SOME RESULTS ON H-AZUMAYA ALGEBRAS

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Let R be a commutative ring with identity, and let H be a finite Hopf algebra over R. In [3], F. Long defined the notion of an H-Azumaya algebra over R as a generalization of an Azumaya algebra over R. In this paper, we shall give some elementary properties of H-Azumaya algebras.

0. Preliminaries. Throughout this paper, R is a fixed commutative ring with identity, each $\otimes$ is taken over R and each map is R-linear unless otherwise stated. Moreover H is a commutative cocommutative Hopf algebra over R, ε and Δ denote the counit and comultiplication maps of H respectively, and the action of Δ is denoted by $\Delta(h) = \sum_{(h)} h^{(1)} \otimes h^{(2)}$.

An R-algebra A is called an *H-module algebra* if A is an H-module such that the H-action map $\nu: H \otimes A \longrightarrow A$ is an R-algebra map, that is, for $h \in H$, a, $b \in A$,

(0.1) $$h(ab) = \sum_{(h)} (h^{(1)}a)(h^{(2)}b) \text{ and } h(1) = \varepsilon(h)\, 1.$$

Similarly an R-algebra A is called an *H-comodule algebra* if A is an H-comodule via $\chi: A \longrightarrow A \otimes H$ such that χ is an R-algebra map, that is, for a, $b \in A$,

(0.2) $$\chi(ab) = \sum_{(a),(b)} a^{(0)}b^{(0)} \otimes a^{(1)}b^{(1)} \text{ and } \chi(1_A) = 1_A \otimes 1_H,$$

where $\chi(a) = \sum_{(a)} a^{(0)} \otimes a^{(1)}$. An R-algebra A is called an *H-dimodule algebra* if A is an H-module algebra and an H-comodule algebra such that the following diagram commutes

(0.3) $$\begin{array}{ccc} H \otimes A & \xrightarrow{\ \nu\ } & A \\ {\scriptstyle 1\otimes\chi}\downarrow & & \downarrow{\scriptstyle \chi} \\ H \otimes A \otimes H & \xrightarrow{\ \nu\otimes 1\ } & A \otimes H \end{array}$$

For an H-module algebra A and an H-comodule algebra B, the *smash product* $A \# B$ is equal to $A \otimes B$ as an R-module but with multiplication

(0.4) $$(a_1 \# b_1)(a_2 \# b_2) = \sum_{(b_1)} a_1(b_1^{(1)}a_2) \# b_1^{(0)}b_2.$$

Moreover by [3, Th. 3. 3], if A and B are H-dimodule algebras, then $A\#B$ is an H-dimodule algebra with the structure

$$h(a\#b)=\sum_{(h)}(h^{(1)}a)\#(h^{(2)}b) \tag{0.5}$$

and

$$\chi(a\#b)=\sum_{(a),(b)}a^{(0)}\#b^{(0)}\otimes a^{(1)}b^{(1)}. \tag{0.6}$$

For an H-dimodule algebra A, we define $\bar{A}$ to be the R-module A with multiplication given by

$$\bar{a}\cdot\bar{b}=\overline{\sum_{(a)}(a^{(1)}b)a^{(0)}} \tag{0.7}$$

and with H-actions inherited from A. Then $\bar{A}$ is really an H-dimodule algebra ([3, Th. 3. 5]).

1. *H*-Azumaya algebra. In the following we shall always assume that A is an H-dimodule algebra.

Definition 1. 1. An R-module M is called an *H-dimodule left A-module* if the following conditions are satisfied.

(1) M is an H-dimodule and a left A-module.

(2) $h(am)=\sum_{(h)}(h^{(1)}a)(h^{(2)}m)$ $\quad(h\in H,\ a\in A,\ m\in M)$.

(3) $\chi(am)=\chi(a)\chi(m)$.

An H-dimodule right A-module is defined similarly.

Clearly the H-dimodule algebra A is an H-dimodule left A-module and an H-dimodule right A-module.

Now we define two maps

$$F:\ A\#\bar{A}\longrightarrow \mathrm{End}\,(A)$$
$$G:\ \bar{A}\#A\longrightarrow \mathrm{End}\,(A)^{\mathrm{op}}$$

by

$$F(a\#\bar{b})(c)=\sum_{(b)}a(b^{(1)}c)b^{(0)}$$
$$G(\bar{a}\#b)(c)=\sum_{(c)}(c^{(1)}a)c^{(0)}b. \qquad (a,\ b,\ c\in A)$$

By [3, Prop. 4. 1], F and G are H-dimodule algebra maps. Then A is a left $A\#\bar{A}$-module via

$$(a\#\bar{b})x=\sum_{(b)}a(b^{(1)}x)b^{(0)} \tag{1.1}$$

and a right $\bar{A}\#A$-module via

$$x(\bar{a}\#b)=\sum_{(x)}(x^{(1)}a)x^{(0)}b. \tag{1.2}$$

Definition 1.2. ([3, Def. 4.2]). An H-dimodule algebra A is said to be *H-Azumaya* if it is a finitely generated projective faithful R-module and both F and G are isomorphisms.

Lemma 1.3. *A is an H-dimodule left $A \# \bar{A}$-module.*

Proof. It is clear that A is an H-dimodule and left $A \# \bar{A}$-module. Moreover since H is commutative and cocommutative, we have

$$\begin{aligned} h((a \# \bar{b})x) &= h\left(\textstyle\sum_{(b)} a(b^{(1)}x)b^{(0)}\right) && \text{(by (1.1))} \\ &= \textstyle\sum_{(h),(b)} (h^{(1)}a)(h^{(2)}(b^{(1)}x))(h^{(3)}b^{(0)}) && \text{(by (0.5))} \\ &= \textstyle\sum_{(h),(b)} (h^{(1)}a)(b^{(1)}(h^{(3)}x))(h^{(2)}b^{(0)}) && \\ &= \textstyle\sum_{(h)} ((h^{(1)}a) \# \overline{(h^{(2)}b)})(h^{(3)}x) && \text{(by (0.3), (1.1))} \\ &= \textstyle\sum_{(h)} (h^{(1)}(a \# \bar{b}))(h^{(2)}x) && \text{(by (0.1)).} \end{aligned}$$

This shows Def. 1.1 (2). Next, since H is commutative and cocommutative we have

$$\begin{aligned} \chi((a \# \bar{b})x) &= \textstyle\sum_{(b)} \chi(a)\chi(b^{(1)}x)\chi(b^{(0)}) && \text{(by (1.1))} \\ &= \textstyle\sum_{(a),(x),(b)} a^{(0)}(b^{(2)}x^{(0)})b^{(0)} \otimes a^{(1)}x^{(1)}b^{(1)} && \text{(by (0.2), (0.3))} \\ &= \left(\textstyle\sum_{(a),(b)} (a^{(0)} \# \overline{b^{(0)}}) \otimes (a^{(1)}b^{(1)})\right)\left(\textstyle\sum_{(x)} x^{(0)} \otimes x^{(1)}\right) && \text{(by (0.6))} \\ &= \chi(a \# \bar{b})\chi(x) && \text{(by (0.6)).} \end{aligned}$$

This shows Def. 1.1 (3). Hence A is an H-dimodule left $A \# \bar{A}$-module.

Similarly we have the following

Lemma 1.3′. *A is an H-dimodule right $\bar{A} \# A$-module.*

Definition 1.4. Let M be an H-dimodule left $A \# \bar{A}$-module, and N an H-dimodule right $\bar{A} \# A$-module. Given a subset S of A, we set

$$M^S = \{m \in M \mid (s \# \bar{1})m = (1 \# \bar{s})m \quad \text{for all} \quad s \in S\},$$
$${}^S N = \{n \in N \mid n(\bar{1} \# s) = n(\bar{s} \# 1) \quad \text{for all} \quad s \in S\}.$$

Lemma 1.5. *Let M be an H-dimodule left $A \# \bar{A}$-module. Then the map ϕ: $\mathrm{Hom}_{A \# \bar{A}}(A, M) \longrightarrow M^A$ defined by $\phi(f) = f(1)$ is an R-module isomorphism.*

Proof. For $f \in \mathrm{Hom}_{A \# \bar{A}}(A, M)$, $a \in A$ and $m \in M^A$, we have

$$(a \# \bar{1})f(1) = f((a \# \bar{1})1) = f(a) = f((1 \# \bar{a})1) = (1 \# \bar{a})f(1).$$

Therefore, ϕ is well defined and monic. Now, for any $m \in M^A$, we put $f_m(x) = (x \# \bar{1})m$ $(x \in A)$. Then

$$\begin{aligned}(a \# \bar{b})f_m(x) &= (\textstyle\sum_{(b)} a(b^{(1)}x) \# \overline{b^{(0)}})m && \text{(by (0. 4), (0. 7))}\\ &= \textstyle\sum_{(b)}(a(b^{(1)}x) \# \bar{1})(1 \# \overline{b^{(0)}})m && \text{(by (0. 4), (0. 7))}\\ &= \textstyle\sum_{(b)}(a(b^{(1)}x) \# \bar{1})(b^{(0)} \# \bar{1})m && \text{(by } m \in M^A)\\ &= (\textstyle\sum_{(b)}(a(b^{(1)}x)b^{(0)}) \# \bar{1})m && \text{(by (0. 4), (0. 7))}\\ &= f_m((a \# \bar{b})x).\end{aligned}$$

Therefore f_m is in $\mathrm{Hom}_{A \# \bar{A}}(A, M)$, that is, ϕ is an epimorphism.

Similarly, we have

Lemma 1. 5′. *Let N be an H-dimodule right $\bar{A} \# A$-module. Then the map ϕ' :* $\mathrm{Hom}_{\bar{A} \# A}(A, N) \longrightarrow {}^A N$ *defined by $\phi'(f) = f(1)$ is an R-module isomorphism.*

Corollary 1. 6. (1) $\mathrm{Hom}_{A \# \bar{A}}(A, A) \cong A^A$ *as R-modules.*

(2) $\mathrm{Hom}_{\bar{A} \# A}(A, A) \cong {}^A A$ *as R-modules.*

Now we shall generalize the notion of a separable algebra in the next

Theorem 1. 7. *If π : $A \# \bar{A} \longrightarrow A$ is defined by $\pi(a \# \bar{b}) = ab$, then the following are equivalent.*

(1) *A is left $A \# \bar{A}$-projective.*

(2) *There exists an element θ in $A \# \bar{A}$ such that $\pi(\theta) = 1$ and $(a \# \bar{1})\theta = (1 \# \bar{a})\theta$ for all $a \in A$.*

Proof. First, we claim that π is a left $A \# \bar{A}$-module epimorphism. In fact,

$$\begin{aligned}\pi((x \# \bar{y})(a \# \bar{b})) &= \pi(\textstyle\sum_{(y)} x(y^{(2)}a) \# \overline{(y^{(1)}b)y^{(0)}}) && \text{(by (0. 4), 0. 7))}\\ &= \textstyle\sum_{(y)} x(y^{(2)}a)(y^{(1)}b)y^{(0)}\\ &= \textstyle\sum_{(y)} x(y^{(1)}(ab))y^{(0)} && \text{(by (0. 1))}\\ &= (x \# \bar{y})\pi(a \# \bar{b}).\end{aligned}$$

(1) $\Longrightarrow$ (2). Since A is left $A \# \bar{A}$-projective, there exists a left $A \# \bar{A}$-module homomorphism $j : A \longrightarrow A \# \bar{A}$ such that $\pi j = 0$. If we put $\theta = j(1)$, then θ satisfies the condition (2).

(2) $\Longrightarrow$ (1). We put $\theta = \sum_i a_i \# \bar{b}_i$ and defined a map $j : A \longrightarrow A \# \bar{A}$ by $j(a) = (a \# \bar{1})\theta$. Then

$$(x \# \bar{y})j(a) = \textstyle\sum_{i,(y)} x(y^{(2)}(aa_i)) \# \overline{(y^{(1)}b_i)y^{(0)}} \qquad \text{(by (0. 4), (0. 7))}.$$

Noting that $(y \# \bar{1})\theta = (1 \# \bar{y})\theta$ for all $y \in A$, we have

(1. 3) $$\sum_i y a_i \# \bar{b}_i = \sum_{i,(y)} y^{(2)} a_i \# \overline{(y^{(1)} b_i) y^{(0)}} \qquad \text{(by (0. 4), (0. 7))}$$

and

$$\begin{aligned} j((x \# \bar{y})a) &= \sum_{i,(y)} x(y^{(1)} a)\, y^{(0)} a_i \# \bar{b}_i && \text{(by (1. 1))} \\ &= \sum_{i,(y)} (x(y^{(1)} a) \# \bar{1})(y^{(0)} a_i \# \bar{b}_i) && \text{(by (0. 4), (0. 7))} \\ &= \sum_{i,(y)} (x(y^{(3)} a) \# \bar{1})(y^{(2)} a_i \# \overline{(y^{(1)} b_i) y^{(0)}} && \text{(by (1. 3))} \\ &= \sum_{i,(y)} x(y^{(3)} a)(y^{(2)} a_i) \# \overline{(y^{(1)} b_i) y^{(0)}} && \text{(by (0. 4), (0. 7))} \\ &= \sum_{i,(y)} x(y^{(2)}(a a_i)) \# \overline{(y^{(1)} b_i) y^{(0)}} && \text{(by (0. 1))} \\ &= (x \# \bar{y}) j(a) && \text{(by (0. 4), (0. 7)).} \end{aligned}$$

Therefore j is a left $A \# \bar{A}$-module homomorphism, and $\pi j = 1$. Hence A is left $A \# \bar{A}$-projective.

Similarly, we have the following

Theorem 1. 7'. *If $\pi' : \bar{A} \# A \longrightarrow A$ is defined by $\pi'(\bar{a} \# b) = ab$, then the following are equivalent.*

(1) A is right $\bar{A} \# A$-projective.

(2) There exists an element θ' in $\bar{A} \# A$ such that $\pi'(\theta') = 1$ and $\theta'(\bar{1} \# a) = \theta'(\bar{a} \# 1)$ for all $a \in A$.

Theorem 1. 8. *The following conditions are equivalent.*

(1) A is an R-progenerator and $F : A \# \bar{A} \longrightarrow \mathrm{End}(A)$ is an isomorphism.

(2) A is a left $A \# \bar{A}$-progenerator and $A^A = R$.

Proof. (1) $\Longrightarrow$ (2). Since A is an R-progenerator and the map $F : A \# \bar{A} \longrightarrow \mathrm{End}(A)$ is an isomorphism, A is a left $A \# \bar{A}$-progenerator by [1, Cor. I. 3. 4]. Hence, by Cor. 1. 6 (1) we have

$$R \cong \mathrm{Hom}_{\mathrm{End}(A)}(A, A) \cong \mathrm{Hom}_{A \# \bar{A}}(A, A) \cong A^A.$$

(2) $\Longrightarrow$ (1). This is clear by [1, Cor. I. 3. 4].

Similarly we have

Theorem 1. 8'. *The following conditions are equivalent.*

(1) A is an R-progenerator and $G : \bar{A} \# A \longrightarrow \mathrm{End}(A)^{op}$ is an isomorphism.

(2) A is a right $\bar{A} \# A$-progenerator and ${}^A A = R$.

Remark 1. 9. Let $Z(A)$ be the center of A, and let $Z(A)^H = \{z \in Z(A) \mid hz = \varepsilon(h)z \text{ for all } h \in H\}$. Then for any $z \in Z(A)^H$, $a \in A$, we have

$$(1 \# \bar{a})z = \sum_{(a)}(a^{(1)}z)a^{(0)} = \sum_{(a)}(\varepsilon(a^{(1)})z)a^{(0)} = z(\sum_{(a)}\varepsilon(a^{(1)})a^{(0)})$$
$$= za = az = (a \# \bar{1})z.$$

Therefore $R \subseteq Z(A)^H \subseteq A^A$. On the other hand, since

$$G_{\bar{z}\#1}(a) = \sum_{(a)}(a^{(1)}z)\,(a^{(0)}1) = za = az = G_{\bar{1}\#z}(a),$$

we have $\bar{z} \# 1 = \bar{1} \# z$ provided G is an isomorphism. Especially if G is an isomorphism and A is an R-progenerator, then z is in R. Hence if A satisfies the condition in Th. 1.8 or Th. 1.8′, we have $R = Z(A)^H$.

As a combination of Th. 1.8, Th. 1.8′ and Remark 1.9, we readily obtain the following

Theorem 1.10. *The following conditions are equivalent.*

(1) *A is H-Azumaya.*

(2) *A is left $A \# \bar{A}$-progenerator, right $\bar{A} \# A$-progenerator and $A^A = R = Z(A)^H = {}^A A$.*

2. Examples. In this section we shall give two examples of H-Azumaya algebras for which the Morita equivalence is also valid.

Let R-**MOD** be the category of H-dimodules and H-dimodule homomorphisms and let $A \# \bar{A}$-**MOD** (resp. **MOD**-$\bar{A} \# A$) be the category of H-dimodule left $A \# \bar{A}$-(resp. right $\bar{A} \# A$-) modules and H-dimodule left $A \# \bar{A}$-(resp. right $\bar{A} \# A$-) module homomorphisms.

2.1. Let G be a group of order 2, and $H = RG$, the group algebra of G over R. If A is an H-dimodule algebra, then for any $a \in A$, we have

$$\chi(a) = a_0 \otimes e + a_1 \otimes \sigma$$

where χ is the comodule structure map of A and $G = \{e, \sigma\}$ $(\sigma^2 = e)$. Therefore for any $a, b \in A$, we have

(2.1) $\quad a = a_0 + a_1 \quad$ (unique) $\quad (a_0, a_1 \in A)$,

(2.2) $\quad (ab)_0 = a_0 b_0 + a_1 b_1, \quad (ab)_1 = a_0 b_1 + a_1 b_0$,

(2.3) $\quad (\sigma a)_0 = \sigma(a_0), \quad (\sigma a)_1 = \sigma(a_1)$.

Throughout this subsection, we shall assume that $H = RG$, A is an H-dimodule algebra and that M (resp. N) is an H-dimodule left $A \# \bar{A}$-(resp. right $\bar{A} \# A$-) module.

Now, for $\mathrm{Hom}_{A \# \bar{A}}(A, M)$ and $\mathrm{Hom}_{\bar{A} \# A}(A, N)$ we define

$$(2.4)\quad \begin{cases} (\sigma f)(a) = \sigma f(\sigma a) & (f \in \mathrm{Hom}_{A\sharp\bar{A}}(A,\ M),\ a \in A), \\ \chi(f) = f_0 \otimes e + f_1 \otimes \sigma & \text{where } f_i(a) = (a \sharp \bar{1}) f(1_i)\ (i = 1,\ 2), \end{cases}$$

$$(2.5)\quad \begin{cases} (\sigma f)(a) = \sigma(f(\sigma a)) & (f \in \mathrm{Hom}_{\bar{A}\sharp A}(A,\ N),\ a \in A), \\ \chi(f) = f_0 \otimes e + f_1 \otimes \sigma & \text{where } f_i(a) = f(1)_i(1 \sharp \bar{a})\ (i = 1,\ 2), \end{cases}$$

respectively.

Proposition 2.1. $\mathrm{Hom}_{A\sharp\bar{A}}(A,\ M)$ *(resp.* $\mathrm{Hom}_{\bar{A}\sharp A}(A,\ N)$*) is an H-dimodule concerning the structure* (2.4) *(resp.* (2.5)*) and* $\mathrm{Hom}_{A\sharp\bar{A}}(A, M) \cong M^A$ *(resp.* $\mathrm{Hom}_{\bar{A}\sharp A}(A,\ N) \cong {}^A N$*) as H-dimodules, where the H-dimodule structure of* M^A (resp. ${}^A N$) *inherits from* M *(resp.* N*).*

Proof. First, we prove that M^A is an H-subdimodule of M. Let $m \in M^A$, $a \in A$. Then $\sigma((\sigma a \sharp \bar{1})m) = \sigma((1 \sharp \overline{\sigma a})m)$, and so σm is in M^A. Since $(a_i \sharp \bar{1})m = (1 \sharp \bar{a}_i)m\ (i = 1,\ 2)$, we have $\chi((a_i \sharp \bar{1})m = \chi((1 \sharp a_i)m)$. By (2.1) we obtain $m_0,\ m_1 \in M^A$. Thus M^A is an H-subdimodule of M. Similarly ${}^A N$ is an H-subdimodule of N.

Since M is an H-dimodule left $A \sharp \bar{A}$-module, $f(1) \in M^A$ and $f(1)_i \in M^A$ by Lemma 1.5. Now we can easily seen that $\mathrm{Hom}_{A\sharp\bar{A}}(A, M)$ is an H-dimodule. It remains therefore to show that the map $\phi : \mathrm{Hom}_{A\sharp\bar{A}}(A, M) \longrightarrow M^A$ defined by $\phi(f) = f(1)$ is an H-dimodule isomorphism. But this is easy by the definition of f_i and Lemma 1.5. Similarly $\mathrm{Hom}_{\bar{A}\sharp A}(A,\ N) \cong {}^A N$ as H-dimodules.

Remark 2.2. The Morita theory for $Z/2Z$-graded case (resp. G-graded case) is developed in [2] (resp. [5]). In [2] (resp. [5]), if we define the G-action on A by $\sigma a = a$ $(\sigma \in G,\ a \in A)$ then each $Z/2Z$-graded (resp. G-graded) Azumaya algebra is an H-Azumaya by [3, p. 588]. Therefore the following theorem is a generalization of the Morita theory for $Z/2Z$-graded case. (Recently in [4], M. Orzech announced that a Morita theory for G-dimodules is developed by M. Beattie.)

Theorem 2.3. *If* A *is H-Azumaya, then each of the following pairs of functors establishes an isomorphism of categories*:

$$(1)\quad \begin{array}{ll} \mathscr{F} : R\text{-}\mathbf{MOD} \longrightarrow A \sharp \bar{A}\text{-}\mathbf{MOD} & \mathscr{F}(X) = A \otimes X, \\ \mathscr{G} : A \sharp \bar{A}\text{-}\mathbf{MOD} \longrightarrow R\text{-}\mathbf{MOD}, & \mathscr{G}(Y) = Y^A. \end{array}$$

$$(2)\quad \begin{array}{ll} \mathscr{K} : R\text{-}\mathbf{MOD} \longrightarrow \mathbf{MOD}\text{-}\bar{A} \sharp A, & \mathscr{K}(X) = X \otimes A, \\ \mathscr{L} : \mathbf{MOD}\text{-}\bar{A} \sharp A \longrightarrow R\text{-}\mathbf{MOD}, & \mathscr{L}(Y) = {}^A Y. \end{array}$$

Proof. We shall prove only (1), and leave (2) to the reader. Since

A is an R-progenerator and $A \# \bar{A} \cong \mathrm{End}(A)$, by Morita theory [1, Prop. I. 3. 3] there hold

$$\mathrm{Hom}_{A\#\bar{A}}(A, \ A \# \bar{A}) \cong \mathrm{Hom}(A, \ R) \text{ and } R \cong \mathrm{Hom}(A, \ R) \otimes_{A\#\bar{A}} A.$$

Thus we have

$$\begin{aligned} X \cong \mathrm{Hom}(A, \ R) \otimes_{A\#\bar{A}} (A \otimes X) &\cong \mathrm{Hom}_{A\#\bar{A}}(A, \ A \# \bar{A}) \otimes_{A\#\bar{A}} (A \otimes X) \\ &\cong \mathrm{Hom}_{A\#\bar{A}}(A, \ A \# \bar{A} \otimes_{A\#\bar{A}} (A \otimes X)) \\ &\qquad \text{(by [1, I. 2. 7])} \\ &\cong \mathrm{Hom}_{A\#\bar{A}}(A, \ A \otimes X) \cong (A \otimes X)^A \\ &\qquad \text{(by Prop. 2. 1)}, \end{aligned}$$

and similarly

$$\begin{aligned} Y \cong A \otimes \mathrm{Hom}_{A\#\bar{A}}(A, A \# \bar{A}) \otimes_{A\#\bar{A}} Y &\cong A \otimes \mathrm{Hom}_{A\#\bar{A}}(A, A \# \bar{A} \otimes_{A\#\bar{A}} Y) \\ &\cong A \otimes \mathrm{Hom}_{A\#\bar{A}}(A, \ Y) \cong A \otimes Y^A \\ &\qquad \text{(by Prop. 2. 1)}. \end{aligned}$$

2. 2. Let R be a commutative algebra over $GF(2)$. Let $H = R \oplus R\delta$ be a free R-module with a free basis $\{1, \ \delta\}$. Then H is a Hopf algebra with the following algebra and coalgebra structure

$$\delta^2 = 0, \qquad \varepsilon(\delta) = 0, \qquad \Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta.$$

If A is an H-dimodule algebra, then for any $a, \ b \in A$, we have

(2. 6) $\delta(ab) = (\delta a)b + a(\delta b)$,

(2. 7) $\chi(a) = a \otimes 1 + a_1 \otimes \delta, \qquad \chi(a_1) = a_1 \otimes 1 \quad (a_1 \in A)$,

(2. 8) $(ab)_1 = ab_1 + a_1 b$,

(2. 9) $\chi(\delta a) = \delta a \otimes 1 + \delta a_1 \otimes \delta, \qquad (\delta a)_1 = \delta a_1$.

Throughout this subsection, we shall assume that $H = R \oplus R\delta$, A is an H-dimodule algebra and that M (resp. N) is an H-dimodule left $\bar{A} \# A$-(resp. right $\bar{A} \# A$-) module.

Now, for $\mathrm{Hom}_{A\#\bar{A}}(A, \ M)$ and $\mathrm{Hom}_{\bar{A}\#A}(A, \ N)$ we define

(2. 10) $\begin{cases} \delta(f)(a) = \delta(f(a)) + f(\delta a) & (f \in \mathrm{Hom}_{A\#\bar{A}}(A, \ M), \ a \in A), \\ \chi(f) = f \otimes 1 + f_1 \otimes \delta, & \text{where } f_1(a) = (a \# \bar{1}) f(1)_1, \end{cases}$

(2. 11) $\begin{cases} (\delta f)(a) = \delta(f(a)) + f(\delta a) & (f \in \mathrm{Hom}_{\bar{A}\#A}(A, \ N), \ a \in A), \\ \chi(f) = f \otimes 1 + f_1 \otimes \delta, & \text{where } f_1(a) = f(1)_1 (\bar{1} \# a), \end{cases}$

respectively.

Proposition 2.4. $\mathrm{Hom}_{A\#\bar{A}}(A, M)$ *(resp.* $\mathrm{Hom}_{\bar{A}\#A}(A, N)$*) is an H-dimodule concerning the structure* (2.10) *(resp.* (2.11)*) and* $\mathrm{Hom}_{A\#\bar{A}}(A, M) \cong M^A$(resp. $\mathrm{Hom}_{\bar{A}\#A}(A, N) \cong {}^A N$) *as H-dimodules, where the H-dimodule structure of* M^A *(resp.* ${}^A N$*) inherits from* M *(resp.* N*).*

Proof. First, we prove that M^A is an H-subdimodule of M. Let $m \in M^A$, $a \in A$. Then by $\delta((a \# \bar{1})m) = \delta((1 \# \bar{a})m)$ and $(\delta a \# \bar{1})m = (1 \# \delta a)m$, we have $m \in M^A$. Moreover, by $\chi((a \# \bar{1})m) = \chi((1 \# \bar{a})m)$ and $(a_1 \# \bar{1})m = (1 \# \bar{a}_1)m$, we have $m_1 \in M^A$. Therefore M^A is an H-subdimodule of M. Similarly ${}^A N$ is an H-subdimodule of N.

Next, we show that $\mathrm{Hom}_{A\#\bar{A}}(A, M)$ is an H-module and an H-comodule. Let $a, b, x \in A$, $f \in \mathrm{Hom}_{A\#\bar{A}}(A, M)$. Then

$$\begin{aligned}(a \# \bar{b})((\delta f)(x)) &= (a \# \bar{b})(\delta((x \# \bar{1})f(1)) + (\delta x \# \bar{1})f(1)) \\ &= (a \# \bar{b})((\delta x \# \bar{1})f(1) + (x \# \bar{1})\delta(f(1)) + (\delta x \# \bar{1})f(1)) \\ &\qquad \text{(by Def. 1.1(2), (0.5), (2.6))} \\ &= (a \# \bar{b})(x \# \bar{1})\,\delta(f(1))\end{aligned}$$

and

$$\begin{aligned}(\delta f)((a \# \bar{b})x) &= \delta(f((a \# \bar{b})x)) + f(\delta((a \# \bar{b})x)) \\ &= \delta((a \# \bar{b})(x \# \bar{1})f(1)) + f((\delta(a \# \bar{b})x + (a \# \bar{b})\delta x) \\ &\qquad \text{(by Def. 1.1(2))} \\ &= (a \# \bar{b})(x \# \bar{1})\,\delta(f(1)) \\ &\qquad \text{(by (0.5), (2.6), } f \in \mathrm{Hom}_{A\#\bar{A}}(A.\ M)).\end{aligned}$$

Therefore $f \in \mathrm{Hom}_{A\#\bar{A}}A, M)$. Moreover,

$$\begin{aligned}(a \# \bar{b})(f_1(x)) &= \textstyle\sum_{(b)}(a(b^{(1)}x) \# \overline{b^{(0)}})(f(1))_1 && \text{(by (0.4), (0.6))} \\ &= \textstyle\sum_{(b)}(a(b^{(1)}x) \# \bar{1})(1 \# \overline{b^{(0)}})(f(1))_1 && \text{(by (0.4), (0.6))} \\ &= \textstyle\sum_{(b)}(a(b^{(1)}x) \# \bar{1})(b^{(0)} \# \bar{1})(f(1))_1 && \text{(by } f(1)_1 \in M^A)) \\ &= \textstyle\sum_{(b)}(a(b^{(1)}x)b^{(0)} \# \bar{1})(f(1))_1 && \text{(by (0.4), (0.6))} \\ &= f_1(\textstyle\sum_{(b)} a(b^{(1)}x)b^{(0)}) = f_1((a \# \bar{b})x).\end{aligned}$$

Hence we have $f_1 \in \mathrm{Hom}_{A\#\bar{A}}(A, M)$. Then it is easy to see that $\mathrm{Hom}_{A\#\bar{A}}(A, M)$ is an H-module and an H-comodule. It remains therefore to show that $\mathrm{Hom}_{A\#\bar{A}}(A, M)$ is an H-dimodule. In fact, we have

$$\begin{aligned}(\delta f_1)(x) &= \delta(f_1(x)) + f_1(\delta x) = \delta((x \# \bar{1})(f(1))_1) + (\delta x \# \bar{1})(f(1))_1 \\ &= (x \# \bar{1})\,\delta((f(1))_1) && \text{(by (0.5), Def. 1.1(2))} \\ &= (x \# \bar{1})((\delta f)(1))_1 = (\delta f)_1(x),\end{aligned}$$

and $\chi(\delta f) = f \otimes 1 + f_1 \otimes \delta$. Hence $\mathrm{Hom}_{A \# \bar{A}}(A, M)$ is an H-dimodule.

Finally, we show that the map $\phi : \mathrm{Hom}_{A \# \bar{A}}(A, M) \longrightarrow M^A$ defined by $\phi(f) = f(1)$ is an H-dimodule isomorphism. In fact,

$$\phi(\delta f) = (\delta f)(1) = \delta f(1) + f(\delta 1) = \delta f(1) = \delta\phi(f)$$

and

$$\begin{aligned}\chi(\phi(f)) &= \chi(f(1)) = f(1) \otimes 1 + (f(1))_1 \otimes \delta = f(1) \otimes 1 + f_1(1) \otimes \delta \\ &= (\phi \otimes 1)\chi(f).\end{aligned}$$

Hence ϕ is an H-dimodule isomorphism by Lemma 1. 5. Similarly $\mathrm{Hom}_{\bar{A} \# A}(A, N) \cong {}^A N$ as H-dimodules.

By Prop. 2. 3, Morita theory [1. Cor. I. 3. 4] and the proof of Th. 2. 2, we have the following

Theorem 2. 5. *If A is H-Azumaya, then each of the following pairs of functors establishes an isomorphism of categories*:

$$(1)\quad \begin{aligned} &\mathscr{F} : R\text{-}\mathbf{MOD} \longrightarrow A \# \bar{A}\text{-}\mathbf{MOD}, && \mathscr{F}(X) = A \otimes X, \\ &\mathscr{G} : A \# \bar{A}\text{-}\mathbf{MOD} \longrightarrow R\text{-}\mathbf{MOD}, && \mathscr{G}(Y) = Y^A. \end{aligned}$$

$$(2)\quad \begin{aligned} &\mathscr{K} : R\text{-}\mathbf{MOD} \longrightarrow \mathbf{MOD}\text{-}\bar{A} \# A, && \mathscr{K}(X) = X \otimes A, \\ &\mathscr{L} : \mathbf{MOD}\text{-}\bar{A} \# A \longrightarrow R\text{-}\mathbf{MOD}, && \mathscr{L}(Y) = {}^A Y. \end{aligned}$$

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