ON s-UNITAL RINGS. II

Dedicated to Professor Tominosuke Otsuki on his 60th birthday

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This is a natural sequel to [11]. The notation and terminology employed there will be used here, and $S_l(R)$ will denote the socle of $_RR$. In this paper, several results obtained by Yue Chi Ming [13], [14] and V. Gupta [5] for rings with identity will be carried over to s-unital rings and, in addition, some of our previous results obtained in [3], [4], [8], [10] and [11] will be improved.

1. In general, an element a of a multiplicative semigroup S is called a *semi-unit* if there exists an element a^* (called a *semi-inverse* of a) such that $a^2a^* = a$, $a^{*2}a = a^*$ and $aa^* = a^*a$. It is known that if $a^2a' = a = a''a^2$ for some a', $a'' \in S$ then a has a uniquely determined semi-inverse a^* and $a^*b = ba^*$ provided $ab = ba$ (cf. [2, Lemma 1]). Moreover, if a is a left (or right) π-regular element of a ring of bounded index n then a^n is a semi-unit (see [2, Theorem 4]). Needless to say, in case R contains 1, a left (or right) regular element of R is a unit if and only if it is a semi-unit.

The next is contained in [11, Theorem 4].

Theorem 1. If *R is left s-unital, then the following conditions are equivalent*:

1) *Every irreducible left R-module is s-injective.*
2) *Every s-unital left R-module M is semisimple*; $\mathrm{rad}(_RM) = 0$.
3) *Every homomorphic image of $_RR$ is semisimple.*
4) *Every left ideal of R is an intersection of maximal left ideals.*

If a left s-unital ring R satisfies one of the equivalent conditions in Theorem 1, then R is called a *left V-ring.* A left s-unital ring R will be called a *left V'-ring* (resp. *left p-V'-ring*) if every irreducible, singular left R-module is s-injective (resp. p-injective).

A left ideal $\mathfrak{l}$ of a ring R is said to be *semi-modular* if for each $a \in R$ there exists some $c \in R$ such that $a - ac \in \mathfrak{l}$. Obviously, every modular left ideal in the sense of [6] is semi-modular.

Proposition 1 (cf. [11, Proposition 4]). (1) *A left V-ring R is a left p-V-ring if and only if every maximal left ideal is semi-modular.*

(2) *If* R *is a left* V'*-ring and every maximal left ideal is semi-modular, then* R *is a left* p-V'*-ring.*

(3) R *is a left* V*-ring (resp. left* p-V*-ring) if and only if* R *is a left* V'*-ring (resp. left* p-V'*-ring) and every minimal left ideal is s-injective (resp. p-injective).*

Proof. (1) First, assume that every maximal left ideal of R is semi-modular. Let $g: R \longrightarrow M$ be an extension of a non-zero R-homomorphism f of Ra into an irreducible left R-module M. Since $\mathfrak{m} = \mathrm{Ker}\ g$ is semi-modular, there exists an element c such that $a - ac \in \mathfrak{m}$. Hence, $(xa)f = (xac)g = xa \cdot cg$ for all $x \in R$. Conversely, we assume that R is a left p-V-ring. Let $\mathfrak{m}$ be a maximal left ideal, and a an element of R. Considering the R-homomorphism $f: Ra \longrightarrow R/\mathfrak{m}$ defined by $xa \longmapsto xa + \mathfrak{m} (x \in R)$, we can find an element $c \in R$ such that $a + \mathfrak{m} = ac + \mathfrak{m}$, which means that $\mathfrak{m}$ is semi-modular.

(2) This is evident by the proof of (1).

(3) If an irreducible left R-module is not singular then it is isomorphic to some minimal left ideal. This proves (3).

Recently, in [5], V. Gupta introduced the notion of a left weakly π-regular ring as a generalization of those of a fully left idempotent ring and of a strongly π-regular ring; R is called a *left weakly* π*-regular ring* if for each $a \in R$ there exists a natural number n such that $a^n \in (Ra^n)^2$, i. e., $a^n = ea^n$ with some $e \in Ra^nR$.

The right analogues of the above notions will be defined in an obvious way.

2. Our first lemma contains several easy statements, which will be used frequently in the subsequent study.

Lemma 1. (1) *If* a *is a left regular element of* R *and* $ea = a$ *for some* $e \in R$, *then* e *is a right identity of* R. *If, in addition,* a *is right regular then* e *is the identity of* R.

(2) *If* R *is left unital and every* Ra *is a left annihilator, then* R *is (right unital and) left s-unital.*

(3) *If* R *is right s-unital and* $(a|$ *is a direct summand of* ${}_RR$, *then* $a = aa'a$ *for some* $a' \in R$.

(4) *If a proper left ideal* $\mathfrak{l}$ *of a left s-unital ring* R *contains* $l(a)$ *with some* $a \in R$, *then* $\mathfrak{l}$ *is contained in a maximal left ideal.*

Proof. (1) Since $xe - x \in l(a) = 0$ for all $x \in R$, e is a right identity of R. If furthermore $r(a) = 0$ then $ae = a$ implies that e is a left identity of R.

(2) In fact, $Ra \cdot r(Ra) = 0$ implies $a \cdot r(Ra) \subseteq r(R) = 0$, so $a \in l(r(Ra)) = Ra$.

(3) Let $R = (a| \oplus \mathfrak{k}$ with a left ideal $\mathfrak{k}$, and $ae = a$. Since $e = u + k$ with some $u \in (a|$ and $k \in \mathfrak{k}$, we obtain $a - au = ak \in (a| \cap \mathfrak{k} = 0$. Hence, $a = au = au^2 \in aRa$.

(4) Choose $e \in R$ with $ea = a$. Since $x - xe \in l(a)$ for all $x \in R$, $\mathfrak{l}$ is a modular left ideal. It is well-known that $\mathfrak{l}$ is contained in a maximal left ideal (see, e. g. [6, Proposition I. 3. 2]).

Corollary 1 ([10, Lemma 1 (a)]). *If $\mathfrak{a}$ is a proper ideal of a left s-unital ring R, then $\mathfrak{a}$ is contained in a maximal left ideal.*

Proof. For any $a \in R \backslash \mathfrak{a}$, it is easy to see that $\mathfrak{a} + l(a) \neq R$. Hence, the statement follows from Lemma 1 (4).

A left annihilator in R is called a *maximal left annihilator* if it is maximal among the left annihilators different from R.

Theorem 2 (cf. [12, Lemma 2], [13, Theorem 9] and [14, Theorem 2]). *The following conditions are equivalent*:

1) *R is a regular ring.*

2) *Every left R-module is p-injective.*

3) *Every $(a|$ is p-injective.*

4) *R is left s-unital and every semisimple homomorphic image of ${}_RR$ is p-injective.*

5) *R is left s-unital, every $|a)$ is a right annihilator, and every singular homomorphic image of ${}_RR$ is p-injective.*

6) *R is s-unital, every Ra is either $l(b)$ with some $b \in R$ or a direct summand of ${}_RR$, and every singular homorphic image of ${}_RR$ is p-injective.*

7) *R is a semiprime s-unital ring, and every finitely generated left ideal is either a maximal left annihilator or a direct summand of ${}_RR$.*

Proof. Obviously, 1) $\Longrightarrow$ 7). As was noted in the introduction of [4], 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3) $\Longrightarrow$ 1), and 1) $\Longrightarrow$ 4) $-$ 6).

4) $\Longrightarrow$ 3) Obviously, R is a left p-V-ring. Let a be an arbitrary non-zero element of R, and b a non-zero element of Ra. Then there exists a left subideal $\mathfrak{l}'$ of Ab which is maximal with respect to excluding b. Since $Rb/\mathfrak{l}'$ is an irreducible left R-module, there exists an element $c \in Rb$ such that $xb + \mathfrak{l}' = xbc + \mathfrak{l}'$ for all $x \in R$. Let $\mathfrak{l} = \{x \in Ra \mid xc \in \mathfrak{l}'\}$. Then, $\mathfrak{l}' = \mathfrak{l} \cap Rb$, $Ra/\mathfrak{l}$ is R-isomorphic to $Rb/\mathfrak{l}'$ and $b \notin \mathfrak{l}$,

namely, $\mathfrak{l}$ is a maximal left subideal of Ra and excludes b. Hence, Ra is semisimple, so that Ra is p-injective.

5) $\Longrightarrow$ 1) Let $|a)=r(S)$ with a subset S of R, and $ea=a$, Choose a left ideal $\mathfrak{k}$ such that $\mathfrak{l}=l(a)\oplus\mathfrak{k}$ is essential in ${}_RR$, and consider the R-homomorphism $f: Ra \longrightarrow R/\mathfrak{l}$ defined by $xa \longmapsto x+\mathfrak{l}$ $(x\in R)$. Then we can find an element $c\in R$ such that $x+\mathfrak{l}=xac+\mathfrak{l}$ for all $x\in R$. Setting $e-ac=u+k$ with $u\in l(a)$ and $k\in\mathfrak{k}$, for any $s\in S$ there holds $se=su+sk$. Since $sk=se-su\in l(a)\cap\mathfrak{k}=0$, it follows $k\in r(S)=|a)$. Hence, $a=ea=(ac+u+k)a=aca+kea\in aRa$,

6) $\Longrightarrow$ 1) In case Ra is a direct summand of ${}_RR$, by Lemma 1 (3) we have $a=aa'a$ with some $a'\in R$. Next, we consider the case $Ra=l(b)$. Let $ae=a$, and $\mathfrak{k}$ a left ideal such that $\mathfrak{l}=l(b)\oplus\mathfrak{k}$ is essential in ${}_RR$. As above, we can find then an element $c\in R$ such that $x+\mathfrak{l}=xbc+\mathfrak{l}$ for all $x\in R$. Since $e-ebc\in\mathfrak{l}$, we set $e-ebc=a'a+k$, where $a'a\in Ra=l(b)$ and $k\in\mathfrak{k}$. Then, $a=aa'a+ak$, and $a-aa'a=ak\in Ra\cap\mathfrak{k}=0$. Hence, $a=aa'a$.

7) $\Longrightarrow$ 1) First, we shall prove that R is left non-singular. Suppose $Z=Z_l(R)$ contains a non-zero element z. Since Z contains no non-zero idempotents, Rz can not be a direct summand of ${}_RR$ (Lemma 1 (3)), so that Rz is a maximal left annihilator $l(t)$ $(t\neq 0)$. Moreover, Rz is essential in ${}_RR$. In fact, if not, $Rz\cap Rw=0$ for some non-zero $w\in R$. Recalling that $Rz\oplus Rw(\supset Rz)$ is a direct summand of ${}_RR$, we see that Rz is also a direct summand of ${}_RR$, which is impossible again by Lemma 1 (3). Hence, t is in Z. If $Rz\neq Z$ then there exists some $z'\in Z$ such that $Rz+Rz'=R$, which means $Z=R$. By [11, Theorem 1], there exists an element e such that $ze=z$ and $z'e=z'$. Then e is obviously a right identity of $R=Z$, which is contradictory. Hence, $Rz=Z$. But then $(Rt)^2\subseteq Z\cdot Rt=0$, which contradicts the semiprimeness of R. Thus, we have seen $Z=0$. Now, assume that Ra is a maximal left annihilator. Since $Z=0$, there exists a non-zero $b\in R$ such $Ra\oplus Rb$ is a direct summand of ${}_RR$. Then Ra is also a direct summand of ${}_RR$, and hence R is a regular ring by Lemma 1 (3).

Next, we shall prove the following which includes [7, Theorem 2] and [8, Theorem].

Theorem 3. *The following conditions are equivalent:*

1) $R=\bigoplus_{\lambda\in\Lambda}R_\lambda$, *where* R_λ *is the complete ring of linear transformations of finite rank of a vector space over a division ring.*

2) R *is a semiprime ring and every left ideal is a left annihilator.*

3) R *is a semiprime ring and every* Ra *and every maximal left ideal*

are left annihilators.

4) *R is a left s-unital semiprime ring and every maximal left ideal is a left annihilator.*

5) *R is a left s-unital, left non-singular ring and every maximal left ideal is a left annihilator.*

6) *R is a regular ring and every maximal left ideal is a left annihilator.*

7) *R is a right s-unital, left V-ring, and every maximal left ideal is a left annhilator.*

8) *R is a left p-V-ring and every maximal left ideal is a left annihilator.*

9) *R is a fully left idempotent ring and every maximal left ideal is a left annihilator.*

Proof. Obviously, 2) $\Longrightarrow$ 3) and 6) $\Longrightarrow$ 9) $\Longrightarrow$ 4). By [6, Theorem IV. 16. 3], 1) $\Longrightarrow$ 2), 6) and 7). By Lemma 1 (2) 3) $\Longrightarrow$ 4), and by Proposition 1 (1) and [11, Proposition 6] 7) $\Longrightarrow$ 8) $\Longrightarrow$ 9).

4) $\Longrightarrow$ 5) Assume $Z = Z_l(R) \neq 0$. Take a left ideal $\mathfrak{k}$ such that $Z \oplus l(Z) \oplus \mathfrak{k}$ is essential in $_RR$. Then $\mathfrak{k} \subseteq r(Z) = l(Z)$, so that $\mathfrak{k} = 0$, which means that $Z \oplus l(Z)$ is essential in $_RR$. If $Z \oplus l(Z) \neq R$ then, by Corollary 1 and the hypothesis, $Z \oplus l(Z) \subseteq l(z)$ for some non-zero $z \in Z$. But $z \in r(Z) \cap r(l(Z)) = l(Z) \cap r(l(Z)) = 0$, a contradiction. Hence $Z \oplus l(Z) = R$, whence it follows $r(l(Z)) = Z$. Again by Corollary 1, $l(Z)$ is contained in a maximal left ideal $l(w)$ with some non-zero $w \in r(l(Z)) = Z$. Since Rw ($\cong R/l(w)$) is a minimal left ideal, $Rw = Re$ with an idempotent $e \in Z$, a contradiction.

5) $\Longrightarrow$ 1) Since no maximal left annihilators are essential in $_RR$, every maximal left ideal is a direct summand of $_RR$. Hence, by [10, Lemma 1 (b)], $_RR$ is completely reducible, and R is a left V-ring. Accordingly, every left ideal of R is a left annihilator. Now, let R_λ be an arbitrary homogeneous component of $_RR$. Then R_λ is a simple ring and every left ideal of R_λ is a left annihilator in R_λ. Hence, by [6, Theorem IV. 16. 3], R_λ is the complete ring of linear transformations of finite rank of a vector space over a division ring.

3. The main theme of this section will concern left V'-rings and left p-V'-rings, and the next will play an important role in our study.

Lemma 2 (cf. [13, Lemma 1]). *Let R be a left p-V'-ring. If a left ideal $\mathfrak{l}$ of R contains $RaR + l(a)$ with some $a \in R$ then $\mathfrak{l}$ is a direct* *summand of* ${}_RR$.

Proof. There exists a left ideal $\mathfrak{k}$ such that $\mathfrak{l} \oplus \mathfrak{k}$ is essential in ${}_RR$. Assume $\mathfrak{l} \oplus \mathfrak{k} \neq R$. Then, by Lemma 1 (4), $\mathfrak{l} \oplus \mathfrak{k}$ is contained in a maximal left ideal $\mathfrak{m}$. Since $\mathfrak{m}$ is essential in ${}_RR$, the irreducible, singular left R-module $R/\mathfrak{m}$ is p-injective. We consider here the R-homomorphism $f: Ra \longrightarrow R/\mathfrak{m}$ defined by $xa \longmapsto x+\mathfrak{m}\ (x \in R)$. Then there exists some $b \in R$ such that $x+\mathfrak{m} = xab+\mathfrak{m}$ for all $x \in R$. But, this yields a contradiction $\mathfrak{m} = R$.

Corollary 2 (cf. [13, Propositions 3 and 6]). *Let* R *be a left* p-V'-*ring.*

(1) $Z_l(R) \cap J(R) = 0$.

(2) *If* a *is a regular element of* R *then* R *has an identity and* $RaR = R$.

(3) *If* $\mathfrak{l}$ *is an essential left ideal of* R *then* $\mathfrak{l}^2 = \mathfrak{l}$.

(4) *If* R *is semiprime, then* R *is fully left idempotent, and is a semiprimitive, right non-singular ring.*

Proof. (1) Let $a \in Z_l(R) \cap J(R)$, and $a = ea$. By Lemma 2, the essential left ideal $RaR + l(a)$ is a direct summand of ${}_RR$, and hence $R = RaR + l(a) = J(R) + l(a)$. Let $e = u + v$ with some $u \in J(R)$ and $v \in l(a)$, and u' the quasi-inverse of u. Then $0 = a - ua = a - (u + u' - u'u)a = a$.

(2) Since R contains an identity by Lemma 1 (1), this is given in [13, Proposition 3 (ii)], and easily seen by Lemma 2.

(3) Let a be an arbitrary element of $\mathfrak{l}$, and $a = ea$. Since $R = \mathfrak{l} + \mathfrak{l}R + l(a)$ by Lemma 2, it follows $a = ea \in (\mathfrak{l} + \mathfrak{l}R + l(a))a = \mathfrak{l}a + \mathfrak{l}Ra \subseteq \mathfrak{l}^2$.

(4) If R is not fully left idempotent, there exists some $a \in R$ with $(Ra)^2 \neq Ra$. Let $\mathfrak{l}$ be a maximal left subideal of Ra containing $(Ra)^2$. Since R is semiprime, $(Ra)^2$ is essential in ${}_RRa$, so that $Ra/\mathfrak{l}$ is an irreducible, singular left R-module. Then there exists some $b \in Ra$ with $x + \mathfrak{l} = xb + \mathfrak{l}$ for all $x \in Ra$. But this implies a contradiction $\mathfrak{l} = Ra$. The latter assertion is given in [11, Proposition 7].

Combining Corollary 2 (4) with [9, Theorem 17], we readily obtain

Corollary 3 (cf. [13, Corollary 8]). *If* R *is a semiprime left Goldie, left* p-V'-*ring, then* R *is a finite direct sum of simple rings.*

In case R is s-unital, we obtain the following characterizations of a left V'-ring.

Theorem 4. *If R is s-unital, then the following conditions are equivalent*:

1) *R is a left V'-ring.*

2) *R is a left p-V'-ring and every singular, s-unital left R-module is semisimple.*

2′) *$Z_l(R) \cap J(R) = 0$ and every singular, s-unital left R-module is semisimple.*

2″) *$Z_l(R) \cap S_l(R) = 0$ and every singular, s-unital left R-module is semisimple.*

3) *R is a left p-V'-ring and every singular homomorphic image of ${}_RR$ is semisimple.*

3′) *$Z_l(R) \cap J(R) = 0$ and every singular homomorphic image of ${}_RR$ is semisimple.*

3″) *$Z_l(R) \cap S_l(R) = 0$ and every singular homomorphic image of ${}_RR$ is semisimple.*

4) *R is a left p-V'-ring and every essential left ideal is an intersection of maximal left ideals.*

4′) *$Z_l(R) \cap J(R) = 0$ and every essential left ideal is an intersection of maximal left ideals.*

4″) *$Z_l(R) \cap S_l(R) = 0$ and every essential left ideal is an intersection of maximal left ideals.*

Proof. Obviously, 2) $\Longrightarrow$ 3) $\Longrightarrow$ 4), 2′) $\Longrightarrow$ 3′) $\Longrightarrow$ 4′), and 2″) $\Longrightarrow$ 3″) $\Longrightarrow$ 4″). Since $Z_l(R)$ contains no non-zero idempotents, there holds $Z_l(R) \cap S_l(R) \subseteq Z_l(R) \cap J(R)$. Combining this with Corollary 2 (1), we see that 2) $\Longrightarrow$ 2′) $\Longrightarrow$ 2″), 3) $\Longrightarrow$ 3′) $\Longrightarrow$ 3″) and 4) $\Longrightarrow$ 4′) $\Longrightarrow$ 4″).

1) $\Longrightarrow$ 2) By Proposition 1 (2), R is a left p-V'-ring. As is noted above, $Z_l(R) \cap S_l(R) = 0$. Now, let M be an arbitrary singular, s-unital left R-module. Given an arbitrary non-zero $u \in M$, there exists an R-submodule Y of M which is maximal with respect to excluding u. Obviously, $Ru + Y$ is the smallest R-submodule of M properly containing Y and $(Ru + Y)/Y$ is an irreducible, singular left R-module. Hence, there exists an R-submodule X of M containing Y such that $M/Y = (Ru + Y)/Y \oplus X/Y$. Then, $u \notin X$ implies $X = Y$, namely, $M = Ru + Y$. This means that Y is a maximal R-submodule of M and $\mathrm{rad}({}_RM) = 0$.

4″) $\Longrightarrow$ 1) Let M be an irreducible, singular left R-module, and $\mathfrak{l}$ a left ideal of R. By [11, Proposition 3], it suffices to prove that every non-zero $f \in \mathrm{Hom}({}_R\mathfrak{l}, {}_RM)$ can be extended to an element of $\mathrm{Hom}({}_RR, {}_RM)$. We may assume here $\mathfrak{l}$ is essential in ${}_RR$. If $\mathfrak{l}' = \mathrm{Ker}\, f (\subset \mathfrak{l})$ is not essential in ${}_RR$, then $\mathfrak{l}' \cap \mathfrak{k} = 0$ for some non-zero left ideal $\mathfrak{k}$. Since $\mathfrak{l}'' = \mathfrak{l} \cap \mathfrak{k} \neq 0$ and $\mathfrak{l}'' \cap \mathfrak{l}' = 0$, $\mathfrak{l}''$ is R-isomorphic to $\mathfrak{l}''f = M$, whence it follows $\mathfrak{l}'' \subseteq Z_l(R) \cap S_l(R) = 0$. This contradiction means that $\mathfrak{l}'$ is essential in ${}_RR$. Accordingly, there exists a maximal left ideal $\mathfrak{m}$ containing $\mathfrak{l}'$ but not $\mathfrak{l}$. Since $\mathfrak{l}/\mathfrak{l}'$ is R-isomorphic to M and $\mathfrak{l} \supset \mathfrak{l} \cap \mathfrak{m} \supseteq \mathfrak{l}'$, we have $\mathfrak{l} \cap \mathfrak{m} = \mathfrak{l}'$. Now, taking this into mind, one can define an extension of f by $l + m \longmapsto lf$ $(l \in \mathfrak{l},\ m \in \mathfrak{m})$.

Corollary 4 (cf. [1, Theorem 1. 1]). *If R is s-unital and left semi-artinian, then the following conditions are equivalent*:

1) *R is a left V'-ring.*

2) *R is left non-singular and every singular, s-unital left R-submodule is semisimple.*

3) *R is left non-singular and every singular homomorphic image of ${}_RR$ is semisimple.*

4) *R is left non-singular and every essential left ideal is an intersection of maximal left ideals.*

In case R is left non-singular, the proof of Theorem 4 enables us to see the following

Corollary 5. *If R is left s-unital and left non-singular then the following conditions are equivalent*:

1) *R is a left V'-ring.*
2) *Every singular, s-unital left R-module is semisimple.*
3) *Every singular homomorphic image of ${}_RR$ is semisimple.*
4) *Every essential left ideal is an intersection of maximal left ideals.*

Corollary 6 (cf. [13, Corollary 4]). *If R is s-unital then the following conditions are equivalent*:

1) *R is a left V-ring.*

2) *R is a left V'-ring and every minimal left ideal is s-injective.*

3) *R is a left p-V-ring, every minimal left ideal is s-injective, and every singular homomorphic image of ${}_RR$ is semisimple.*

4) *R is a left p-V'-ring, every minimal left ideal is s-injective, and every singular homomorphic image of ${}_RR$ is semisimple.*

5) *R is a left p-V'-ring, every minimal left ideal is s-injective, and every essential left ideal is an intersection of maximal left ideals.*

Proof. By Proposition 1 (3), 1) $\Leftrightarrow$ 2). The equivalence of 2) – 5) is obvious by Theorem 4.

Theorem 5 (cf. [3, Theorem], [4, Theorem 1], [13, Theorem 2] and

[14, Theorem 1]). *The following conditions are equivalent*:

1) R *is a strongly regular ring.*

2) R *is a left duo, left V-ring.*

2′) R *is a left duo, left V′-ring.*

3) R *is a left duo, left p-V-ring.*

3′) R *is a left duo, left p-V′-ring.*

4) R *is a reduced ring and every* $(a|$ *is a left annihilator.*

5) R *is a left s-unital, reduced ring and every maximal left ideal is p-injective.*

6) R *is a left s-unital reduced ring and every maximal left ideal is either p-injective or a left annihilator.*

7) R *is a left non-singular ring and every* $(a|$ *is the left annihilator of a left ideal.*

8) R *is a left non-singular, left duo ring, and every* $(a|$ *is closed in* ${}_RR$.

9) *For each* $a \in R$ *there exists one and only one element* a' *such that* $aa'a = a$ *and* $a'aa' = a'$.

10) R *is a reduced ring and in any homomorphic image of* R *each element is either a zero-divisor or a semi-unit.*

Proof. It is easy to see that 1)$\Longrightarrow$9) and 10). Obviously, 2)$\Longrightarrow$2′), 3)$\Longrightarrow$3′), 5)$\Longrightarrow$6), and 1)$\Longrightarrow$7) and 8). By [3, Theorem] and Theorem 2, 1)$\Longrightarrow$2), 3) and 5). By Proposition 1 (2), 2′)$\Longrightarrow$3′), and by [10, Lemma 3], 7)$\Longrightarrow$4) and 8)$\Longrightarrow$4).

3′)$\Longrightarrow$1) Given $a \in R$, $Ra + l(a)$ is essential in ${}_RR$. In fact, if $(Ra + l(a)) \cap \mathfrak{k} = 0$ for a (left) ideal $\mathfrak{k}$ then $\mathfrak{k}a \subseteq Ra \cap \mathfrak{k} = 0$ and $\mathfrak{k} \subseteq l(a)$, which means $\mathfrak{k} = 0$. Hence, by Lemma 2 we have $Ra + l(a) = R$, whence it follows $Ra^2 = Ra \ni a$.

4)$\Longrightarrow$1) As is well-known, $r(a) = r(a^2)$ in the reduced ring R. Since $(a|$ and $(a^2|$ are left annihilators, $(a| = l(r((a|)) = l(r(a)) = l(r(a^2)) = (a^2|$.

6)$\Longrightarrow$1) Let $a \in R$, and $ea = a$. We shall prove $Ra + l(a) = R$, which will yield $a \in Ra^2$. If not, there exists a maximal left ideal $\mathfrak{m}$ containing $Ra + l(a)$ (Lemma 1 (4)). In case $\mathfrak{m}$ is p-injective, considering the canonical injection $i: Ra \longrightarrow \mathfrak{m}$, we can find an element $c \in \mathfrak{m}$ with $a = ac$. Then, $e - ce \in r(a) = l(a)$, and so $0 = (e - ce)a = a - ca$. Hence, $x - xc \in l(a)$ for all $x \in R$, whence it follows a contradiction $\mathfrak{m} = R$. On the other hand, in case $\mathfrak{m} = l(b)$ with some non-zero $b \in R$, we have $b \in r(\mathfrak{m}) \subseteq r(a) = l(a) \subseteq \mathfrak{m} = l(b)$. Thus, $b^2 = 0$, a contradiction.

9)$\Longrightarrow$1) Let $a^2 = 0$, $aa'a = a$ and $a'aa' = a'$. Then, setting $a'' =$

$a' + aa'$, we have $aa''a = a$ and $a''aa'' = a''$. By the uniqueness of a', it follows $aa' = 0$ and $a = 0$. Hence R is a reduced ring.

10) $\Longrightarrow$ 1) Let a be an arbitrary non-zero element of R. Obviously, $\mathfrak{a} = \bigcup_i r(a^i) = \bigcup_i l(a^i)$ is an ideal, which excludes a. If $\bar{a}\bar{b} = 0$ (resp. $\bar{b}\bar{a} = 0$) in $\bar{R} = R/\mathfrak{a}$ then $a^{n+1}b = 0$ (resp. $a^n ba = 0 = a^{n+1}b$) for some n, whence it follows $\bar{b} = 0$. Hence, $\bar{a}$ is a semi-unit. Then, $\bar{a}^2\bar{c} = \bar{a}$ with some c, so that $a^{m+1}c = a^m$ for some m. Hence, recalling that R is of bounded index 1, a is a semi-unit by [2, Theorem 4].

4. First, we claim that all the results in [5, §3] are still valid for rings without identity.

Lemma 3 (cf. [5, Propositions 3. 1 and 3. 3]). *Let R be a left weakly π-regular ring.*

(1) *The center of R is a π-regular ring.*

(2) *$J(R)$ is a nil ideal.*

(3) *$Z_r(R)$ is a nil ideal.*

(4) *If a is a left regular (resp. regular) element of R then R contains a right identity (resp. the identity) and $R = RaR$.*

Proof. (1) is an easy consequence of [2, Lemma 1].

(2) Let $a \in J(R)$, and $a^n = ea^n$ with $e \in Ra^nR$. Let e' be the quasi-inverse of e. Then $a^n = (e + e' - e'e)a^n = 0$.

(3) Let $a \in Z_r(R)$, and $a^n = ea^n$ with $e \in Ra^nR$. Since $r(e)$ is essential in R_R and $r(e) \cap a^nR = 0$, it follows $a^nR = 0$ and $a^n \in R(a^nR)a^n = 0$.

(4) Let $l(a) = 0$ (resp. $l(a) = r(a) = 0$), and $a^n = ea^n$ with $e \in Ra^nR$. By Lemma 1 (1), e is a right identity (resp. the identity) of R and $RaR \supseteq Re = R$.

Corollary 7 (cf. [5, Proposition 3. 2]). *Let R be a reduced ring.*

(1) *If R is left weakly π-regular then R is right weakly π-regular, and conversely.*

(2) *R is a prime left weakly π-regular ring if and only if R is a simple ring with* 1.

Proof. (1) Let $ea^n = a^n$ with $e \in Ra^nR$. Then $(a^ne - a^n)^2 = 0$, so $a^ne = a^n$.

(2) Since every non-zero element of a prime reduced ring is regular, this is a consequence of Lemma 3 (4).

We shall conclude our study with the following

Theorem 6 (cf. [5, Theorem 3. 4] and [14, Theorem 3]). *The following conditions are equivalent*:

1) *R is artinian semiprimitive.*

2) *R is a semiprime left Goldie, left V'-ring, and every essential left ideal is an ideal.*

3) *R is a semiprime left Goldie, left p-V'-ring, and every essential left ideal is an ideal.*

4) *R is a semiprime left Goldie, left weakly π-regular ring, and every essential left ideal is an ideal.*

5) *R is a left s-unital ring with a left regular element and every maximal left ideal is the left annihilator of an idempotent right ideal.*

6) *R is a left s-unital ring with a left regular element and the right annihilator of any maximal left ideal is a non-zero s-injective right ideal.*

7) *R is a left s-unital ring with a left regular element and the right annihilator of any maximal left ideal is a non-zero p-injective right ideal.*

8) *R is a left s-unital ring with a left regular element and the right annihilator of any maximal left ideal contains a non-nilpotent right ideal.*

Proof. Obviously, 1) $\Longrightarrow$ 2) – 7), 5) $\Longrightarrow$ 8), and 6) $\Longrightarrow$ 7) $\Longrightarrow$ 8). We claim here that if any one of the conditions 2)–8) is satisfied then R has a right identity (Lemma 1 (1)). Especially, 2) $\Longrightarrow$ 3). Now, assume one of the conditions 3) and 4). Given an arbitrary left ideal $\mathfrak{l}$, there exists a left ideal $\mathfrak{k}$ such that $\mathfrak{l} \oplus \mathfrak{k}$ is essential in ${}_RR$. As is well-known, any essential left ideal of the semiprime left Goldie ring R contains a regular element. Hence, by Corollary 2 (2) and Lemma 3 (4) we have $\mathfrak{l} \oplus \mathfrak{k} = R$, which implies 1). Finally, we shall prove that 8) $\Longrightarrow$ 1). Let $\mathfrak{m}$ be an arbitrary maximal left ideal, $\mathfrak{r}$ a non-nilpotent right ideal contained in $r(\mathfrak{m})$, and $ab \neq 0$ $(a, b \in \mathfrak{r})$. Then, $\mathfrak{m} = l(a) = l(b) = l(ab)$. If $\mathfrak{m}$ is essential in ${}_RR$ then $\mathfrak{m} \cap Ra$ contains a non-zero element xa, and it follows a contradiction $x \in l(ab) = l(a)$. Hence, $\mathfrak{m}$ is a direct summand of ${}_RR$, so that R is artinian semiprimitive by [10, Lemma 1 (b)].

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