TWO THEOREMS ON LEFT s-UNITAL RINGS

Dedicated to Professor Eiji Inaba on his 65th birthday

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Throughout R will represent a ring, and N the set of all nilpotent elements of R. Following [7], R is called a *left s-unital ring* if for each $x\in R$ there exists an element e with $ex=x$. The purpose of this note is to present two theorems concerning left s-unital rings which include [2, Satz] and [5, Theorem 2. 1] respectively.

As was shown in [7, Theorem 1], if $x_1, \cdots, x_n$ are arbitrary elements of a left s-unital ring R then there exists $e\in R$ such that $ex_i=x_i$ for all x_i. As an application of this fact, we can simplify the formulation of [2, Satz] as follows:

Theorem 1 (cf. [8, Theorem]). *Let A be an ideal of R such that R/A is left s-unital, and B a subgroup of the additive group of R. If $R=A+B$ and $AB=BA=0$, then $R=A\oplus B'$ with some ideal B' of R.*

Proof. Since $B^2R=B\cdot BR\subseteq B(A+B)=B^2$ and $RB^2\subseteq B^2$, $B'=B^2$ is an ideal of R. Now, let $r=a+b$ be an arbitrary element of R ($a\in A$, $b\in B$), and choose an element $e\in B$ with $b-eb=a^*\in A$. Then $r=(a+a^*)+eb\in A+B'$. Next, let $a'=b_1c_1+\cdots+b_nc_n$ be an arbitrary element of $A\cap B'$ (b_i, $c_i\in B$). Then there exists some $e'\in B$ such that $b_i-e'b_i\in A$ for all i, and so $a'=(b_1-e'b_1)c_1+\cdots+(b_n-e'b_n)c_n+e'a'=0$. Thus, we have proved $R=A\oplus B'$.

Next, we consider the following conditions:

1) For each $x\in R$ there exists a positive integer n such that $x^{n+1}-x\in N$.

2′) $x-y\in N$ implies that $x^2=y^2$ or both x and y are contained in the centralizer $V_R(N)$ of N in R.

3) $xR\subseteq N$ (or equivalently $Rx\subseteq N$) for each $x\subseteq N$.

Recently, in their paper [5], S. Ligh and J. Luh proved that if R contains a left identity and the conditions 1), 2′) and 3) hold then R is commutative. However, as they noted there, the last is not true in general for rings without identity. In our second theorem we shall show that the last is still true for left s-unital rings and the condition 3) is dispensable. In advance of proving the theorem, we consider further the following conditions:

1′) For each $x \in R$ there exist positive integers m, n such that $x^{n+m} - x^m \in N$.

2) $x - y \in N$ and $y - z \in N$ imply that $x^2 = z^2$ or $xy = yx$.

2*) $x - y \in N$ implies that $x^2 = y^2$ or $xy = yx$.

Needless to say, 2*) is a consequence of 2) or 2′). Moreover, we have the following:

Lemma 1. (1) *If* 2*) *is satisfied, then* $x^2 \in V_R(N)$ *for each* $x \in R$, *especially every idempotent of* R *is central, and* N *is an ideal of* R.

(2) *If* N *is an ideal of* R *then* 1) and 1′) *are equivalent.*

Proof. (1) Let $x \in R$, and $y \in N$ with $y^m = 0$. If $xy \neq yx$ then $(x+y)^2 = x^2$ and $0 = (x+y)(x+y)^2 - (x+y)^2(x+y) = (x+y)x^2 - x^2(x+y) = yx^2 - x^2y$, proving the first assertion. Since $(xy)^2$ is in $V_R(N)$, one will easily see that $(xy)^{2m} = (xyx)^m y^m = 0$, and similarly $(yx)^{2m} = 0$. We have therefore seen that $Ry \subseteq N$ and $yR \subseteq N$. Now, we assume further that $x^m = 0$. Then, noting that $(xy)^{2m} = (yx)^{2m} = 0$, it is easy to see that $(x+y)^{4m} = 0$. Thus, N is an ideal of R.

(2) It suffices to show that 1′) implies 1). Since R/N is a reduced periodic ring, 1) is an easy consequence of [1, Theorem 4 and Lemma 1].

In virtue of Lemma 1, we see that 2′) implies 2) and 1) + 2) is equivalent to 1′) + 2).

Lemma 2. *Let* R *be a left s-unital ring. If* N *is an ideal of* R *and* 1) *is satisfied, then for each finite subset* F *of* R *there exists a subring* S *of* R *with a left identity such that* $F \subseteq S$ *and* $(S+N)/N$ *is finite.*

Proof. Let $F = \{x_1, \cdots, x_m\}$, and choose an element c such that $cx_i = x_i$ for all i. If T is the subring generated by F and c, then $\bar{T} = (T+N)/N$ is a finite ring (cf. [6]). As is well known, the identity element of $\bar{T}$ can be lifted to an idempotent e of T. We write $e = x_1p_1 + \cdots + x_mp_m + cp_{m+1}$, where p_i is a (non-commutative) polynomial in $x_1, \cdots, x_m$, c with integer coefficients. By 1), $c^{n+1} - c \in N$ for some positive integer n. This together with $cx_i = x_i$ proves that $c^n \equiv c^n e = c^n x_1 p_1 + \cdots + c^n x_m p_m + c^{n+1} p_{m+1} \equiv e \pmod{N}$, so that $(c^{2n} - c^n)^r = 0$ with some positive integer r. Now, we consider the polynomial $f(X) = \sum_{k=0}^{r-1} \binom{2r}{k} (1-X)^k X^{2r-k}$. Obviously, $f = f(c^n)$ has a meaning as an element of R, and it is well-known that $f^2 = f \equiv c^n \pmod{N}$. Moreover, one will easily see that $fx_i = c^{2rn}x_i = x_i$. Accordingly, f is a left identity of the subring S of T generated by F and f.

Lemma 3 (cf. [4, Theorem 2]). *Let R be a ring with a left identity. If 2) is satisfied and R/N is finite (cf. Lemma 1), then R is commutative.*

Proof. First, we claim that R has the identity. In fact, by Lemma 1 (1) every idempotent of R is central, and so any left identity of R is the identity. Next, we shall prove that N is a commutative ring. Suppose there exist $x, y \in N$ such that $xy \neq yx$. Since $x+y \equiv x \equiv 0$ (mod N) and $(x+y)x \neq x(x+y)$, 2) implies $0=(x+y)^2=x^2$, and similarly, $0=(x+y)^2=y^2$. From those it follows $xy+yx=0$. While, noting that $1+x+y \equiv 1+x$ (mod N), 2) implies also $(1+x+y)^2=(1+x)^2$, whence it follows $2(x+y)=2x$, namely, $2y=0$. This yields a contradiction $xy=-yx=yx$. Finally, we shall show that N is in the center of R, which will complete the proof by Herstein's theorem (see [3, p. 221]). By [6, Lemma], $R/N = R_1/N \oplus \cdots \oplus R_m/N$ where R_i/N is a finite field. Suppose there exist $r \in R$ and $s \in N$ such that $rs \neq sr$. Then there exists some $r_j \in R_j$ such that $r_j s \neq s r_j$. Since $2r_j=(1+r_j)^2-1-r_j \in V_R(N)$ by Lemma 1 (1), there holds $2(r_j s - s r_j)=0$. We shall distinguish between the following two cases:

Case 1: $R_j/N = \mathrm{GF}(2^k)$. Since $r_j^2 s = s r_j^2$, one will easily see that $0=(r_j^{2^k}-r_j)s - s(r_j^{2^k}-r_j) = s r_j - r_j s$, which is a contradiction.

Case 2: $R_j/N = \mathrm{GF}(p^k)$, $p \neq 2$. Since $p(r_j s - s r_j) = (p r_j)s - s(p r_j) = 0$ and $2(r_j s - s r_j)=0$, we readily obtain $r_j s - s r_j = 0$, a contradiction.

Now, we are at the position to state our second theorem, whose proof is only a combination of Lemmas 1, 2 and 3.

Theorem 2. *Let R be a left s-unital ring. If 1) and 2), or equivalently 1′) and 2), are satisfied, then R is commutative.*

Remark. In Theorem 2, if 2*) is assumed instead of 2) then N need not be commutative. However, given $x, y \in N$, we can prove that $xy=yx$ or the subring generated by x, y is nilpotent of index 3.

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