SOME APPLICATIONS OF THE CHARACTER ANALOGUE OF THE POISSON SUMMATION FORMULA

TAKESHI KANO

Suppose that $f(x)$ is a function defined and of bounded variation on the interval $[a, b]$. Then there holds the well-known Poisson summation formula

$$(1)\quad \frac{1}{2}\sum_{a\leq n\leq b}{}' \{f(n+0)+f(n-0)\} = \int_a^b f(x)dx + 2\sum_{n=1}^{\infty}\int_a^b f(x)\cos(2\pi nx)\,dx$$

where, and in what follows analogously, the prime ′ indicates that if a is an integer then the first term of the sum is $(1/2)f(a+0)$, and if b is an integer then the last term is $(1/2)f(b-0)$. This formula has many important applications in number theory and there are known some variants or genealizations of it. B. C. Berndt [1, 2; cf. also 3] has recently obtained the following "character analogue"[1)] of (1) as a particular case of his result.

Theorem A. *Let $f(x)$ be of bounded variation throughout $[a, b]$, and $\chi(n)$ be a primitive, non-principal Dirichlet character* (mod k). *Then if χ is even, i.e. $\chi(-1)=1$,*

$$(2)\quad \frac{1}{2}\sum_{a\leq n\leq b}\chi(n)\{f(n+0)+f(n-0)\} = \frac{2}{k}G(\chi)\sum_{n=1}^{\infty}\bar{\chi}(n)\int_a^b f(u)\cos\frac{2\pi nu}{k}du$$

and if χ is odd, i.e. $\chi(-1)=-1$,

$$(3)\quad \frac{1}{2}\sum_{a\leq n\leq b}\chi(n)\{f(n+0)+f(n-0)\} = \frac{-2i}{k}G(\chi)\sum_{n=1}^{\infty}\bar{\chi}(n)\int_a^b f(u)\sin\frac{2\pi nu}{k}du$$

where

$$G(\chi)=\sum_{n=0}^{k-1}\chi(n)\,e^{\frac{2\pi in}{k}}.$$

The main purpose of the present paper is to give some applications of Theorem A, which are related to our former result [6].

1. Let us put

1) It should be noticed that similar results have been obtained by A. P. Guinand (cf. e. g. [4]).

$$Q(x)=\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{x}{n}\qquad (x>0).$$

Then S. L. Segal [7] has obtained the Bessel-series expression

$$(4)\qquad \int_0^y Q(x)\,dx=\frac{\pi}{2}y-\frac{1}{2}+\left(\frac{\pi}{2}y\right)^{\frac{1}{2}}\sum_{n=1}^{\infty}n^{-\frac{1}{2}}J_1(2\sqrt{2\pi ny}).$$ [2]

He pointed out that if both sides of (4) could be diffesentiated term-by-term, then we would have the formula

$$(5)\qquad Q(x)=\frac{\pi}{2}+\pi\sum_{n=1}^{\infty}J_0(2\sqrt{2\pi nx}).$$

However, the present author has shown that the right-hand side of (5) diverges for all positive x, and hence the formula (5) is invalid. [3]

We shall show in this paper that this kind of phenomenon is an exceptional case in the sense that if χ is non-principal and primitive, then

$$(6)\qquad \sum_{n=1}^{\infty}\frac{\chi(n)}{n}\sin\frac{x}{n}$$

converges for all positive x and the Bessel-series expression like (5) for (6) can be obtained. In fact we can prove the following theorem.

Theorem 1. *Let* χ *be a primitive, non-principal character* (mod k). *If* χ *is even, then*

$$(7)\qquad \sum_{n=1}^{\infty}\frac{\chi(n)}{n}\sin\left(\frac{x}{n}\right)=\frac{\pi}{k}G(\chi)\sum_{n=1}^{\infty}\bar{\chi}(n)J_0(2\sqrt{2\pi nx/k}),$$

$$(8)\qquad \sum_{n=1}^{\infty}\frac{\chi(n)}{n}\cos\left(\frac{x}{n}\right)=\frac{2}{k}G(\chi)\sum_{n=1}^{\infty}\bar{\chi}(n)\{k_0(2\sqrt{2\pi nx/k})-\frac{\pi}{2}Y_0(2\sqrt{2\pi x/k})\}.$$

If χ *is odd, then*

$$(9)\qquad \sum_{n=1}^{\infty}\frac{\chi(n)}{n}\sin\left(\frac{x}{n}\right)=\frac{-2i}{k}G(\chi)\sum_{n=1}^{\infty}\bar{\chi}(n)\{K_0(2\sqrt{2\pi nx/k})+\frac{\pi}{2}Y_0(2\sqrt{2\pi nx/k})\},$$

$$(10)\qquad \sum_{n=1}^{\infty}\frac{\chi(n)}{n}\cos\left(\frac{x}{n}\right)=\frac{-\pi i}{k}G(\chi)\sum_{n=1}^{\infty}\bar{\chi}(n)J_0(2\sqrt{2\pi nx/k}).$$

Before proving this theorem, we shall state the following result due to Berndt [1].

2) About notations and definitions of Bessel functions we follow [9].

3) This notwithstanding, the series of the right hand side of (5) is summable (C, 1) for all positive x.

Theorem B. *Let* χ *be a primitive, non-principal character* (mod k). *If* χ *is even, then*

$$(11)\qquad 2\sum_{n=1}^{\infty}\chi(n)\sin^2\left(\frac{x}{n}\right)=G(\chi)\sum_{n=1}^{\infty}\overline{\chi}(n)\left(\frac{\pi x}{kn}\right)^{\frac{1}{2}}J_1(4\sqrt{\pi nx/k}),$$

and if χ *is odd, then*

$$(12)\qquad (2i)\sum_{n=1}^{\infty}\chi(n)\sin\left(\frac{x}{n}\right)=G(\chi)\sum_{n=1}^{\infty}\overline{\chi}(n)\left(\frac{2\pi x}{kn}\right)^{\frac{1}{2}}J_1(2\sqrt{2\pi nx/k}).$$

We remark that there are two ways to obtain (7). The first one is to differenciate the both sides of (11) term-by-term and then replace $2x$ by x. This formal procedure will be assured if the right-hand side of (11) is compact uniformly convergent in $x>0$. This is what we shall prove in this section. The other way to obtain (7) will be explained in section 2.

From the Hankel asymptotic expansion for $J_0(x)$, we know that it will suffice to prove

$$(13)\qquad \sum_{n=1}^{\infty}\overline{\chi}(n)\,n^{-\frac{1}{4}}\,{\cos \atop \sin}\,(2\pi\, x\sqrt{n})$$

being convergent compact uniformly in $x>0$ when χ is non-principal, primitive and even. This fact is a particular case of the following more general result.

Theorem 2. *Let* χ *be a primitive, non-principal character* (mod k). *If* χ *is even, then for any fixed* $\sigma>0$ *the series*

$$(14)\qquad L(s,\chi,\alpha)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}e^{2\pi i\alpha\sqrt{n}}\qquad (s=\sigma+it,\ \alpha>0)$$

converges compact uniformly in α (*and* t).

Proof. Let us put

$$S_N(\alpha)=\sum_{n=1}^{N}\chi(n)\,{\cos \atop \sin}(2\pi\alpha\sqrt{n}),$$
$$S_N^*(\alpha)=\sum_{n=1}^{N}{}'\,\chi(n)\,{\cos \atop \sin}\,(2\pi\,\alpha\sqrt{n}).$$

Then obviously

$$S^*_N(\alpha)=S_N(\alpha)+O(1),$$

where the constant implied by the O is absolute. We shall estimate this

$S^*_N(\alpha)$ in the sequel.

Let us put in the formula (3)

$$f(u) = \frac{\cos}{\sin}(2\pi\alpha\sqrt{u}).$$

Then we have

$$S^*_N(\alpha) = \frac{2}{k}\,G(\chi)\sum_{n=1}^{\infty}\bar{\chi}(n)\int_1^N \frac{\cos}{\sin}(2\pi\alpha\sqrt{u})\cos\left(2\pi n\frac{u}{k}\right)du.$$

We set

$$\text{(15)}\qquad I_N(\alpha) = \int_1^N 2\cos(2\pi\alpha\sqrt{u})\cos\left(2\pi n\frac{u}{k}\right)du$$
$$= \int_1^N \left\{\cos 2\pi\left(\alpha\sqrt{u} + \frac{nu}{k}\right) + \cos 2\pi\left(\alpha\sqrt{u} - \frac{nu}{k}\right)\right\}du$$
$$= J_1 + J_2,$$

and similarly

$$\text{(16)}\qquad \tilde{I}_N(\alpha) = \int_1^N 2\sin(2\pi\alpha\sqrt{u})\cos\left(2\pi n\frac{u}{k}\right)du$$
$$= \int_1^N \left\{\sin 2\pi\left(\alpha\sqrt{u} + \frac{nu}{k}\right) + \sin 2\pi\left(\alpha\sqrt{u} - \frac{nu}{k}\right)\right\}du$$
$$= \tilde{J}_1 + \tilde{J}_2.$$

First we consider the integral J_2. If we write $u = t^2$, then we have

$$\text{(17)}\quad J_2 = \int_1^{\sqrt{N}} 2t\cos 2\pi\left(\alpha t - \frac{n}{k}t^2\right)dt$$
$$= -\frac{k}{n}\int_1^{\sqrt{N}}\left(\alpha - \frac{2n}{k}t\right)\cos 2\pi\left(\alpha t - \frac{n}{k}t^2\right)dt$$
$$+ \frac{k}{n}\int_1^{\sqrt{N}}\cos 2\pi\left(\alpha t - \frac{n}{k}t^2\right)dt$$
$$= -\frac{k}{2\pi n}\int_1^{\sqrt{N}}\left(\sin 2\pi\left(\alpha t - \frac{n}{k}t^2\right)\right)' dt + \frac{k\alpha}{n}\int_1^{\sqrt{N}}\cos 2\pi\left(\alpha t - \frac{n}{k}t^2\right)dt$$
$$= \frac{k}{2\pi n}\left\{\sin 2\pi\left(\frac{nN}{k} - \alpha\sqrt{N}\right) - \sin 2\pi\left(\frac{n}{k} - \alpha\right)\right\}$$
$$+ \frac{\alpha k}{n}\int_1^{\sqrt{N}}\cos 2\pi\left(\alpha t - \frac{n}{k}t^2\right)dt.$$

Also we have

$$(18)\qquad J_1 = \frac{k}{2\pi n}\left\{\sin 2\pi\left(\frac{nN}{k} + \alpha\sqrt{N}\right) - \sin 2\pi\left(\frac{n}{k} + \alpha\right)\right\} - \frac{\alpha k}{n}\int_1^{\sqrt{N}} \cos 2\pi\left(\alpha t + \frac{n}{k}t^2\right) dt.$$

Now we need to estimate the integrals

$$I_N = \int_1^{\sqrt{N}} \cos 2\pi\left(\alpha t \pm \frac{n}{k}t^2\right) dt.$$

For this purpose we use the following lemma (cf. [8 ; p. 61]).

Lemma 1. *Let $F(x)$ be a real function, twice differentiable, and let $F''(x) \geq r > 0$, or $F''(x) \leq -r < 0$, throughout $[a, b]$. Then*

$$\left|\int_a^b e^{iF(x)}dx\right| \leq \frac{8}{\sqrt{r}}.$$

Thus we know

$$(19)\qquad I_N = O\left(\sqrt{\frac{k}{n}}\right).$$

Hence from (15), (17), (18), (19) we obtain

$$(20)\qquad I_N(\alpha) = \frac{k}{\pi n}\left\{\cos(2\pi\alpha\sqrt{N})\sin\left(2\pi\frac{nN}{k}\right) - \cos(2\pi\alpha)\sin\left(\frac{2\pi n}{k}\right)\right\} + O\left(\alpha\left(\frac{k}{n}\right)^{\frac{3}{2}}\right).$$

Similarly we obtain

$$(21)\qquad \tilde{I}_N(\alpha) = \frac{k}{\pi n}\left\{\sin(2\pi\alpha\sqrt{N})\sin\left(2\pi\frac{nN}{k}\right) - \sin(2\pi\alpha)\sin\left(\frac{2\pi n}{k}\right)\right\} + O\left(\alpha\left(\frac{k}{n}\right)^{\frac{3}{2}}\right).$$

If $k \mid N$, i.e. $N = kM$ ($M \in \mathrm{N}$), then (20) and (21) become

$$(22)\qquad \begin{aligned} I_N(\alpha) &= -\frac{k}{\pi}\cos(2\pi\alpha)\frac{1}{n}\sin\left(\frac{2\pi n}{k}\right) + O\left(\alpha\left(\frac{k}{n}\right)^{\frac{3}{2}}\right), \\ \tilde{I}_N(\alpha) &= -\frac{k}{\pi}\sin(2\pi\alpha)\frac{1}{n}\cos\left(\frac{2\pi n}{k}\right) + O\left(\alpha\left(\frac{k}{n}\right)^{\frac{3}{2}}\right). \end{aligned}$$

Consequently in this case we have

$$S_N^*(\alpha) = S_{kM}^*(\alpha) = \frac{1}{k} G(\chi) \sum_{n=1}^{\infty} \bar{\chi}(n) \left\{ -\frac{k}{\pi} \frac{\cos}{\sin}(2\pi\alpha) \frac{1}{n} \sin\left(\frac{2\pi n}{k}\right) + O\left(\alpha\left(\frac{k}{n}\right)^{\frac{3}{2}}\right)\right\}$$

$$= -\frac{G(\chi)}{\pi} \frac{\cos}{\sin}(2\pi\alpha) \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sin\left(\frac{2\pi n}{k}\right) + O\left(|G(\chi)| \cdot \alpha k^{\frac{3}{2}} \sum_{n=1}^{\infty} \frac{|\bar{\chi}(n)|}{n^{\frac{3}{2}}}\right).$$

Observing that $G(\chi) = O(\sqrt{k})$ and $\sum_{n=1}^{\infty} \frac{|\bar{\chi}(n)|}{n^{\frac{3}{2}}} = O(1)$, we obtain

$$S_{kM}^*(\alpha) = -\frac{G(\chi)}{\pi} \frac{\cos}{\sin}(2\pi\alpha) \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sin\left(\frac{2\pi n}{k}\right) + O(\alpha k^2). \tag{23}$$

For a general N, we write $N = kM + r$ $(0 \leq r \leq k-1)$. Then from (23),

$$S_N^*(\alpha) = \sum_{n=1}^{N}{}' \chi(n) \frac{\cos}{\sin}(2\pi\alpha\sqrt{n}) = \sum_{n=1}^{kM+r}{}' = \sum_{n=1}^{kM}{}' + \sum_{n=kM+1}^{kM+r} = S_{kM}^*(\alpha) + O\left(\sum_{j=1}^{r}{}' 1\right)$$
$$= S_{kM}^*(\alpha) + O(r) = S_{kM}^*(\alpha) + O(k).$$

Therefore we obtain

$$S_N(\alpha) = -\frac{G(\chi)}{\pi} \frac{\cos}{\sin}(2\pi\alpha) \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sin\left(\frac{2\pi n}{k}\right) + O(\alpha k^2) + O(k) + O(1). \tag{24}$$

Here it is known (cf. [4]) that the series

$$\sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sin\left(\frac{2\pi n}{k}\right)$$

is convergent when χ is an even, non-principal primitive character (mod k). Therefore we finally have the estimate

$$S_N(\alpha) = O_k(1) + O(1) + O(\alpha k^2), \tag{25}$$

where $O_k(1)$ denotes the bound depending possibly on k only. This means that $S_N(\alpha)$ is uniformly bounded in $\alpha \in [a, b]$ $(a > 0)$ and N. Hence by partial summation, we obtain

$$\sum_{n=M}^{N} \frac{\chi(n)}{n^s} e^{2\pi i \alpha\sqrt{N}} = O_k(M^{-\sigma}) + O_k(N^{-\sigma}),$$

which proves that (14) converges compact uniformly in α (and t).

We remark that formula (10) can be derived from (12) in the same way as above.

2. In this section we shall first prove formula (7) by a method different from that of the preceding section. Formulae (8), (9) and (10) can also be obtained by this method without appealing to Theorem B.

We require the following identities (cf. [5] and [9]).

Lemma 2. *For any positive* x, *we have*

(a) $\displaystyle\int_0^\infty \cos t \cdot \sin \frac{x}{t} \frac{dt}{t} = \int_0^\infty \sin t \cdot \cos \frac{x}{t} \frac{dt}{t} = \frac{\pi}{2} J_0(2\sqrt{x}),$

(b) $\displaystyle\int_0^\infty \cos t \cdot \cos \frac{x}{t} \frac{dt}{t} = K_0(2\sqrt{x}) - \frac{\pi}{2} Y_0(2\sqrt{x}),$

(c) $\displaystyle\int_0^\infty \sin t \cdot \sin \frac{x}{t} \frac{dt}{t} = K_0(2\sqrt{x}) + \frac{\pi}{2} Y_0(2\sqrt{x}),$

(d) $\displaystyle\int_0^\infty \cos t \cdot \sin \frac{x}{t} \frac{dt}{t} = -\sqrt{x}\left\{\frac{\pi}{2} Y_1(2\sqrt{x}) + K_1(2\sqrt{x})\right\}.$

Proof of identity (7). Put in (2)

$$f(u) = \begin{cases} \dfrac{1}{u} \sin \dfrac{x}{u} & (u \geqq 1), \\ 0 & (0 \leqq u < 1). \end{cases}$$

Then if χ is even and $b > 0$ is an integer, we have

$$S = \frac{1}{2} \sum_{0 \leqq n \leqq b}{}' \chi(n)\{f(n+0) + f(n-0)\} = \sum_{n=1}^{b} \frac{\chi(n)}{n} \sin \frac{x}{n} + \frac{\chi(b)}{2} f(b)$$
$$= \frac{2G(\chi)}{k} \sum_{n=1}^{\infty} \bar{\chi}(n) \int_0^b \cos\left(\frac{2\pi n u}{k}\right) \sin\left(\frac{x}{u}\right) \frac{du}{u}.$$

By the substitution $u = kt/(2\pi n)$, we have from identity (a)

$$S = \frac{2G(\chi)}{k} \frac{k}{2\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \int_0^{\frac{2\pi n b}{k}} \cos t \cdot \sin\left(\frac{2\pi n x/k}{t}\right) \cdot \frac{2\pi n}{k} \frac{dt}{t}$$
$$= \frac{2G(\chi)}{k} \sum_{n=1}^{\infty} \bar{\chi}(n) \int_0^{\frac{2\pi n b}{k}} \cos t \cdot \sin\left(\frac{2\pi n x/k}{t}\right) \frac{dt}{t}$$
$$= \frac{2G(\chi)}{k} \sum_{n=1}^{\infty} \bar{\chi}(n) \left\{\int_0^\infty - \int_{\frac{2\pi n b}{k}}^\infty\right\} \cos t \cdot \sin\left(\frac{2\pi n x/k}{t}\right) \frac{dt}{t}$$
$$= \frac{2G(\chi)}{k} \sum_{n=1}^{\infty} \bar{\chi}(n) \left\{\frac{\pi}{2} J_0(2\sqrt{2\pi n x/k}) - \int_{\frac{2\pi n b}{k}}^\infty \cos t \cdot \sin\left(\frac{2\pi n x/k}{t}\right) \frac{dt}{t}\right\}.$$

Therefore our task is to prove that

$$\lim_{b\to+\infty}\sum_{n=1}^{\infty}\int_{\frac{2\pi nb}{k}}^{\infty}\cos t\cdot\sin\left(\frac{2\pi nx/k}{t}\right)\frac{dt}{t}=0. \tag{26}$$

This fact is an immediate consequence of the following lemma.

Lemma 3. *Let* $0<A<B,\ y>0,\ y\leqslant A.$ *Then*

$$\int_A^B \cos t \sin\frac{y}{t}\frac{dt}{t}=O\left(\frac{1}{A^2}\right).$$

The proof of this lemma is easily carried out by making partial integration twice. In quite the same way, (10) also follows from (a), and (8) and (9) from (b) and (c) respectively. Moreover, we can derive the following identity from (d) by the same method.

Theorem 3. *Let* χ *be an even, non-principal primitive character* (mod k). *Then*

$$\sum_{n=1}^{\infty}\chi(n)\sin\left(\frac{x}{n}\right)=-\frac{1}{2}G(\chi)\sum_{n=1}^{\infty}\bar{\chi}(n)\left(\frac{x}{2kn}\right)^{\frac{1}{2}}\times \tag{27}$$

$$\times\left\{\frac{2}{\pi}K_1(2\sqrt{2\pi nx/k})+Y_1(2\sqrt{2\pi nx/k})\right\}.$$

We notice that when χ is odd Berndt [1] has obtained the identity

$$\sum_{n=1}^{\infty}\chi(n)\sin\left(\frac{x}{n}\right)=-\frac{i}{2}G(\chi)\sum_{n=1}^{\infty}\bar{\chi}(n)\left(\frac{2\pi x}{kn}\right)^{\frac{1}{2}}J_1(2\sqrt{2\pi nx/k}). \tag{28}$$

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DEPARTMENT OF MATHEMATICS,
OKAYAMA UNIVERSITY

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