WEAKLY π-REGULAR RINGS AND GROUP RINGS

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1. Introduction. The well-known notion of von Neumann regularity in the theory of rings has given rise to a number of generalizations. In this paper we further generalize these generalizations ; we define the notion of (right, left) weak π-regularity. The system of weakly π-regular rings is a wide class of rings which strictly includes π-regular rings, weakly regular rings and hence locally artinian rings (see [5]) and perfect rings. In § 2 we give the definition and an example. In §3 we study the elementary properties of weakly π-regular rings generalizing some results in [8], [11]. We also try to establish the relation between weakly π-regular rings and artinian rings. In § 4 we study the corresponding problem in group rings for weakly π-regular rings. We prove that the group algebra KG over a field K satisfying a nontrivial polynomial identity is right (resp. left) weakly π-regular if and only if G is locally finite.

2. Definition. Throughout this paper all rings are assumed to be associative with identity but not necessarily commutative. The Jacobson radical of a ring A will be denoted by $J(A)$, and the right (resp. left) singular ideal of A by $Z_r(A)$ (resp. $Z_l(A)$).

A ring A is called a *π-regular* ring if for every $a \in A$ we have $a^n \in a^n A a^n$ for some natural number $n = n(a)$. While, A is called a *right* (resp. *left*) *π-regular* ring if for every $a \in A$ we have $a^n \in a^{n+1} A$ (resp. $a^n \in Aa^{n+1}$) for some natural number $n = n(a)$. Recently, F. Dischinger [4] has proved that every one-sided π-regular ring is strongly π-regular, namely, right and left π-regular. Following [11], A is called a *right* (resp. *left*) *weakly regular* ring if for each $a \in A$ we have $a \in (aA)^2$ (resp. $a \in (Aa)^2$). A is a *weakly regular* ring if it is right and left weakly regular. Now, we introduce the following definition.

Definition 2. 1. A ring A is said to be *right* (resp. *left*) *weakly π-regular* if for every $a \in A$ there exists a natural number $n = n(a)$ such that $a^n \in (a^n A)^2$ (resp. $a^n \in (Aa^n)^2$), i. e., $a^n = a^n x$ (resp. $a^n = xa^n$) with some x in the (two-sided) ideal (a^n) generated by a^n. A ring is *weakly π-regular* if it is right and left weakly π-regular.

Obviously, every π-regular ring and every weakly regular ring are weakly π-regular. First, we state the following easy proposition, which will be used freely in the subsequent study.

Proposition 2.2. *Let I be an ideal of a ring A. If A is right weakly π-regular then so are both A/I and I. On the other hand, if A/I is right weakly π-regular and I is right weakly regular then A is right weakly π-regular.*

Example 2.3. Let A_1 be a weakly regular ring which is not strongly π-regular ([3]). Let A_2 be a π-regular ring which is neither right weakly regular nor left weakly regular ([6, Example 1] and [9, Example 1, p. 64]). Then, $A_1 \oplus A_2$ is a weakly π-regular ring which is neither π-regular nor strongly π-regular, right weakly regular or left weakly regular.

3. Weakly π-regular rings.

Proposition 3.1. *If I is a proper ideal of a right weakly π-regular ring A, then every element of I is a left zero divisor in A. Especially, every right weakly π-regular ring without non-zero left zero divisors is simple.*

Proof. Let $a \in I$, and $a^n = a^n x$ for some natural number n and some $x \in (a^n)$. If a is not a left zero divisor (right regular element) then $a^n(1-x) = 0$ implies a contradiction $1 = x \in (a^n) \subseteq I$.

In general the right weak π-regularity does not coincide with the left one. But we have the following

Proposition 3.2. *Let A be a reduced ring.*

(1) *A is right weakly π-regular if and only if it is left weakly π-regular.*

(2) *A is a prime weakly π-regular ring if and only if it is simple.*

Proof. (1) is obvious by the fact that in a reduced ring $xy = 0$ is equivalent with $yx = 0$. (2) is only a combination of Proposition 3.1 and the fact that every prime reduced ring has no non-zero zero divisors.

Proposition 3.3. *Let A be a right weakly π-regular ring.*

(1) *The center of A is a π-regular ring.*

(2) *$J(A)$ is a nil ideal.*

(3) *$Z_l(A)$ is a nil ideal.*

Proof. (1) This is obvious by [1, Lemma 1].

(2) Let $a \in J(A)$, and $a^n = a^n x$ with $x \in (a^n)$. Since x is in $J(A)$, $1 - x$ has the inverse u. Hence, $a^n = a^n(1-x)u = 0$.

(3) Let $a \in Z_l(A)$, and $a^n = a^n x$ with $x \in (a^n)$. Since the left annihilator $l(x)$ of $x \in Z_l(A)$ is an essential left ideal of A and $l(x) \cap Aa^n = 0$, it follows $Aa^n = 0$, i. e., $a^n = 0$.

Theorem 3. 4. *Let A be a semiprime right Goldie ring such that every essential right ideal of A is an ideal. Then, A is right weakly π-regular if and ond only if A is semisimple artinian.*

Proof. It suffices to prove the only if part. By [7, Theorem 3. 9], every essential right ideal of the semiprime right Goldie ring A contains a regular element. Since every essential right ideal of A is two-sided, it is whole of A by Proposition 3. 1. Now, let I be an arbitrary proper right ideal of A. Then there exists a right ideal I' of A such that $I+I' = I \oplus I'$ is essential, and hence $I \oplus I' = A$. Thus, A is semisimple artinian.

Corollary 3. 5. *Let A be a reduced ring with finite right Goldie dimension such that every essential right ideal of A is an ideal. Then, A is a right weakly π-regular ring if and only if it is semisimple artinian.*

Proof. If A is right weakly π-regular, then $Z_r(A) = 0$ by Proposition 3. 2 (1) and Proposition 3. 3 (3). Accordingly, A is a semiprime right Goldie ring (see [7, p. 206], and then A is semisimple artinian by Theorem 3. 4. The converse is trivial.

Corollary 3. 6. *Let A be a right noetherian ring such that every essential right ideal of $A/J(A)$ is an ideal. Then, A is right weakly π-regular if and only if it is right artinian.*

Proof. It suffices to prove the only if part. Since $A/J(A)$ is a semiprime right Goldie ring, it is semisimple artinian by Theorem 3. 4. Now, noting that $J(A)$ is nilpotent by Proposition 3. 3 (2) and Levitzki theorem (see e. g. [7, Theorem 3. 4]), one will easily see that the right A-module A has a composition series.

4. **Weakly π-regular group rings.** If A is a ring and G is a group, AG will denote the group ring of G over A. The ideal ωG of AG generated by $\{1-g \mid g \in G\}$ is called the *fundamental ideal* of AG. As is well-known, $AG/\omega G \cong A$. We set $\Delta = \{g \in G \mid (G: C_G(g)) < \infty\}$, which is a normal subgroup of G. Given $\alpha = \sum a_g g \in AG$, we set $\operatorname{supp}(\alpha) = \{g \in G \mid a_g \neq 0\}$. For further properties of group rings we

refere [2] and [10].

Theorem 4.1. *If AG is right weakly π-regular, then A is right weakly π-regular and G is a torsion group.*

Proof. It remains only to prove the latter assertion. Let $g \in G$, and $(1-g)^n = (1-g)^n x$ with $x \in ((1-g)^n)$. Supose g is not of finite order. Then $1-g$ is regular by [2, Proposition 6]. Hence, $(1-g)^n (1-x) = 0$ yields a contradiction $1 = x \in \omega G$.

Corollary 4.2. *Let G be an abelian group. If AG is right weakly π-regular, then A is right weakly π-regular and G is locally finite.*

Theorem 4.3. *Let A be a ring. If Φ is a family of subgroups of G such that*

1) *AH is right weakly π-regular for each $H \in \Phi$;*
2) *every finite subset of G is contained in some $H \in \Phi$,*

then AG is right weakly π-regular. In particular, if G is locally finite and if AH is right weakly π-regular for each finite subgroup H of G, then AG is right weakly π-regular.

Proof. Given $\alpha \in AG$, by 2) we can find some $H \in \Phi$ containing supp (α). Then, by 1) there exists a positive integer n such that $\alpha^n \in (\alpha^n AH)^2 \subseteq (\alpha^n AG)^2$.

Corollary 4.4. *Let A be a right artinian ring, and G an infinite locally finite group. Then, AG is not right artinian but right weakly π-regular.*

Proof. By [2, Theorem 1], AG is not right artinian. On the other hand, AH being right artinian for each finite subgroup H of G, AG is right weakly π-regular by Theorem 4.3.

Now, we shall give our main theorem of this section.

Theorem 4.5. *Assume that the group algebra KG over a field K satisfies a nontrivial polynomial identity. Then, KG is right weakly π-regular if and only if G is locally finite.*

Proof. In virtue of Corollary 4.4, it suffices to prove the only if part. By Theorem 4.1 and [10, Theorem 5.5], G is a torsion group and $(G: \Delta) < \infty$. Now, let H be a finitely generated subgroup of G. Then $(H: H \cap \Delta) < \infty$, and by [10, Lemma 6.1] $H \cap \Delta$ is a finitely generated subgroup of Δ. Since the center $Z(H \cap \Delta)$ of $H \cap \Delta$ is of finite index by [10, Lemma 2. 2], $(H: Z(H \cap \Delta)) < \infty$, and $Z(H \cap \Delta)$ is a finitely generated torsion group again by [10, Lemma 6. 1]. Hence, $Z(H \cap \Delta)$ is finite, and eventually H is finite.

Finally, we propose the following

Problem. Let KG be right weakly π-regular. Is G necessarily locally finite ?

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REFERENCES

[1] G. AZUMAYA: Strongly π-regular rings, J. Fac. Sci. Hokkaido Univ., Ser. I, **13** (1954), 34—39.

[2] I. G. CONNELL: On the group ring, Canad. J. Math. **15** (1961), 650—685.

[3] J. H. COZZENS: Homological properties of the ring of differential polynomials, Bull. Amer. Math. Soc. **76** (1970), 75—79.

[4] F. DISCHINGER: Sur les anneaux fortement π-réguliers, C. R. Acad. Sci. Paris **283 A** (1976), 571—573.

[5] J. W. FISHER and R. L. SNIDER: Prime von Neumann regular rings and primitive group algebras, Proc. Amer. Math. Soc. **44** (1974), 244—250.

[6] J. W. FISHER and R. L. SNIDER: On the von Neumann regularity of rings with regular prime factor rings, Pacific J. Math. **54** (1974), 135—144.

[7] A. W. GOLDIE: Semiprime rings with maximum condition, Proc. London Math. Soc. **10** (1960), 201—220.

[8] V. GUPTA: Generalized regular rings and semiprimary group rings, Ph. D. Thesis, I. I. T., Delhi, 1975.

[9] W. K. NICHOLSON: Rings whose elements are quasi-regular or regular, Aequationes Math. **9** (1973), 64—70.

[10] D. S. PASSMAN: Infinite Group Rings, Marcel Dekker, New York, 1971.

[11] V. S. RAMAMURTHI: Weakly regular rings, Canad. Math. Bull. **16** (1973), 317—321.

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