NOTE ON COMMUTATIVITY OF RINGS

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Throughout R will represent a ring. Following [4], R is called a *left* (resp. *right*) *s-unital ring* if for each $a \in R$ there exists some $e \in R$ such that $ea = a$ (resp. $ae = a$). Given $a, b \in R$, $[a, b]$ will denote the commutator $ab - ba$. An element $a \in R$ is defined to be *almost central* if for each $b \in R$ there exist positive integers n and m such that

$$(1) \qquad (ab)^k = a^k b^k, \qquad k = n,\ n+1,\ n+2\,;$$

$$(2) \qquad (ba)^h = b^h a^h, \qquad h = m,\ m+1,\ m+2\,.$$

In this note, we shall prove the following:

Theorem. *Let R be (left and right) s-unital. If $a \in R$ is almost central, then for each $b \in R$ there exists a positive integer s such that $a^s[a, b] = 0 = [a, b]a^s$.*

As application of the theorem, we shall improve also the main results of [1], [2] and [3] (Corollary 2).

In advance of proving our theorem, we state two lemmas.

Lemma 1.[1] (a) *If F is a finite subset of an s-unital ring R, then there exists an element e such that $ea = ae = a$ for all $a \in F$.*

(b) *If a left s-unital ring R contains a regular element a, then R contains* 1.

Proof. (a) By [4, Theorem 1], there exist elements e' and e'' such that $e'a = a$ and $ae'' = a$ for all $a \in F$. Then, one will easily see that the element $e = e' + e'' - e''e'$ has the property requested.

(b) Choose an element e with $ea = a$. Since $(be - b)a = 0$ for all $b \in R$, e is a right identity of R. Accordingly, we obtain also $a(eb - b) = 0$, namely, $eb = b$.

Lemma 2. *Let $a \in R$ be almost central, and $b \in R$.*

(a) *Assume that R is left (resp. right) s-unital. If $ab = 0$ (resp. $ba = 0$) then $ba^s = 0$ (resp. $a^s b = 0$) with some positive integer s.*

(b) *Assume that R is a ring without non-zero nil right (resp. left)*

1) This lemma is due to Prof. H. Tominaga who kindly permitted us to cite it here. We are indebted to him for his helpful suggestions and advices.

ideals. If $a^s b = 0$ *(resp.* $ba^s = 0$*) for some positive integer* s, *then* $ab = 0$ *(resp.* $ba = 0$*). In particular, if* R *is right (resp. left) s-unital and* a *is nilpotent then* $a = 0$.

Proof. (a) By [4, Theorem 1], there exists an element e such that $ea = a$ and $eb = b$. Since $(ba)^2 = 0$, there holds $b^{m+1}a^{m+1} = 0$ by (2). Now, choose a positive integer p such that

$$\{(b+e)\,a\}^h = (b+e)^h a^h, \qquad h = p,\ p+1,\ p+2\,.$$

Noting that $ab = 0$, we have $\{(b+e)a\}^h = ba^h + a^h = (b+e)a^h$. Hence,

$$\{(b+e)^h - (b+e)\}\,a^h = 0, \qquad h = p,\ p+1,\ p+2$$

and $b^{m+1}\,a^{m+p+1} = 0$. If $b^t a^{m+p+1} = 0$ for some $t > 1$, then

$$\begin{aligned} 0 &= b^{t-2}[\,\{(b+e)^{p+1} - (b+e)\}\,a^{p+1}\,a^m - \{(b+e)^p - (b+e)\}\,a^p a^{m+1}] \\ &= b^{t-2}\,(b+e)^p b a^{m+p+1} = b^{t-1}a^{m+p+1}. \end{aligned}$$

This means $ba^{m+p+1} = 0$.

(b) Suppose $s > 1$. For any $c \in R$, there is an integer $t > 1$ such that $\{a(a^{s-2}bc)\}^t = a^t(a^{s-2}bc)^t = a^{t-2}a^s bc\,(a^{s-2}bc)^{t-1} = 0$. Hence, $a^{s-1}bR$ is a nil right ideal, whence it follows $a^{s-1}b = 0$. This means evidently $ab = 0$.

Proof of Theorem. By Lemma 1 (a), there exists an element e such that $ea = ae = a$ and $eb = be = b$. The first two equations of (1) induce $a^n[a, b^n]b = 0$, and the last two equations of (1) do $a^{n+1}[a, b^{n+1}]b = 0$. By Lemma 2 (*a*), we have then $[a, b^n]\,ba^q = 0$ and $[a, b^{n+1}]ba^q = 0$ for some positive integer q. Hence, $[a, b]b^{n+1}a^q = [a, b^{n+1}]\,ba^q - b[a, b^n]\,ba^q = 0$. Again by Lemma 2 (a), it follows $a^r[a, b]b^{n+1} = 0$ for some positive integer r. Considering $b + e$ instead of b, we see that $a^s[a, b](b+e)^{p+1} = 0$ for some $s \geq r$ and some $p > 0$. Since $a^s[a, b]\,b^n = a^s[a, b]\,(b+e)^{p+1}b^n = 0$, we obtain eventually $a^s[a, b] = a^s[a, b](b+e)^{p+1} = 0$. Now, our assertion is evident by Lemma 2 (a).

Corollary 1. (a) *If* R *is an s-unital ring without non-zero nil one-sided ideals, then every almost central element is central.*

(b) *If* R *is s-unital and a regular element* $a \in R$ *is almost central, then* R *contains* 1 *and* a *is central.*

(c) *If* R *contains* 1, *and both* a *and* $a + 1$ *are almost central, then* a *is central.*

Proof. (a) Let $a \in R$ be almost central, and b an arbitrary element of R. Combining Theorem with Lemma 2 (b), we readily obtain

$[a, b]a = 0$. Since $[a, b]ca = [a, bc]a - b[a, c]a = 0$ for any $c \in R$, we see that $[a, b]R[a, b] = 0$. This means that $[a, b]R$ is nilpotent, and hence $[a, b] = 0$.

(b) By Lemma 1 (b), R contains 1. Furthermore, by Theorem and the hypothesis, $[a, b] = 0$ for any $b \in R$.

(c) Let b be an arbitrary element of R. By Theorem, $a^s[a, b] = 0$ and $(a+1)^t[a+1, b] = 0$ for some non-negative integers s, t. If $s > 0$, then $0 = a^{s-1}(a+1)^t[a+1, b] = a^{s-1}[a, b]$. Hence, $[a, b] = 0$.

Corollary 2 (cf. [1, Theorems 1, 2], [2, Theorem] and [3, Theorem]). *Assume that for each* a, $b \in R$ *there exists a positive integer* n *such that*

$$(ab)^k = a^k b^k, \qquad k = n,\ n+1,\ n+2.$$

(a) *If* R *is s-unital, then* R *is commutative.*

(b) *If* R *is semiprimitive, then* R *is commutative.*

Proof. (a) Let a and b be arbitrary elements of R, and choose an element e with $ea = ae = a$ and $eb = be = b$ (Lemma 1 (a)). By Theorem, $a^s[a, b] = 0$ and $(a+e)^t[a+e, b] = 0$ for some positive integers s, t. If $s > 1$, then $0 = a^{s-1}(a+e)^t[a+e, b] = a^{s-1}[a, b]$. This means $a[a, b] = 0$. Thus, we obtain $[a, b] = (a+e)[a+e, b] - a[a, b] = 0$.

(b) As is shown in the proof of [1, Theorem 1], R is a subdirect sum of division rings. Hence R is commutative by (a).

Remark 1. Let a, $b \in R$. Assume that $a^s[a, b] = 0 = [a, b]a^s$ for some positive integer s (cf. Theorem). Then, by $[a^k, b] = a^{k-1}[a, b] + [a^{k-1}, b]a$, one will easily see that $a^s[a^k, b] = 0 = [a^k, b]a^s$ for any positive integer k. Hence, $a^{s+k}b = a^s b a^k$ and $ba^{s+k} = a^k b a^s$, in particular, $a^{2s}b = a^s b a^s = ba^{2s}$. Moreover, if $s > k > 0$ then $a^{s+k}ba^{s-k} = a^s b a^s = a^{2s}b = ba^{2s}$.

Remark 2. Let K be a field, and $R = \sum_{i \geq j} Ke_{ij}$ where e_{ij}'s are matrix units of $(K)_4$. Then every element of the radical $J = \sum_{i > j} Ke_{ij}$ is almost central. We see therefore that almost central quasi-regular elements of R need not be central, and that Corollary 2 (a) is not true in general for rings without 1.

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