ON THE FIXED POINT SET OF S^1-ACTIONS ON THE COMPLEX FLAG MANIFOLDS

Dedicated to Professor Ken'iti Koseki on his 60th birthday

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Introduction. In this paper we consider S^1-actions on a closed manifold X whose cohomology ring is generated by elements of degree 2.

We adopt this notation: $G=S^1$ is the circle group. Λ is the field of rational numbers Q or the ring of integers Z. X is always a closed (topological) manifold. We use sheaf-theoretic cohomology, and assume that $H^*(X;\Lambda)$ is generated by elements of degree 2; that is, $H^*(X;\Lambda) \cong\Lambda[\alpha_1, \cdots, \alpha_n]/(p_1, \cdots, p_m)$, where $p_1, \cdots, p_m$ are homogeneous polynomials, and deg $\alpha_i=2$ $(i=1, \cdots, n)$. If G acts on X, $X\times_G E_G$ is the bundle associated to a universal principal S^1-bundle $E_G\longrightarrow B_G$. The equivariant cohomology ring of X is defined by $H_G^*(X;\Lambda)=H^*(X\times_G E_G:\Lambda)$.

The natural projection π: $X\times_G E_G\longrightarrow B_G$ makes $H_G^*(X;Q)$ into an algebra over $H^*(B_G;Q)$. Let R be the quotient field of $H^*(B_G;Q)$ and A the localization of $H_G^*(X;Q)$ as an $H^*(B_G;Q)$-module at the zero ideal. Under more general situation Wu-Yi Hsiang proved in [5] the fundamental fixed theorem: the ideal of relations between a set of generators for A has a finite number of zeros in 1-1 correspondence with the connected components of the fixed point set F. In our case $H_G^*(X;Q) \cong Q[t, x_1, \cdots, x_n]/J$, where $J=(p_1-tf_1, \cdots, p_m-tf_m)$, and f_i is a homogeneous polynomial such that $\deg f_i=\deg p_i-1$ $(i=1, \cdots, m)$. In §2, we show that the zero points of J and the connected components of F correspond bijectively. If $m=n$, the multiplicity of a zero point of J equals to the dimension of the cohomology ring over Q of the corresponding component (Theorem 2. 6). This result generalizes [4] Theorem 3. 1. P. Tomter [7] also implies such a result.

In §3, we show that the bundle lifting is closely related to the zero points of J.

In §4, we consider the case in which X is a complex flag manifold $U(n)/T^n$. We give some examples of S^1-actions on X. For $n=3$, we show a list of all possible fixed point sets.

S^1-actions on the connected sum $CP(3)\#CP(3)$ gives another examples in our case. In §5, we consider the case in which $X=CP(3)\#CP(3)$, and show a list of all possible fixed point sets.

1. In this section we give some algebraic preliminaries.

Let $\mathfrak{A}=(p_1, \cdots, p_n)$ be a homogeneous ideal of the polynomial ring $Q[x_1, \cdots, x_n]$ such that the radical $\sqrt{\mathfrak{A}}=(x_1, \cdots, x_n)$. Let J be a homogeneous ideal of the polynomial ring $Q[t, x_1, \cdots, x_n]$ generated by the following n forms:

$$p_1-tf_1, \cdots, p_n-tf_n.$$

Lemma 1. 1. *J is an unmixed ideal, and the number of the zero points (in the projective space) of J is at most $\Pi_{i=1}^n \deg p_i$.*

proof. If $t\in\sqrt{J}$, then $\sqrt{J}=(t, x_1, \cdots, x_n)$, but this is impossile. Hence we have $t\notin\sqrt{J}$. Then t is not algebraic in $Q[t, x_1, \cdots, x_n]/J$ over Q. Since $\sqrt{\mathfrak{A}}=(x_1, \cdots, x_n)$, it is clear that $Q[t, x_1, \cdots, x_n]/J$ is a finitely generated module over $Q[t]$. It follows that the projective dimension of J is zero, and J is unmixed ([8], Ch. VII, Lemma 2). Since the number of the zero points of J is finite, a theorm of Bezout (cf. [6]) implies that there are zero points of J in the number at most $\Pi_{i=1}^n \deg p_i$.

q. e. d.

Let $\xi^{(j)}=(1, \xi_1^{(j)}, \cdots, \xi_n^{(j)})$ be the zero point of J $(j=1, \cdots, k)$, and assume $\xi^{(j)}\in Q^{n+1}$. Let I_j be the homogeneous ideal generated by the coefficients of $g(t, x_1+\xi_1^{(j)}t, \cdots, x_n+\xi_n^{(j)}t)$ $(g\in J)$ with respect to t, and $m_j=\dim_Q Q[x_1, \cdots, x_n]/I_j$ $(j=1, \cdots, k)$. Let q_j be the homogeneous ideal such that $g\in q_j$ is equivalent to $g(t, x_1+\xi_1^{(j)}t, \cdots, x_n+\xi_n^{(j)}t)\in I_j[t]$. Then q_j is a primary ideal. The u-resultant $R(u)$ of n forms p_i-tf_i $(i=1, \cdots, n)$ is decomposed as follows:

$$R(u)=c\cdot\Pi_{j=1}^k(u+\xi_1^{(j)}u_1+\cdots+\xi_n^{(j)}u_n)^{\rho_j}$$

where $u, u_1, \cdots, u_n$ are indeterminates, c a constant and ρ_j the multiplicity of $\xi^{(j)}$[6]. Then we have the following

Theorem 1. 2. *The following conditions are equivalent;*

(1) $\sum_{j=1}^k m_j=\Pi_{i=1}^n \deg p_i$.

(2) $J=\bigcap_{j=1}^k q_j$ *(a reduced primary decomposition).*

(3) $m_j=\rho_j$ $(j=1, \cdots, k)$.

Proof. (3)$\Longrightarrow$(1) is immediate from Bezout's theorem. To prove (2)$\Longrightarrow$(3), let ${}^a: Q[t, x_1, \cdots, x_n]\longrightarrow Q[x_1, \cdots, x_n]$ denotes a homomorphism defined by

$${}^af(x_1, \cdots, x_n)=f(1, x_1, \cdots, x_n)$$

for every $f \in Q[t, x_1, \cdots, x_n]$. Then we see that $\dim_Q Q[x_1, \cdots, x_n]/{}^a q_j = \rho_j$ [6]. (3) follows from the equalities:

$$\dim_Q Q[x_1, \cdots, x_n]/{}^a q_j = \dim_Q Q[x_1, \cdots, x_n]/{}^a I_j[t]$$
$$= \dim_Q Q[x_1, \cdots, x_n]/I_j = m_j .$$

Finally, suppose that (1) holds. Since $\sqrt{J} = \cap_{j=1}^k (x_1 - \xi_1^{(j)} t, \cdots, x_n - \xi_n^{(j)} t)$ by the assumption, and J is unmixed by (1. 1), we have the reduced primary decomposition $J = \cap_{j=1}^k \bar{q}_j$ where $\sqrt{\bar{q}_j} = (x_1 - \xi_1^{(j)} t, \cdots, x_n - \xi_n^{(j)} t)$. By the definition of q_j we see $\cap_{j=1}^k q_j \supset J$, and $\sqrt{q_j} = (x_1 - \xi_1^{(j)} t, \cdots, x_n - \xi_n^{(j)} t)$, and hence $q_j \supset \bar{q}_j$ $(j=1, \cdots, k)$. Then we have ${}^a q_j \supset {}^a \bar{q}_j$, this implies that

$$\sum_{j=1}^k m_j = \sum_{j=1}^k \dim_Q Q[x_1, \cdots, x_n]/{}^a q_j$$
$$\leq \sum_{j=1}^k \dim_Q Q[x_1, \cdots, x_n]/{}^a \bar{q}_j = \sum_{j=1}^k \rho_j$$
$$= \Pi_{i=1}^n \deg p_i .$$

It follows that ${}^a q_j = {}^a \bar{q}_j$ $(j=1, \cdots, k)$. Since $t \notin \sqrt{J}$ this implies that $q_j = \bar{q}_j$ $(j=1, \cdots, k)$. q. e. d.

We say a commutative graded algebra $A = A_0 \oplus \cdots \oplus A_m$ over Q satisfies duality if $A_m \cong Q$ and the multiplication $A_i \otimes A_{m-i} \longrightarrow A_m \cong Q$ is a duality pairing $(0 \leq i \leq m)$.

Remark. For a graded algebra $Q[x_1, \cdots, x_n]/\mathfrak{A}$ where $\mathfrak{A} = (p_1, \cdots, p_n)$ and $\deg x_i = 1$ $(i=1, \cdots, n)$, the following conditions are equivalent:

(1) $\sqrt{\mathfrak{A}} = (x_1, \cdots, x_n)$.

(2) $\{p_1, \cdots, p_n\}$ is a prime sequence.

(3) $Q[x_1, \cdots, x_n]/\mathfrak{A}$ satisfies duality.

If one of the conditions of (1. 2) is satisfied, we have the following

Proposition 1. 3. *$Q[x_1, \cdots, x_n]/I_j$ satisfies duality $(j=1, \cdots, k)$.*

Proof. Let $\phi = \oplus_{j=1}^k \phi_j : Q[t, x_1, \cdots, x_n]/J \longrightarrow \oplus_{j=1}^k Q[x_1, \cdots, x_n]/I_j \otimes Q[t]$, where $\phi_j : Q[t, x_1, \cdots, x_n]/J \longrightarrow Q[x_1, \cdots, x_n]/I_j \otimes Q[t]$ is defined by

$$\phi_j(f) = f(t, x_1 + \xi_1^{(j)} t, \cdots, x_n + \xi_n^{(j)} t)$$

for every $f \in Q[t, x_1, \cdots, x_n]$. If $\phi(f) = 0$, $\phi_j(f) = 0$ $(j=1, \cdots, k)$. Then $f \in \cap_{j=1}^k q_j = J$. Thus we have the following exact sequence:

$$0 \longrightarrow Q[t, x_1, \cdots, x_n]/J \longrightarrow \oplus_{j=1}^k Q[x_1, \cdots, x_n]/I_j \otimes Q[t].$$

Since $Q[x_1, \cdots, x_n]/\mathfrak{A}$ satisfies duality it is only necessary to show that ϕ is surjective in high degrees [2]. We show that for each homogeneous polynomial $f \in Q[x_1, \cdots, x_n]$, there exists $g \in Q[t, x_1, \cdots, x_n]$ such that $\phi_j(g) = t^N f \pmod{I_j}$, and $\phi_h(g) = 0 \pmod{I_h,\ \text{for}\ h \neq j}$ for sufficiently large N which is not depend on f and g. Since $\sqrt{I_j} = (x_1, \cdots, x_n)$, if $\deg f$ is sufficiently large, then $f = 0 \pmod{I_j}$, and we can take $g = 0$. So we suppose that, if $\deg f > l$ there is $g \in Q[t, x_1, \cdots, x_n]$ satisfying the above condition. Let $\deg f = l$, we define $g \in Q[t, x_1, \cdots, x_n]$ by

$$g = f(x_1 - \xi_1^{(j)} t, \cdots, x_n - \xi_n^{(j)} t) \cdot \Pi_{h \neq j} (x_{i(h)} - \xi_{i(h)}^{(h)})^{N_0}$$

where $i(h)$ is such that $\xi_{i(h)}^{(h)} \neq \xi_{i(h)}^{(j)}$, and N_0 a sufficiently large number. Then we have $\phi_h(g) = 0 \pmod{I_h}$ for $h \neq j$, and

$$\begin{aligned} \phi_j(g) &= f \cdot \Pi_{h \neq j} (x_{i(h)} + (\xi_{i(h)}^{(j)} - \xi_{i(h)}^{(h)}) t)^{N_0} \\ &= c t^{(k-1) N_0} \cdot f + \textstyle\sum_{i=1}^{(k-1) N_0 - 1} c_i t^i f_i \end{aligned}$$

where c, c_i $(i = 1, \cdots, (k-1) N_0 - 1)$ are constants, and $\deg f_i > l$. Now we complete the proof by the induction on $\deg f$. q. e. d.

2. Let G be the circle group S^1 and X a closed (topological) manifold with a S^1-action. We assume the following:

(2. 1) $H^*(X; \Lambda) \cong \Lambda[\alpha_1, \cdots, \alpha_n]/(p_1, \cdots, p_m)$ $(\Lambda = Q$ or $Z)$

where $\deg \alpha_i = 2$ $(i = 1, \cdots, n)$ and p_i is a homogeneous polynomial of $\alpha_1, \cdots, \alpha_n$ $(i = 1, \cdots, m)$. Then we have the following

Lemma 2. 2. $H_G^*(X; \Lambda) \cong \Lambda[t, x_1, \cdots x_n]/J$, *where* $J = (p_1 - t f_1, \cdots, p_m - t f_m)$ *and* $f_i \in \Lambda[t, x_1, \cdots, x_n]$ *is a homogeneus polynomial such that* $\deg f_i = \deg p_i - 1$ $(i = 1, \cdots, m)$.

Proof. Let $\xi : X \times E_G \times C^1/G \longrightarrow X \times_G E_G$ be the associated complex line bundle of the principal S^1-bundle $X \times E_G \longrightarrow X \times_G E_G$. Consider the Gysin sequence

$$\longrightarrow H_G^q(X; \Lambda) \xrightarrow{\cup t} H_G^{q+2}(X; \Lambda) \longrightarrow H^{q+2}(X; \Lambda) \longrightarrow H_G^{q+1}(X; \Lambda) \longrightarrow$$

where $\cup t$ is the cup product of the Euler class of ξ. Since $H^{odd}(X; \Lambda) = 0$, X is totally non-homologous to zero in $X \times_G E_G \longrightarrow B_G$, and hence $H_G^{odd}(X; \Lambda) = 0$, so the given exact sequence reduces to

(2. 3) $$0 \longrightarrow H_G^{2q}(X; \Lambda) \xrightarrow{\cup t} H_G^{2q+2}(X; \Lambda) \xrightarrow{i^*} H^{2q+2}(X; \Lambda) \longrightarrow 0,$$

where $i : X \longrightarrow X \times_G E_G$ is the inclusion of a fiber. For each $i = 1, \cdots, n$, let $x_i \in H_G^2(X; \Lambda)$ be such that $i^*(x_i) = \alpha_i$. Then $H_G^*(X; \Lambda)$ is generated by $x_1, \cdots, x_n$ and t. By (2. 3), the ideal J of relations contains m relations $p_1 - t f_1, \cdots, p_m - t f_m$, where $\deg f_i = \deg p_i - 1$ $(i = 1, \cdots, m)$. If $f =$

$f(t, x_1, \cdots, x_n) \in J$, then $f(0, \alpha_1, \cdots, \alpha_n) = i^*(f) = 0$, and

$$f(0, x_1, \cdots, x_n) = g_1 p_1 + \cdots + g_m p_m$$

for some $g_1, \cdots, g_m \in A[x_1, \cdots, x_n]$. We have

$$f - g_1 p_1 - \cdots - g_m p_m = th$$

for some $h \in A[t, x_1, \cdots, x_n]$, and hence

$$f = t(g_1 f_1 + \cdots + g_m f_m + h) \qquad (\text{mod } J)$$

Deviding by t we have

$$g_1 f_1 + \cdots + g_m f_m + h \in J.$$

It is shown by the induction with respect to $\deg f$ that

$$g_1 f_1 + \cdots + g_m f_m + h = k_1(p_1 - tf_1) + \cdots + k_m(p_m - tf_m)$$

for some $k_1, \cdots, k_m \in A[t, x_1, \cdots, x_n]$. Thus we have

$$f = G_1(p_1 - tf_1) + \cdots + G_m(p_m - tf_m),$$

where $G_i = g_i + tk_i$ $(i=1, \cdots, m)$. q. e. d.

Let $F = X^G$ be the fixed point set of the S^1-action on X, and let $F_1, \cdots, F_k$ be connected components of F. There is an exact sequence

(2.4) $0 \longrightarrow H_G^*(X; Q) \xrightarrow{\phi^*} H_G^*(F; Q) \longrightarrow H^*(X/G, F; Q) \longrightarrow 0$

where ϕ is the inclusion of $F \times_G E_G$ into $X \times_G E_G$ [2]. For each $i=1, \cdots, n$, we set

$$\phi^*(x_i) = \textstyle\sum_{j=1}^{k} (b_{ij} + c_{ij}t),$$

where $b_{ij} \in H^2(F_j; Q)$ and $c_{ij} \in Q (j=1, \cdots, k)$. Since $\dim_Q H^*(X/G; Q)$ and $\dim_Q H^*(F; Q)$ are finite, there is an integer N such that $H^q(X/G, F; Q) = 0$ for $q > N$. It follows that $H^*(F_j; Q)$ is generated by $b_{1j}, \cdots, b_{nj}$, and that $(c_{1j}, \cdots, c_{nj}) \neq (c_{1l}, \cdots, c_{nl})$ if $j \neq l$. We denote by I_j the ideal generated in $Q[x_1, \cdots, x_n]$ by the coefficients of $f(t, x_1 + c_{1j}t, \cdots, x_n + c_{nj}t)$ $(f \in J)$ with respect to t. We define q_j similarly as in § 1.

Proposition 2.5.
(1) $H^*(F_j; Q) \cong Q[x_1, \cdots, x_n] / I_j \qquad (j=1, \cdots, k)$.
(2) $J = \cap_{j=1}^{k} q_j$ *is the reduced primary decomposition, where* $\sqrt{q_j} = (x_1 - c_{1j}t, \cdots, x_n - c_{nj}t)$.

Proof. By the definition of I_j, it is obvious that I_j is contained in the ideal of relations. Let f be a relation. Since $(c_{1j}, \cdots, c_{nj}) \neq (c_{1l}, \cdots, c_{nl})$ if $j \neq l$, there is an integer $i(l)$ such that $c_{i(l)j} \neq c_{i(l)l}$, for each $l \neq j$. We set

$$g=f(x_1-c_{1j}t, \cdots, x_n-c_{nj}t) \cdot \Pi_{l\neq j}(x_{i(l)}-c_{i(l)l}t)^N$$

where N is an integer such that $H^q(F;Q)=0$ for $q\geqq 2N$. Then we have $\phi^*(g(t, x_1, \cdots, x_n))=0$, and hence $g\in J$. The coefficient of the highest degree with respect to t in the polynomial $g(t, x_1+c_{1j}t, \cdots, x_n+c_{nj}t)$ is a multiple of f by some non-zero constant. Thus we have $f\in I_j$. This complete the proof of (1).

Since $\dim_Q Q[x_1, \cdots, x_n]/I_j=\dim_Q H^*(F_j;Q)<\infty$, we have $\sqrt{I_j}=(x_1, \cdots, x_n)$ and hence I^j is primary. It follows that $I_j[t]$ is also primary and $\sqrt{I_j[t]}=(x_1, \cdots, x_n)$. By the definition of q_j, it follows that q_j is primary with radical $(x_1-c_{1j}t, \cdots, x_n-c_{nj}t)$. On the other hand we have $J=\bigcap_{j=1}^k q_j$, since ϕ^* is a monomorphism.

By (1. 2) and (2. 5) we obtain the following

Theorem 2. 6. *Let X be a closed manifold with a S^1-action, and assume that $H^*(X;Q)\cong Q[x_1, \cdots, x_n]/(p_1, \cdots, p_n)$ where* deg $x_i=2$ *and p_i is a homogeneous polynomials $(i=1, \cdots, n)$. Then, there is a $1-1$ correspondence between the zero points of J and the connected components of the fixed point set F, in such a way that, the multiplicity of a zero point of J equals to the dimension of the cohomology ring over Q of the corresponding component of F.*

3. In this section we assume that the cohomology ring of X satisfies (2. 1) with $\Lambda=Z$.

Let $p: E\longrightarrow X$ be the principal T^n-bundle induced from a universal principal T^n-bundle over B_G^n by $\mu=\mu_1\times\cdots\times\mu_n$ where $\mu_i: X\longrightarrow B_G$ denotes the map such that $\mu_i^*(t)=\alpha_i (i=1, \cdots, n)$. There is a bundle lifting $\Phi: G\times E\longrightarrow E$ of the given S^1-action on X [3]. For each $g\in p^{-1}(F_j)$ and $s\in G$ there is $t=(t_1, \cdots, t_n)\in T^n$ such that $\Phi(s, g)=g.t$. The correspondence $s\longrightarrow t$ defines a continuos homomorphism $h_g: G \longrightarrow T^n$. Since $\mathrm{Hom}(G, T^n)=Z^n$ has the discrete topology, h_g do not depend on the choices of $g\in p^{-1}(F_j)$. Thus, for each $j=1, \cdots, k$, we have a homomorphism $h_j: G\longrightarrow T^n$. We set $A_j=(a_{1j}, \cdots, a_{nj})$ where $h_j(s)=(s^{a_{1j}}, \cdots, s^{a_{nj}})\in T^n$ for every $s\in G (j=1, \cdots, k)$.

Let η_i: $E\times_{T^n}C^1\longrightarrow X$ be complex line bundle, where $(t_1, \cdots, t_n\in T^n$ acts on C^1 by the multiplication of $t_i (i=1, \cdots, n)$. Similarly, we define $\xi_i: (E\times_G E_G)\times C^1/T^n\longrightarrow X\times_G E_G$. By the definition of p it is easy to see that the Euler class of η_i is $\alpha_i (i=1, \cdots, n)$. Let $x_i\in H_G^2(X;Q)$ be the Euler class of ξ_i. Since η_i is the induced bundle $i^*(\xi_i)$, where $i: X$

$\subset X \times_G E_G$, we have $i^*(x_i) = \alpha_i$. Let τ be the canonical line bundle over B_G. Then we have

$$\xi_i | F_j \times B_G = p_1^*(\tau^{a_{ij}}) \otimes p_2^*(\eta_i | F_j)$$

where $p_1 : F_j \times B_G \longrightarrow B_G$ and $p_2 : F_j \times B_G \longrightarrow F_j$ are projections, and $\tau^{a_{ij}}$ denotes a_{ij} times tensor product of $\tau (i=1, \cdots, n : j=1, \cdots, k)$. It follows that

$$\phi^*(x_i) = \sum_{j=1}^{k} (b_{ij} + a_{ij} t),$$

where $\phi : F \times B_G \subset X \times_G E_G$, and b_{ij} is the Euler class of $\eta_i | F_j$. By (2.2) we see that $H_G^*(X; Z)$ is generated by $x_1, \cdots, x_n$ and t. From (2.6), we have the following:

Proposition 3.1. *Under the above notations, the zero points of the ideal J of relations are* $(1, a_{11}, \cdots, a_{n1}), \cdots, (1, a_{1k}, \cdots, a_{nk})$.

The system of vectors $A_1, \cdots, A_k$ is determined up to the translations by integral vectors. That is, if $\Phi' : G \times E \longrightarrow E$ is another bundle lifting of the given S^1-action on X with corresponding coordinates $A'_1, \cdots, A'_k$, there is an integral vector $(b_1, \cdots, b_n)$ such that

$$\Phi'(s, g) = \Phi(s, g) \cdot (s^{b_1}, \cdots, s^{b_n})$$

for every $s \in G$ and every $g \in E$ [3]. This implies that $A'_j = A_j + (b_1, \cdots, b_n)$ $(j = 1, \cdots, k)$.

Let p be a prime. We consider the restricted Z_p-action on X, where $Z_p \subset S^1$. Since E_{Z_p} can be taken to coincide with E_G, there is a commutative diagram:

$$\begin{array}{ccc} X \times_{Z_p} E_G & \xrightarrow{q} & X \times_G E_G \\ \downarrow & & \downarrow \pi \\ B_{Z_p} & \xrightarrow{\bar{q}} & B_G \end{array}$$

where $q : X \times_{Z_p} E_G \longrightarrow X \times_G E_G$ is a principal S^1-bundle. Let $\nu : (X \times_{Z_p} E_G) \times C^1/G \longrightarrow X \times_G E_G$ be the complex line bundle associated with q, where S^1 acts on C^1 by the multiplication. Then $\nu = \pi^*(\tau_p)$ and hence the Euler class of ν equals to zero mod p. Hence the Gysin sequence associated with ν reduces to

$$0 \longrightarrow H_G^q(X; Z_p) \xrightarrow{q^*} H_{Z_p}^q(X; Z_p) \longrightarrow H_G^{q-1}(X; Z_p) \longrightarrow 0$$

Now we suppose that $H^*(X;Z)$ has no p-torsion. Then X is totally non-homologous to zero in $X\times_G E_G$, and hence $H_G^{odd}(X;Z_p)=0$. Thus we have $q^*: H_G^{even}(X;Z_p)\cong H_{Z_p}^{even}(X;Z_p)$. Then it is clear that $H_{Z_p}^{even}(X;Z_p)\cong Z_p[t,x_1,\cdots,x_n]/J$ for odd p, where x_i means mod p reduction of $x_i\in H_G^2(X;Z)$. If $p=2$, we replace t by t^2. Let F', $F'_j(j=1,\cdots,h)$ be the fixed point set of the Z_p-action on X and its connected components respectively, and $\phi': F'\times_{Z_p}E_G \longrightarrow X\times_{Z_p}E_G$ the inclusion map. There is an exact sequence:

$$(3.2)\qquad 0\longrightarrow H_{Z_p}^*(X;Z_p)\xrightarrow{(\phi')^*} H_{Z_p}^*(F';Z_p)\longrightarrow H^*(X/Z_p,F';Z_p)\longrightarrow 0$$

where $(\phi')^*$ is epimorphic in high degrees [2]. Since $H^{odd}(F';Z_p)=0$ [2], we set

$$(\phi')^*(x_i)=\begin{cases}\sum_{j=1}^h (b'_{ij}+a'_{ij}t) & (p:\text{odd})\\ \sum_{j=1}^h (b'_{ij}+a'_{ij}t^2) & (p=2)\end{cases}$$

where t (resp. t^2) means $\bar{q}^*(t)$. Let I'_j be the homogeneous ideal generated by the coefficients of $g(t,x_1+a'_{1j}t,\cdots,x_n+a'_{nj}t)$ $(g\in J)$ with respect to t (if $p=2$, we replace t by t^2 similarly as in the above). By the similar argument as in the proof of (2.5), we have the following

Theorem 3.3. *If* $H^*(X;Z)\cong Z[\alpha_1,\cdots,\alpha_n]/(p_1,\cdots,p_m)$ *has no* p*-torsion, then*

$$H^*(F'_j;Z_p)\cong Z_p[x_1,\cdots,x_n]/I'_j \qquad (j=1,\cdots,h).$$

Remark. By a slightly delicate argument as in (2.2) we see that

$$H_{Z_p}^*(X;Z_p)\cong\begin{cases} Z_p[s,t,x_1,\cdots,x_n]/(s^2,p_1-tf_1,\cdots,p_m-tf_m) & (p:\text{odd})\\ Z_2[t,x_1,\cdots,x_n]/(p_1-t^2f_1,\cdots,p_m-t^2f_m) & (p=2).\end{cases}$$

Moreover we have the following proposition corresponding [2], Ch. VII, Proposition 5.3.

Proposition 3.4. *Under the condition of* (3.3), *two components* F_i *and* F_j *of* F *are contained in the same component of* F' *if and only if* $(a_{1i},\cdots,a_{ni})\equiv(a_{1j},\cdots,a_{nj})$ (mod p).

Proof. Suppose that F'_1 is the component of F' which contains F_i. Then there is a commutative diagram:

$$\begin{array}{ccc} H_G^2(X;Z) & \xrightarrow{\phi_j^*} & H_G^2(F_i;Z) \\ \downarrow & & \downarrow \\ H_{Z_p}^2(X;Z_p) \xrightarrow{(\phi_1')^*} & H_{Z_p}^2(F_1';Z_p) & \rightarrow H_{Z_p}^2(F_i;Z_p) \end{array}$$

where the vertical maps denotes the mod p reductions. We have $\phi_i{}^*(x_l) = b_{li} + a_{li}t$ and $(\phi_1')^*(x_l) = b_{l1}' + a_{l1}'t$ $(l=1, \cdots, n)$. It follows that $a_{li} \equiv a_{l1}'$ mod p $(l=1, \cdots, n)$. This implies that $(a_{1i}, \cdots, a_{ni}) \equiv (a_{1j}, \cdots, a_{nj})$ (mod p) if F_i and F_j are comtained in the same component of F'. The converse follows by (3. 3). q. e. d.

4. Let X be a complex flag manifold $U(n)/T^n$, where $U(n)$ is the n-th unitary group and T^n is a maximal torus. We denote by $p: U(n) \longrightarrow X$ the natural projection. Then p is a principal T^n-bundle. Let $\eta_i: U(n) \times_{T^n} C^1 \longrightarrow X$ be the complex line bundle as in §3 $(i=1, \cdots, n)$. We see that

$$H^*(X;Z) \cong Z[\alpha_1, \cdots, \alpha_n]/(\sigma_1, \cdots, \sigma_n)$$

where $\sigma_i = \sigma_i(\alpha_1, \cdots, \alpha_n)$ is the i-th fundamental symmetric function, and α_i is the Euler class of $\eta_i (i=1, \cdots, n)$ [1]. There are following examples of complex analytic S^1-actions on X.

Let $A = (a_1, \cdots, a_n)$ be an integral vector, and let $h_A: G \longrightarrow T^n$ be the homomorphism defined by

$$h_A(s) = \begin{pmatrix} s^{a_1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & 0 & & s^{a_n} \end{pmatrix}$$

for every $s \in G$. Then we have an S^1-action $\Phi_A: G \times U(n) \longrightarrow U(n)$ by taking $\Phi_A(s, g) = h_A(s) \cdot g$ for every $s \in G$ anc every $g \in U(n)$. Since Φ_A commutes with the right action of T^n on U(n), there is an S^1-action $\overline{\Phi}_A: G \times X \longrightarrow X$ on X, such that Φ_A is a bundle lifting of $\overline{\Phi}_A$. Then we have

$$\xi_1 \oplus \cdots \oplus \xi_n = \pi^*(\tau^{a_1} \oplus \cdots \oplus \tau^{a_n})$$

where $\xi_i: (U(n) \times {}_G E_G) \times C^1/T^n \longrightarrow X \times {}_G E_G$ denotes the complex line bundle defined similarly as in §3 $(i = 1, \cdots, n)$. This implies that, for each $i = 1, \cdots, n$

$$\begin{aligned}\sigma_i(x_1, \cdots, x_n) &= c_i(\xi_1 \oplus \cdots \oplus \xi_n)\\ &= \pi^*(c_i\ (\tau^{a_1} \oplus \cdots \oplus \tau^{a_n}))\\ &= \pi^*(\sigma_i(a_1 t, \cdots, a_n t))\\ &= t^i \sigma_i(a_1, \cdots, a_n)\end{aligned}$$

where c_i denotes the i-th chern class of corresponding bundles. According to the proof of (2. 2), we see $H_G^*(X; Z) \cong Z[t, x_1, \cdots, x_n]/J$ where

$$J = (\sigma_1 - t\sigma_1(a_1, \cdots, a_n), \cdots, \sigma_n - t^n\sigma_n(a_1, \cdots, a_n)).$$

Let $S(n)$ be the symmetric group of permutations of n symbols. Then the zero point set of J is $\{(1, a_{\pi(1)}, \cdots, a_{\pi(n)}) \mid \pi \in S(n)\}$.

Now let

$$\begin{aligned}&a_1 = \cdots = a_{n_1} < a_{n_1+1} = \cdots = a_{n_1+n_2} < \cdots\\ &< a_{n_1+\cdots+n_{m-1}+1} = \cdots = a_{n_1+\cdots+n_{m-1}+n_m},\end{aligned}$$

where $n_1 + \cdots + n_m = n$. It is easily seen that the fixed point set is a disjoint union of $n!/(n_1! \times \cdots \times n_m!)$ copies of $U(n_1)/T^{n_1} \times \cdots \times U(n_m)/T^{n_m}$.

Now we consider the fixed point set of S^1-actions on $U(3)/T^3$. In this case J has three homogeneous generators $x_1 + x_2 + x_3 - tf_1$, $x_1x_2 + x_2x_3 + x_3x_1 - tf_2$ and $x_1x_2x_3 - tf_3$ where f_1, f_2 and f_3 are suitable homogeneous polynomials of degree 0, 1 and 2 respectively. Let $A_j = (a_{1j}, a_{2j}, a_{3j})$ $(j = 1, \cdots, k)$ be the zero points of ${}^aJ$. By the remark following (3. 1) we assume that $A_1 = (0, 0, 0)$. Then after suitable substitution we may assume that J posseses three generators

$$\begin{cases} g_1 = x_1 + x_2 + x_3\\ g_2 = x_1^2 + x_1x_2 + x_2^2 - t(\alpha x_1 + \beta x_2)\\ g_3 = x_1^3 - t(ax_1^2 + bx_1x_2 + t(cx_2 + dx_2))\end{cases}$$

where α, β, a, b, c and d are constants. Then we have

$$I_1 = (x_1 + x_2 + x_3,\ x_1^2 + x_1x_2 + x_2^2,\ \alpha x_1 + \beta x_2,\ cx_1 + dx_2,\ ax_1^2 + bx_1x_2).$$

Consider the matrix

$$M = \begin{pmatrix} \alpha & \beta\\ c & d\end{pmatrix}$$

There are following cases:

i) Assume that M is non-singular. Then we have $I_1 = (x_1, x_2, x_3)$, and hence $H^*(F_1; Q) \cong Q$. Thus we have $F_1 \sim_Q \mathrm{pt}$[1].

1) $X \sim_Q Y$ means that $H^*(X; Q)$ and $H^*(Y; Q)$ are isomorphic.

ii) Assume that rk $M=1$. Then without loss of generality we may suppose that $x_1-\gamma x_2$ is contained in I_1 for some $\gamma \in Q$. It follows that

$$(1+\gamma+\gamma^2)x_2^2=x_1^2+x_1x_2+x_2^2-(x_1+x_2+\gamma x_2)(x_1-\gamma x_2)\in I_1.$$

Since $1+\gamma+\gamma^2>0$, this is equivalent to $x_2^2\in I_1$, and hence

$$\begin{aligned} H^*(F_1;Q) &\cong Q[x_1,x_2,x_3]/(x_1+x_2+x_3,x_1-\gamma x_2,x_2^2) \\ &\cong Q[x]/(x^2). \end{aligned}$$

Thus we have $F_1\sim_Q S^2$.

iii) Assume that $M=0$. Since $x_1^2+x_1x_2+x_2^2=0$ has no root other than (0, 0), this implies that F is connected. Hence $F_1=F=X$.

Since $\dim_Q H^*(F;Q)=\dim_Q H^*(X;Q)=6$, we obtain

Theorem 4.1. *There are the following possibilities of the fixed point set of non-trivial S^1-actions on $U(3)/T^3$.*

(1) $F\sim_Q S^2+S^2+S^2$ *(disjoint union).*

(2) $F\sim_Q S^2+S^2+2$ *points.*

(3) $F\sim_Q S^2+4$ *points.*

(4) $F\sim_Q 6$ *points.*

Remark. We have shown the examples for (1) and (4). For the type of (2) or (3), we do not know whether corresponding S^1-actions exist or not. But there are examples of J which satisfy the corresponding algebraic conditions for (2) or (3). For instance, let

$$J=(x_1+x_2+x_3,\ x_1^2+x_1x_2+x_2^2-t(2x_1+x_2),\ x_1^3-tx_1^2),$$

then the zero points of ^aJ are (0, 0, 0), (0, 1, −1), (1, 1, −2) and (1, −1, 0) where corresponding multiplicities are 2, 2, 1 and 1 respectively. Similarly, if we set

$$J=(x_1+x_2+x_3,\ x_1^2+x_1x_2+x_2^2-t(5x_1+x_2),\ x_1^3-t(5x_1^2+2x_1x_2)),$$

it is easy to see that the zero points of ^aJ are (0, 0, 0), (0, 1, −1), (1, −2, 1), (2, −3, 1) and (5, 0, −5) with multiplicties 2, 1, 1, 1 and 1.

5. S^1-actions on the connected sum $X=CP(3)\sharp CP(3)$ gives another examples in our case. Since $H^*(X;Z)\cong Z[\alpha_1,\alpha_2]/(\alpha_1\alpha_2,\alpha_1^3-\alpha_2^3)$, by (2.2) we may assume that J has two generators

$$\begin{cases} g_1=x_1x_2-t(\alpha x_1+\beta x_2) \\ g_2=x_1^3-x_2^3-t(ax_1^2+bx_2^2+t((cx_1+dx_2)) \end{cases}$$

where α,β,a,b,c and d are integers. Let F_1 be the connected component of F which corresponds to the zero point (0, 0) of ${}^aJ$, and consider the matrix

$$M=\begin{pmatrix} \alpha & \beta \\ c & d \end{pmatrix}.$$

There are three possibilities for the rank of M.

i) M is non-singular; that is, rk $M=2$. In this case we have $I_1=(x_1, x_2)$, and hence $H^*(F_1; Q)\cong Q$. Thus we have $F_1 \sim_Q$ pt.

ii) rk $M=1$. In this case we may suppose that $\alpha=c=0$ without loss of generality.

If $a\neq 0$, $I_1=(x_2, x_1^2)$, hence $H^*(F_1; Q)\cong Q[x_1, x_2]/(x_2, x_1^2)\cong Q[x]/(x^2)$, i.e. $F_1 \sim_Q S^2$.

If $a=0$, $I_1=(x_2, x_1^3)$, hence $H^*(F_1; Q)\cong Q[x_1, x_2]/(x_2, x_1^3)\cong Q[x]/(x^3)$, i.e. $F_1 \sim {}_Q CP(2)$.

iii) $M=0$; that is, rk $M=0$.

If $a=b=0$. Then $I_1=(x_1x_2, x_1^3-x_2^3)$, and hence $F_1=F=X$.

If $ab\neq 0$. Then $I_1=(ax_1^2+bx_2^2, x_1x_2)$, and hence $\dim_Q H^*(F_1; Q)=4$.

If $a=0$, $b\neq 0$ (or $a\neq 0$, $b=0$). Then $I_1=(x_2^2, x_1^3)$(resp. $I_1=(x_1^2, x_2^3)$), and hence $\dim_Q H^*(F_1; Q)=4$.

Similarly as (4.1) we obtain the following

Theorem 5.1. *There are the following possibilities of the fixed point set of non-trivial S^1-actions on $CP(3)\, \sharp\, CP(3)$.*

(1) $F\sim {}_Q F_1+2$ *points,*
where $H^*(F_1; Q)\cong Q[x_1, x_2]/(x_1x_2, ax_1^2+bx_2^2)$ *for some* $0\neq a$, $b\in Q$

(2) $F\sim_Q CP(2)+CP(2)$.

(3) $F\sim_Q CP(2)+S^2+$ pt.

(4) $F\sim_Q CP(2)+3$ *points.*

(5) $F\sim_Q S^2+S^2+S^2$.

(6) $F\sim_Q S^2+S^2+2$ *points.*

(7) $F\sim_Q S^2+4$ *points.*

(8) $F\sim_Q 6$ *points.*

Proof. Since $\dim_Q H^*(F; Q)=\dim_Q H^*(X; Q)=6$ [2], it suffices to show that, if a component F_1 of F has cohomological dimension 4 over Q, it follows that F is in the case (1). Suppose that $\dim_Q H^*(F_1; Q)=4$. Then we have

$$J=(x_1x_2, x_1^3-x_2^3-t(ax_1^2+bx_2^2))$$

for some $a, b \in Q$, and there is another component F_2 of F. Let (a_1, a_2) be the zero point of ${}^aJ$ which corresponds to F_2. Since $(a_1, a_2) \neq (0, 0)$ is a root of the equations

$$\begin{cases} x_1x_2 = 0 \\ x_1^3 - x_2^3 - ax_1^2 - bx_2^2 = 0 \end{cases}$$

we see that, only one of a_1, a_2 equals to zero. Let $a_1 = 0$, then $a_2 \neq 0$, and $a_2^3 + ba_2^2 = 0$ implies that $b = -a_2 \neq 0$. It is easy to see that $I_1 = (x_1, x_2)$, and there is another component F_3 of F. Let (b_1, b_2) be the zero point of ${}^aJ$ which corresponds to F_3. As above, we see that $b_2 = 0$ and $a = b_1 \neq 0$. The case $a_2 = 0$ is similar. q. e. d.

Now we construct examples of S^1-actions on $CP(3) \# CP(3)$.

Let S^7 be the unit sphere in C^4, and let $p_1 : S^7 \longrightarrow CP(3)$ be the natural projection. We set $D^6 = \{p_1(z_1, z_2, z_3, z_4) \mid (z_1, z_2, z_3, z_4) \in S^7, |z_4|^2 \geqq 1/2\}$, and define the diffeomorphism $g : p_1^{-1}(\partial D^6) \times S^1 \longrightarrow p_1^{-1}(\partial D^6) \times S^1$ by $g(z_1, z_2, z_3, z_4, s)$
$= (s\bar{z}_1/(\sqrt{2}\,\bar{z}_4), s\bar{z}_2/(\sqrt{2}\,\bar{z}_4), s\bar{z}_3/(\sqrt{2}\,\bar{z}_4), s/(\sqrt{2}\,\bar{z}_4))$
for every $(z_1, z_2, z_3, z_4) \in p_1^{-1}(\partial D^6)$ and every $s \in S^1$, which induces the orientation reversing diffeomorphism $\bar{g} : \partial D^6 \longrightarrow \partial D^6$. Then we may consider $CP(3) \# CP(3)$ to be the attaching space $(CP(3) - \operatorname{Int} D^6) \cup_{\bar{g}} (CP(3) - \operatorname{Int} D^6)$ which is covered by the attaching space $E_1 = (S^7 - p_1^{-1}(\operatorname{Int} D^6)) \times S^1 \cup_g (S^7 - p_1^{-1}(\operatorname{Int} D^6)) \times S^1$. Let $f_1, f_2 : (S^7 - p_1^{-1}(\operatorname{Int} D^6)) \times S^1 \longrightarrow S^5 \times S^3$ be the map defined by

$$f_1(z_1, z_2, z_3, z_4, s) = \left(\frac{\bar{s}z_1}{\sqrt{1-|z_4|^2}}, \frac{\bar{s}z_2}{\sqrt{1-|z_4|^2}}, \frac{\bar{s}z_3}{\sqrt{1-|z_4|^2}}, z_4, \sqrt{1-|z_4|^2}\,s\right)$$

$$f_2(z_1, z_2, z_3, z_4, s) = \left(\frac{s\bar{z}_1}{\sqrt{1-|z_4|^2}}, \frac{s\bar{z}_2}{\sqrt{1-|z_4|^2}}, \frac{s\bar{z}_3}{\sqrt{1-|z_4|^2}}, \sqrt{1-|z_4|^2}\,s, z_4\right)$$

respectively for every $(z_1, z_2, z_3, z_4) \in S^7 - p_1^{-1}(\operatorname{Int} D^6)$ and every $s \in S^1$. It is easy to see that f_1 and f_2 defines a diffeomorphism $f : E_1 \longrightarrow S^5 \times S^3$. Now define T^2-actions on the two copies of $(S^7 - p_1^{-1}(\operatorname{Int} D^6)) \times S^1$ by

$$(z_1, z_2, z_3, z_4, s) \cdot (t_1, t_2) = (t_1 z_1, t_1z_2, t_1z_3, t_1z_4, t_2s)$$

and

$$(z_1, z_2, z_3, z_4, s) \cdot (t_1, t_2) = (t_2z_1, t_2z_2, t_2z_3, t_2z_4, t_1s)$$

for every $(z_1, z_2, z_3, z_4, s) \in (S^7 - p_1^{-1}(\operatorname{Int} D^6)) \times S^1$ and every $(t_1, t_2) \in T^2$ respectively. Then it is clear that f induces the T^2-action on $S^5 \times S^3$ defined by

$$(z_1,\ z_2,\ z_3,\ z_4,\ z_5)\cdot(t_1,\ t_2)$$
$$=(t_1t_2^{-1}z_1,\ t_1t_2^{-1}z_2,\ t_1t_2^{-1}z_3,\ t_1z_4,\ t_2z_5)$$

for every $(z_1, z_2, z_3)\in S^5$, every $(z_4, z_5)\in S^3$ and every $(t_1, t_2)\in T^2$. By the T^2-action, $p: S^5\times S^3\longrightarrow X$ is a principal T^2-bundle, and plays the role of the bundle p defined in §3 for the suitable generators in $H^2(X; Z)$.

Let a_i be an integer $(i=1,\cdots,5)$, and let $\Phi_A: G\times S^5\times S^3\longrightarrow S^5\times S^3$ be the S^2-action on $S^5\times S^3$ defined by

$$\Phi_A(s,\ z_1,\ z_2,\ z_3,\ z_4,\ z_5)$$
$$=(s^{a_1}z_1,\ s^{a_2}z_2,\ s^{a_3}z_3,\ s^{a_4}z_5,\ s^{a_5}z_5)$$

for every $s\in G$ and every $(z_1, z_2, z_3, z_4, z_5)\in S^5\times S^3$. Since Φ_A commutes with the T^2-action on $S^5\times S^3$, there is the induced S^1-action on X. We denote by ξ_1, ξ_2: $(S^5\times S^3\times {}_GE_G)\times C^1/T^2\longrightarrow X\times {}_GE_G$ the complex line bundle defined similarly as in §3. Then we see that the bundle $(\xi_1\otimes\pi^*(\tau^{-a_4}))\oplus(\xi_2\otimes\pi^*(\tau^{-a_5}))$ has everywhere non-zero cross-section, and hence the Euler class
$c_2((\xi_1\otimes\pi^*(\tau^{-a_4}))\oplus(\xi_2\otimes\pi^*(\tau^{-a_5})))=0$, i. e. $(x_1-a_4t)(x_2-a_5t)\in J$ where $H_G^*(X; Z)=Z[t, x_1, x_2]/J$. Similarly we have $(x_1-x_2-a_1t)(x_1-x_2-a_2t)(x_1-x_2-a_3t)\in J$. Thus we see $J=((x_1-a_4t)(x_2-a_4t), (x_1-x_2-a_1t)(x_1-x_2-a_2t)(x_1-x_2-a_3t))$. There are following types:

i) Suppose that $a_1=a_2=a_3=a_4-a_5$. Then Φ_A induces the trivial action on X.

ii) Suppose that $a_1=a_2=a_3\neq a_4-a_5$. Then the fixed point set F consists of disjoint union of two copies of $CP(2)$. The corresponding zero points of ${}^aJ$ are (a_4, a_4-a_1) and (a_1+a_5, a_5) respectively.

iii) Suppose that $a_1=a_2=a_4-a_5\neq a_3$. Then we have $F\approx S^2\times S^2+2$ points, and corresponding zero points of ${}^aJ$ are (a_4, a_5), (a_4, a_4-a_3) and (a_3+a_5, a_5) respectively.

iv) Suppose that $a_1=a_2\neq a_3=a_4-a_5$. Then we have $F\approx S^2+S^2+S^2$, and corresponding zero points are (a_4, a_4-a_1), (a_1+a_5, a_5) and (a_4, a_4-a_3).

v) Suppose that $a_1=a_2\neq a_3$ and $a_4\neq a_i+a_5$ $(i=1, 3)$. Then we have $F\approx S^2+S^2+2$ points, and corresponding zero points are (a_4, a_4-a_1), (a_1+a_5, a_5), (a_4, a_4-a_3) and (a_3+a_5, a_5) respectively.

vi) Suppose that a_1, a_2, a_3 are mutually distinct and $a_4=a_1+a_5$. Then we have $F\approx S^2+4$ points, and corresponding zero points are (a_4, a_5), (a_4, a_4-a_2), (a_4, a_4-a_3), (a_2+a_5, a_5) and (a_3+a_5, a_5) respectively.

vii) Suppose that a_1, a_2, a_3 are mutually distinct and $a_4\neq a_i+a_5$ $(i=1,$

2, 3). Then F consists of 6 isolated points. The corresponding zero points are (a_4, a_4-a_1), (a_4, a_4-a_2), (a_4, a_4-a_3), (a_1+a_5, a_5), (a_2+a_5, a_5) and (a_3+a_5, a_5).

Thus we have shown the examples except (3) and (4). A similar construction shows that there are some examples for (3) where the attaching map g above is changed. But, we can not construct the examples for (4) in this manner.

6. Concluding remarks. Let $\mathfrak{A} = (p_1, \cdots, p_n) \subset Q[x_1, \cdots x_n]$ be a homogeneous ideal such that $\sqrt{\mathfrak{A}} = (x_1, \cdots, x_n)$, and J a homogeneous ideal of $Q[t, x_1, \cdots, x_n]$ which has a generator system $p_1 - tf_1, \cdots, p_n - tf_n$ for some $f_1, \cdots, f_n \in Q[t, x_1, \cdots, x_n]$ and satisfies the conditions of (1.2). Let $\xi^{(1)}, \cdots, \xi^{(k)}$ be the zero points of J with multiplicities $m_1, \cdots, m_k$. There are several cases where $\xi^{(j)}, m_j$ $(j=1, \cdots, k)$ determine J uniquely up to certain equivalence.

i) Let $n=3$, and $p_i = \sigma_i$ $(i=1, 2, 3)$. Then it is shown that, for a fixed $\{(\xi^{(j)}, m_j) \mid j = 1, \cdots, k\}$, corresponding J is determined uniquely.

ii) Let $n=4$ or 5, and $p_i = \sigma_i$ $(i=1, \cdots, n)$. We put $P = \{(1, a_{\pi(1)}, \cdots, a_{\pi(n)}) \mid \pi \in S(n)\}$ for a fixed integral vector $(a_1, \cdots, a_n)$, where $S(n)$ denotes the symmetric group of permutations of n symbols. Then the ideal J above, whose zero point set is P with constant multiplicities, is determined uniquely.

iii) Let $n=2$, $p_1 = x_1x_2$ and $p_2 = x_1^3 - x_2^3$. Then easy calculation shows that, if J' is another ideal whose zero points and their multiplicities are that of J and satisfies the above conditions, then either $J' = J$ or J' is mapped onto J by the permutation $(x_1, x_2) \longrightarrow (x_2, x_1)$.

Let $\mathfrak{A} = (p_1, \cdots, p_n) \subset Z[x_1, \cdots, x_n]$ be a homogeneous ideal such that $\sqrt{\mathfrak{A}} = (x_1, \cdots, x_n)$, and J (resp. $\bar{J}$) a homogeneous ideal generated by $p_j - tf_j$ $(j = 1, \cdots, n)$ in $Z[t, x_1, \cdots, x_n]$ (resp. $Q[t, x_1, \cdots, x_n]$). It is not difficult to show that, if $Z[x_1, \cdots, x_n]/\mathfrak{A}$ is a free Abelian, then $\bar{J} \cap Z[t, x_1, \cdots, x_n] = J$.

It follows that, if $X = U(3)/T^3$ or $X = CP(3) \# CP(3)$, then equivariant cohomology rings $H_G^*(X; Z)$ (and hence $H^*(F; Q)$) are determined by A_j and $\dim_Q H^*(F_j; Q)$ $(j = 1, \cdots, k)$.

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