ by identifying each $p\in\partial(S^2\times D^2)$ with $\tilde{f}(p)\in S(\nu_1)=\partial\overline{(N'-D(\nu_1))}$. Since $\pi_1(SO(3))=\mathbf{Z}_2$ and the map $y(\in S^1)\longrightarrow\rho(y)^{2q}(\in SO(3))$ is homotopic to a constant, hence f is isotopic to the identity. Therefore N'' is diffeomorphic to N'. Thus we obtain a new S^1 action on N', such that component of $F(m_1)$ of the old action is changed by a component of $F(|r|)$ (a component of F if $r=0$) of the new action, where $|r|<m_2<m_1$ (note that $r=0$ implies $m_2=1$). Now repeating this process, we obtain finally an S^1 action on N' such that $F(m)=\emptyset$ for all $m\geqq 2$, hence it is semifree. q. e. d.

By Lemma 5. 1 and 5. 2, to complete the proof of Theorem it suffices to decompose S^1 manifolds with semifree S^1 actions.

Let M be a closed S^1 4-manifold with semifree S^1 action.

First suppose that the fixed point set F contains a 2-sphere S_0 and suppose that $F-S_0$ is not empty. Let F_0 be another component of F. Let p (resp. q) be a point in F_0 (resp. S_0). Let D (resp. D') be an invariant 2-disc smoothly embedded in M with the center p (resp. q). We assume that D' intersects transversally with S_0 at q and D intersects transversally with F_0 at p if F_0 is 2-dimensional, and assume that D and D' are so small that $(D\cup D')\cap F=p\cup q$ and $D\cap D'=\emptyset$.

$\pi(D)$ and $\pi(D')$ are two arcs in M^*. By the same way as in the proof of Lemma 4. 2, we obtain an arc γ in M^* extending $\pi(D)$ and $\pi(D')$ such that $S=\pi^{-1}(\gamma)$ is an invariant 2-sphere smoothly embedded in M and it intersects transversally with S_0 at q and with F_0 at p if F_0 is 2-dimensional. The representation of S^1 on the normal plane of S at q is trivial and that on the normal plane of S at p is trivial or $t^{\pm 1}$ according as F_0 is a 2-sphere or a point. Since S^1 acts on S semifreely, this implies that the self intersection number of S, $S\cdot S$, is equal to 0 or ± 1 according as F_0 is a 2-sphere or a point by the same argument as in the proof of Lemma 4. 3. If $S\cdot S=\pm 1$, then the process of Case(I) in § 3 can be applied to S and M is decomposed as $M'\#\pm CP^2$. If $S\cdot S=0$, then that of Case (II) in § 3 can be applied to S and S_0, and M is decomposed as $M'\#(S^2\times S^2)$ or $M'\# CP^2\#(-CP^2)$. We note that the number of the components of the fixed point set in M' is strictly smaller than that in M and the action on M' is semifree in each case.

Next suppose that F consists of isolated fixed points, $F=\{p_1,\cdots,p_k\}$ (here $k\geqq 2$, as $\dim_{\mathbf{Q}}H(F:\mathbf{Q})=\dim_{\mathbf{Q}}H(M:\mathbf{Q})\geqq 2$). Assume that $k\geqq 3$. In M^*, as in the proof of Lemma 4. 2, we can choose arcs $\gamma_1,\cdots,\gamma_{k-2}$ such that the endpoints of γ_i are $\pi(p_i)$ and $\pi(p_{i+1})$ and the interior of γ_i is contained in $M_0{}^*$ and $S_i=\pi^{-1}(\gamma_i)$ is an invariant 2-sphere smoothly embeeded in M, and S_i intersects transversally with S_{i+1} at $p_{i+1}=S_i\cap S_{i+1}\,(i=1,\cdots,k-2)$ and $S_i\cap S_j=\emptyset$ if $|i-j|\geqq 2$. Let $D(\nu_i)$ be the disc bundle of the equivariant normal bundle of S_i in M. Let K be the equivariant plumbing of $D(\nu_1),\cdots,D(\nu_{k-2})$ such that $D(\nu_i)$ and $D(\nu_{i+1})$ are plumbed at $p_{i+1}(i=1,\cdots,k-3)$ ($\underset{\alpha_1}{\bullet}\!-\!\underset{\alpha_2}{\bullet}\!-\!\underset{\alpha_3}{\bullet}\!-\!\cdots\!-\!\underset{\alpha_{k-2}}{\bullet}$ in the usual diagram). Then K is identified with a closed invariant neighborhood of $S_1\cup\cdots\cup S_{k-2}$ in M. Let D be an invariant closed 4-disc with the center p_k which is disjoint from K. Then $\pi:\overline{M-K\cup D}\longrightarrow\overline{\pi(M-K\cup D)}$ is a principal S^1 bundle. It can be seen that $\pi(K)$ is the 3-disc and $\partial\pi(K)=\pi(\partial K)$ is the 2-sphere S^2. The S^1 bundles $\pi:\partial D\longrightarrow\pi(\partial D)$ and $\pi:\partial K\longrightarrow\pi(\partial K)$ are equivalent to each other, hence it follows that $\partial K\approx S^3$. This implies that the intersection form on $H_2(K:\boldsymbol{Z})$ is unimodular. Choose an orientation o_i of S_i so that the intersection number of S_i and S_{i+1} at p_i may be $-1(i=1,\cdots,k-2)$. Then the intersection matrix is

$$\begin{pmatrix} \alpha_1 & -1 & & & 0 \\ -1 & \alpha_2 & -1 & & \\ & -1 & \alpha_3 & -1 & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & \alpha_{k-2} \end{pmatrix}$$

, where $\alpha_i=S_i\cdot S_i$ is the self intersection number of $S_i(i=1,\cdots,k-2)$. Hence using the notation in the proof of Lemma 4. 4, it follows that $\mathrm{B}(\alpha_1,\cdots,\alpha_{k-2})=\pm 1$, and by the argument in that place at least one α_i is equal to ± 1 or 0. Therefore as before, M can be decomposed into the equivariant connected sum $M'\,\#\pm CP^2$ or $M'\,\#\,(S^2\times S^2)$ or $M'\,\#\,CP^2\,\#\,(-CP^2)$, where the number of the fixed points in M' is strictly smaller than that in M.

Summerizing the above argument, we conclude that M is decomposed as $M\approx M'\,\#\,k(CP^2)\,\#\,m(-CP^2)\,\#\,n(S^2\times S^2)$, where M' is a smifree S^1 manifold and the fixed point set in M' is only one 2-sphere or two isolated points. In each case $\dim_Q H(M' : Q)=\dim_Q H(F : Q)=2$ by the theorem of A. Borel (§ 2), and this implies M' is a homotopy sphere, since it is simply connected. This completes the proof of Theorem.

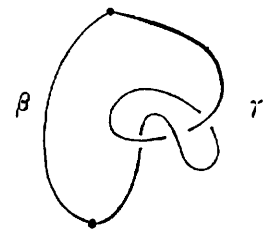
6. Some S^1 actions

In this section, we construct an S^1 action on S^4 and an S^1 action on the connected sum $kCP^2 \sharp m(-CP^2) \sharp n(S^2 \times S^2)$.

(1) An S^1 action on S^4

Let $C+C$ be the complex plane with S^1 action defined by $g(x, y) = (gx, g^2y)$ for $g \in S^1$ and $(x, y) \in C+C$. The one-point compactification of $C+C$, $(C+C)^c = (C+C) \cup \{\infty\}$, is S^4, and we extend the S^1 action on $C+C$ to S^4 naturally. The orbit space $(S^4)^*$ is S^3, and $\beta = \pi(F(2)) = \pi((0+C)^c)$ is an arc in S^3 joining $\pi((0, 0))$ and $\pi(\infty)$. Let γ be an embedded arc in S^3 joining the same points such that the circle $\beta \cup \gamma$ is the trefoil knot in S^3 (Figure (*)).

(*)



We choose γ so that $\pi^{-1}(\gamma)$ may be an invariant 2-sphere smoothly embedded in S^4 intersecting transversally with $F(2) = (0 + C)^C$ at the two fixed points. Since any embedded 2-sphere in S^4 has a trivial normal bundle, the equivariant normal bundle of $\pi^{-1}(\gamma)$ in S^4, ν, is trivial and its disc bundle $D(\nu)$ is diffeomorphic to $S^2 \times D^2$. The restricted S^1 action on its boundary $S(\nu) \approx S^2 \times S^1$ is equivalent to the S^1 action on $S^2 \times S^1$ defined by $g(x, y)=(\rho(g)x, g^2y)$ $(g \in S^1, (x, y) \in S^2 \times S^1)$. Let f be the map from $S^2 \times S^1$ to itself defined by $f(x, y) = (\rho(y)^{-2}x, y)$ $((x, y) \in S^2 \times S^1)$. Let $(S^2 \times D^2)'$ be $S^2 \times D^2$ with S^1 action defined by $g(x, y)= (\rho(g)^5 x, g^2 y)$. Then f is equivariant diffeomorphism from $\partial(S^2 \times D^2)'$ to $S^2 \times S^1$ with the above S^1 action which is equivalent to $S(\nu)$. Hence f induces an equivariant diffeomorphism $\tilde{f}$ from $\partial(S^2 \times D^2)'$ to $S(\nu)$. Deleting from S^4 the open disc bundle $\dot{D}(\nu)$ and then patching $(S^2 \times D^2)'$ along the boundary using $\tilde{f}$, we obtain an S^1 manifold M. Since f is isotopic to the identity, M is diffeomorphic to S^4. This S^1 action is not equivalent to any linear S^1 action on S^4 since the complement of the singular and exceptional orbits $F(5) \cup F(2)$ is not equivariantly diffeomorphic to that of any linear action.

The similar constructions produce arbitrary knots and links in the orbit spaces when the number of fixed points is greater than 2. This implies that any classification of the equivariant diffeomorphism types of simply connected S^1 manifolds leads to the classification of knots and links in S^3 in addition to the classification of the orbit types.

(2) A semifree S^1 action on the connected sum $k(CP^2) \sharp m(-CP^2) \sharp n(S^2 \times S^2)$.

Let S^1 act on CP^2 (resp. $-CP^2$) by the equation $g[z_0, z_1, z_2] = [z_0, z_1, gz_2]$ (resp. $[z_0, z_1, g^{-1}z_2]$), where $g \in S^1$ and $[z_0, z_1, z_2] \in CP^2$. Let S^1 act on $CP^1 \times CP^1$ by the equation $g([z_0, z_1], [z_0, z_1]) = ([z_0, gz_1], [z_0, z_1])$, where $([z_0, z_1], [z_0, z_1]) \in CP^1 \times CP^1$. Now let $P_1, P_2, \cdots, P_k$ be the k copies of CP^1, and let $Q_1, Q_2, \cdots, Q_m$ be the m copies of $-CP_1$ and let $R_1, R_2, \cdots, R_n$ be the n copies of $CP^1 \times CP^1$. Assume that each of $\{P_1, \cdots, P_k, Q_1, \cdots, Q_m, R_1, \cdots, R_n\}$ has the S^1 action defined as above. Put $a_i = [1, 0, 0]$ and $b_i = [0, 1, 0]$ in $P_i (i=1, \cdots, k)$, put $c_j = [1, 0, 0]$ and $d_j = [0, 1, 0]$ in $Q_j (j=1, \cdots, m)$, and put $e_r = ([1, 0], [1, 0])$ and $f_r = ([1, 0], [0, 1])$ in $R_r (r=1, \cdots, n)$. The tangent representations of S^1 are $1+t^1$ at a_i and $b_i (1 \leqq i \leqq k)$, $1+t^{-1}$ at c_j and $d_j (1 \leqq j \leqq m)$, and $1+t$ at e_r and $f_r (1 \leqq r \leqq n)$ under the natural complex structures on the corresponding tangent spaces. Note that there is an orientation reversing equivariant diffeomorphism from the representation space $1+t$ to the representation space $1+t^{-1}$ ($(u, v) \longrightarrow (u, \bar{v})$, $(u, v) \in C+C$ cnd $\bar{v} =$ the conjugate of v). We can form the equivariant connected sum $P_1 \sharp \cdots \sharp P_k \sharp Q_1 \sharp \cdots \sharp Q_m \sharp R_1 \sharp \cdots \sharp R_n$ in such a way as P_i and P_{i+1} are connected at b_i and a_{i+1} $(1 \leqq i \leqq k-1)$, and P_k and Q_1 are connected at b_k and c_1, and Q_j and Q_{j+1} are connected at d_j and c_{j+1} $(1 \leqq j \leqq m-1)$, and Q_m and R_1 are connected at d_m and e_1, and R_r and R_{r+1} are connected at f_r and e_{r+1} $(1 \leqq r \leqq n-1)$. This gives a semifree S^1 action on the connected sum $kCP^2 \sharp m(-CP^2) \sharp n(S^2 \times S^2)$ (note that $S^2 \times S^2$ has an orientation reversing diffeomorphism).

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Added in proof. R. Fintushel has proved the following result; if a simply connected closed 4-manifold M admits a non-trivial locally S_1 action, then M is homotopy eqivalent to a connected sum of copies of S^4, $\pm CP^2$ and $S^2 \times S^2$ (R. Fintushel, Circle actions on simply connected 4-manifolds, Trans. A. M. S. 230, 1977, 141—171). He informed me that he and P. S. Pao have obtained the same result in this paper in the forthcomming paper.