ON GENERALIZED UNISERIAL BLOCKS

KAORU MOTOSE and YASUSHI NINOMIYA

Throughout R will represent a (unital) Artinian algebra over a field K of characteristic $p > 0$, $J(R)$ the radical of R, and G a finite group whose order is divisible by p. In [7, Theorem 6], M. Osima stated that the group algebra KG is uniserial if and only if G is p-nilpotent and a Sylow p-subgroup of G is cyclic. In § 1, by making use of K. Morita [3] we formulate the same for RG (Theorem 1). In § 2, we consider KG for a spliting field K. If a block B of KG has a cyclic defect group D then Dade's theorem [1, Theorem 78. 1] and [8, Lemma 4. 2] enable us to see that the nilpotency index $t(B)$ of $J(B)$ is not greater than $|D|$ (cf. [4, Remark 1]). In Theorem 2, we shall prove that $t(B) = |D|$ if and only if B is a generalized uniserial ring.

1. At first we consider the case R is a simple algebra over K. As was stated in [5, Theorem 8], by making use of [7, Theorem 1] and [3, Theorem 8] (instead of [7, Theorem 6]) we have the following

Lemma 1. *Let R be a simple algebra over K.*

(1) *RG is primary decomposable if and only if G is p-nilpotent.*

(2) *RG is uniserial if and only if G is a p-nilpotent group with a cyclic Sylow p-subgroup.*

Now, we can prove our first theorem.

Theorem 1. *RG is uniserial if and only if R is semisimple and G is a p-nilpotent group with a cyclic Sylow p-subgroup.*

Proof. We assume that RG is uniserial. Since R is a homomorphic image of RG, R is uniserial. Let $R = R_1 \oplus \cdots \oplus R_s$ be a decomposition of R into primary rings, and $\bar{R}_i = R_i/J(R_i)$. Then, $\bar{R}_iG$ being uniserial, G is a p-nilpotent group with a cyclic Sylow p-subgroup (Lemma 1 (2)). Since R_i is primary, R_i is isomorphic to the matrix ring $(S_i)_{n_i}$ with some completely primary ring S_i. Hence, we have $R_iG \cong (S_iG)_{n_i} \cong S_iG \otimes_K (K)_{n_i}$. Then by [5, Lemma 6], S_iG is uniserial. Let P be a Sylow p-subgroup of G. Since S_iP is a homomorphic image of S_iG and $S_iP/J(S_iP) \cong S_i/J(S_i)$, S_iP is a completely primary uniserial ring. If $J(S_j) \neq 0$ for some j, then it is obvious that $J(S_j)$ is not contained in the augmentation ideal Δ of S_jP. Further, since $g - 1 \in \Delta \backslash J(S_j)P$ for any $g \neq 1$ in P, we see that $J(S_j)P$ and Δ are incomparable. This yields a contradiction that S_jP is not uniserial. Thus, R is semisimple. The converse part is also easy by Lemma 1 (2).

2. Let L be an extension field of the p-adic completion of the rationals, and R the complete local ring whose quotient field is L. Let K be the residue class field of R. Throughout the present section, we assume that L is a splitting field for G.

Lemma 2. *If B is a block of KG with a defect group D, then the following conditions are equivalent:*

(1) *D is cyclic and the decomposition matrix of B takes the form*

$$\text{(I)} \qquad \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & 0 & \cdot & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & 1 \end{pmatrix} .$$

(2) *D is cyclic and the Cartan matrix of B is of the form*

$$\text{(II)} \qquad \begin{pmatrix} s+1 & s & \cdot & \cdot & \cdot & s \\ s & s+1 & \cdot & \cdot & \cdot & s \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & s \\ s & & \cdot & \cdot & \cdot & s+1 \end{pmatrix} .$$

(3) *B is a generalized uniserial ring.*

Proof. The implication $(1) \Rightarrow (2)$ is obvious, and $(2) \Rightarrow (3)$ is a consequence of [2, Folgerung 4]. $(3) \Rightarrow (2)$: Since B is a generalized uniserial ring, by [6, Theorem 17] B has only a finite number of indecomposable modules. Hence, D is cyclic. The rest of the proof is evident by [3, Remark, p. 158]. $(2) \Rightarrow (1)$: By Dade's theorem [1, Theorem 68. 1], the Cartan matrix (c_{kl}) of B is of the form

$$
\text{(III)} \qquad
\begin{pmatrix}
\overbrace{\begin{matrix} 2 & & & \\ & 2 & * & \\ & & \ddots & \\ & * & & 2 \end{matrix}}^{a} & \overbrace{\begin{matrix} & & & \\ & * & & \\ & & & \\ & & & \end{matrix}}^{b} \\
\begin{matrix} & & & \\ & & & \\ & * & & \\ & & & \end{matrix} & \begin{matrix} s+1 & s & \cdots & s \\ s & s+1 & \cdots & s \\ \vdots & & \ddots & \vdots \\ & & & s \\ s & \cdots & \cdots & s+1 \end{matrix}
\end{pmatrix}
$$

where the elements in the $*$-parts are 0 or 1. By (2), we have $s = 1$ or $a = 0$. First we consider the case $s = 1$. Let $\{U_i \mid 1 \leqq i \leqq a + b\}$ be a complete set of representatives of isomorphic classes of principal indecomposable B-modules, $\widetilde{U}_i$ the principal indecomposable RG-module such that $K \otimes \widetilde{U}_i \cong U_i$, and Φ_i the character afforded by $\widetilde{U}_i$. Let $\{\chi_j \mid 1 \leqq j \leqq a + b + 1\}$ be a complete set of irreducible complex characters of B. Since $s = 1$, each Φ_i is the sum of distinct two χ_j' s. When $a + b \leqq 2$, it is trivial that the decomposition matrix of B takes the form (I). Hence, we suppose $a + b > 2$ and $\Phi_1 = \chi_1 + \chi_2$. Since $(\Phi_1, \Phi_k) = c_{1k} = 1$ for $k \neq 1$, Φ_k contains χ_1 or χ_2. We may assume here that Φ_2 contains χ_1. If one of Φ_i's $(i \geqq 3)$, say Φ_3, does not contain χ_1, then Φ_3 contains χ_2. Since $(\Phi_2, \Phi_3) = c_{23} = 1$, Φ_2 and Φ_3 contain a character different from χ_1 or χ_2 in common. This yields a contradiction that $\{\chi_j\}$ is not a tree. We have therefore seen that each Φ_i $(1 \leqq i \leqq a + b)$ contains χ_1. Thus, the decomposition matrix of B takes the form (I). Next, we consider the case $s \neq 1$. Then $a = 0$ and the decomposition matrix of B takes the form (I) by [1, Theorem 68. 1].

By Dade's theorem [1, Theorem 68. 1] and [8, Lemma 4. 2], it is easy to see that if a defect group D of B is cyclic then $t(B) \leqq |D|$. Hence, if a Sylow p-subgroup P of G is cyclic then the nilpotency index $t(G)$ of $J(KG)$ is not greater than $|P|$. Now, our attention will be directed towards the case $t(B) = |D|$ and the case $t(G) = |P|$.

Theorem 2. *If a defect group D of a block B of KG is cyclic, then the following conditons are equivalent*:

(1) $t(B) = |D|$.

(2) *B is a generalized uniserial ring.*

Proof.[1] (1)$\Longrightarrow$(2): By [Theorem 68.1], the Cartan matrix (c_{kl}) of B is of the form (III). Therefore we have $|D| = t(B) \leqq \max_k \{\sum_l c_{kl}\} \leqq a + bs + 1 \leqq (a+b)s + 1 = |D|$, whence it follows that $a + bs + 1 = (a+b)s + 1 = |D|$. Hence, we have $s = 1$ or $a = 0$. First we consider the case $s = 1$. Then $a + b = |D| - 1$. Since $a+b$ divides $p-1$, we have $|D| = p$ and $t(B) = \sum_l c_{kl} = p$ for some k. Let U_i, $\tilde{U}_i$, Φ_i $(1 \leqq i \leqq a+b)$, $\chi_j (1 \leqq j \leqq a+b+1)$ be as in the proof of Lemma 2. Since $s = 1$, each Φ_i is the sum of distinct two χ_j's. We suppose $\Phi_k = \chi_1 + \chi_2$. Since $(\Phi_k, \Phi_l) = c_{kl} = 1$ for $l \neq k$, Φ_l contains χ_1 or χ_2. We suppose that m Φ_l's contain χ_1 and n Φ_l's do χ_2. Now, let M, N be RG-submodules of $\tilde{U}_k$ corresponding to χ_1, χ_2 respectively. Since $K \otimes \tilde{U}_k$ is uniserial by $t(B) = p$, we may assume $K \otimes M$ contains $K \otimes N$. Then all composition factors of $K \otimes N$ apear among those of $K \otimes M$. Thus, we have $n = 1$. Rearranging Φ_i's and χ_j's, the decomposition matrix of B takes the form (I). Next, we consider the case $s \neq 1$. Then $a = 0$ and the Cartan matrix of B is of the form (II). Thus, by Lemma 2, B is a generalized uniserial ring. (2)$\Longrightarrow$(1): Since B is a generalized uniserial ring, the Cartan matrix of B is of the form (II) (Lemma 2). Now, let f be an arbitrary primitive idempotent of B. Since $\sum_l c_{kl} = se + 1 = |D|$ for $1 \leqq k \leqq e =$ the number of non isomorphic principal indecomposable modules of B, the length of the unique composition series of Bf is $|D|$. Therefore $J(B)^{|D|-1}f \neq 0$ and $J(B)^{|D|}f = 0$. Hence $t(B) = |D|$.

Corollary. *If G has a cyclic Sylow p-subgroup of order p^a, then the following conditions are equivalent:*

(1) $t(G) = p^a$.

(2) *There exists a generalized uniserial block of defect a.*

REFERENCES

[1] L. DORNHOFF: Group Representation Theory, Part B, Dekker, 1972.

[2] H. KUPISCH: Symmdtrische Algebren mit endlich vielen unzerlegbaren Darstellungen II, J. Reine Angew. Math. **245** (1970), 1—14.

[3] K. MORITA: On group rings over a modular field which pcssess radicals expressible as principal ideals, Sci. Rep. Tokyo Bunrika Daigaku, **A4** (1951), 155—172.

[4] K. MOTOSE: On radicals of principal blocks, to appear.

1) Recently, S. Koshitani gave a different prcof by making use of the result in [H. Kupisch: Projektive Mcduln endlicher Gruppen mit zyklischer p-Sylow Gruppe, J. of Algebra **10** (1968), 1—7].

[5] T. NAGAHARA, T. ONODERA and H. TOMINAGA: On normal basis theorems and strictly Galois extensions, Math. J. Okayama Univ. **8** (1958), 133—143.
[6] T. NAKAYAMA: On Frobeniusean algebra II, Ann. Math. **42** (1941), 1—21.
[7] M. OSIMA: On primary decomposable group rings, Proc. Phys.-Math. Scc. Japan **24** (1942), 1—9.
[8] D. A. R. WALLACE: Lower bounds for the radical of the group algebra of a finite p-soluble group, Proc. Edinburgh Math. Scc. **16** (1968/69), 127—134.

SHINSHU UNIVERSITY

(Received April 30, 1977)