# NOTE ON AN IDEAL OF A POSITIVELY FILTERED RING OVER A COMMUTATIVE RING

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Let $K$ be a commutative ring, and $K[X]$ the polynomial ring over $K$. Then it is known that if $I$ is a proper ideal of $K[X]$ such that $K[X]/I$ is finitely generated and projective as a $K$-module, then $I$ is generated by a *quasi-monic* polynomial (and conversely) ([1; Th. 1.3]). The purpose of this note is to extend this result to some positively filtered rings.

Throughout the present note, all rings are associative, but not necessarily commutative. Every ring has 1, which is preserved by homomorphisms, inherited by subrings and acts as the identity operator on modules.

Let ${}_AM_{A'}$ be a left $A$-, right $A'$-module. If $M_{A'}$ is finitely generated, projective and generator, and $\mathrm{End}(M_{A'}) \simeq A$ under the mapping induced by ${}_AM$, we call ${}_AM_{A'}$ an *invertible module.* It is well known that this is right-left symmetric.

Let $R \supseteq K$ be rings, and $R_0 = K \subseteq R_1 \subseteq R_2 \subseteq \cdots$ an ascending sequence of additive subgroups such that $R = \cup_i R_i$ and $R_i \cdot R_j \subseteq R_{i+j}$ for all $i$, $j \geq 0$. Then we call $R = \cup_i R_i$ a *positively filtered ring over* $K$. If, further, $R = \cup_i R_i$ satisfies the following condition we call $R = \cup_i R_i$ a *(*)-positively filtered ring over* $K$:

(*) Each $R_n/R_{n-1}$ $(n \geq 1)$ is an invertible module as a $K$-bimodule, and $(R_n/R_{n-1}) \otimes_K (R_m/R_{m-1}) \simeq R_{n+m}/R_{n+m-1}$ canonically for all $n$, $m \geq 1$.

We denote this (*)-positively filtered ring over $K$ by $K[R_1]$, and put $R_i = 0$ for $i < 0$. It is easy to see that the latter half of (*) can be replaced by the condition that $R_n$ is $R_1^n = R_1 \cdots R_1$ ($n$ times) for all $n \geq 1$, because the both sides are invertible $K$-bimodules.

In what follows, $R = K[R_1]$ is always a (*)-positively filtered ring over a commutative ring $K$. We shall characterize a left $K$-, right $R$-submodule $I$ of $R$ such that ${}_KR/I$ is finitely generated and projective. We begin with the following easy lemma.

**Lemma 1.** *Let $M$ be an invertible $K$-bimodule, and $\mathfrak{p}$ a maximal ideal of $K$. Then $M/\mathfrak{p}M$ is a simple left $K$-module and a simple right $K$-module.*

**Corollary.** *Let $M_0 = \{0\} \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ be a sequence of $K$-bimodules such that each $M_i/M_{i-1}$ $(i = 1, 2, \cdots, n)$ is an invertible $K$-bi-*

*module. Let $p$ be a maximal ideal of $K$. Put $\overline{M}=M/pM$, and $\overline{M}_i=(M_i+pM)/pM$ $(i=0, 1, \cdots, n)$. Then $\{0\}\subseteq\overline{M}_1\subseteq\cdots\subseteq\overline{M}_n=\overline{M}$ is a composition series of $\overline{M}$ as a left $K$-module and as a right $K$-module.*

*Proof.* For $i\geq 1$, there holds $\overline{M}_i/\overline{M}_{i-1}\rightrightarrows M_i/(M_{i-1}+(M_i\cap pM))$ canonically. But, since ${}_KM_i$ is a direct summand of ${}_KM$, we have $M_i\cap pM=pM_i$. Hence $\overline{M}_i/\overline{M}_{i-1}$ is isomorphic to $M_i/(M_{i-1}+pM_i)$ as a $K$-bimodule. Then the result follows from Lemma 1.

**Lemma 2.** *Let $p$ be a maximal ideal of $K$. Then there are $u_1$, $u_2$, $u_3$, $\cdots$ in $R_1$ such that $R_n=K+u_1K+u_1u_2K+\cdots+u_1u_2\cdots u_nK+pR_n$ for $n=1, 2, 3\cdots$.*

*Proof.* Take $u_1\in R_1$ not contained in $K+pR_1$. Then $R_1=K+u_1K+pR_1$, by Cor. to Lemma 1. Assume that $R_n=K+u_1K+\cdots+u_1\cdots u_nK+pR_n$. Then $R_{n+1}=R_n+u_1\cdots u_nR_1+pR_{n+1}$. Take $u_{n+1}\in R_1$ such that $u_1\cdots u_nu_{n+1}\notin R_n+pR_{n+1}$. Then, by Cor. to Lemma 1, $R_{n+1}=R_n+u_1\cdots u_{n+1}K+pR_{n+1}=K+u_1K+\cdots+u_1\cdots u_{n+1}K+pR_{n+1}$, as desired.

**Corollary.** *Let $p$ be a maximal ideal of $K$, and $I$ a right ideal of $R$ such that $I\not\subseteq pR$. Then $\bar{R}=\bar{I}\oplus\bar{R}_{r-1}$, where $\bar{S}$ is $\{x+pR\,|\,x\in S\}$ for any subset $S$ of $R$, and $r=$* length $\bar{R}/\bar{I}_K$.

*Proof.* First we claim that $R_n\cap pR=pR_n$ for any $n$ (cf. the proof of Cor. to Lemma 1). For any $z\in R\backslash pR$, $e=\deg z$ is defined by $\bar{z}=z+pR\in\bar{R}_e\backslash\bar{R}_{e-1}$. Take $y\in I\backslash pR$ such that $\deg y=r$ is minimal, and then evidently $\bar{I}\cap\bar{R}_{r-1}=0$. Assume that $\deg z=e\geq r$, and let $\bar{z}\equiv\bar{u}_1\cdots u_ec$ mod $\bar{R}_{e-1}$, and $\bar{y}\equiv\bar{u}_1\cdots u_ra$ mod $\bar{R}_{r-1}$, where $c$, $a\in K$. Then, since $\bar{u}_1\cdots u_ra\notin\bar{R}_{r-1}$, we have $\bar{R}_{r-1}+\bar{u}_1\cdots u_raK=\bar{R}_r$ (Lemma 1). Therefore $\bar{u}_1\cdots u_rab\equiv\bar{u}_1\cdots u_r$ mod $\bar{R}_{r-1}$ for some $b\in K$. Hence we may assume that $a=1$. Then $\bar{z}-\bar{y}u_{r+1}\cdots u_ec\in\bar{R}_{e-1}$, and $\bar{y}u_{r+1}\cdots u_ec\in\bar{I}$. By induction, we have $\bar{R}=\bar{I}+\bar{R}_{r-1}$, and so $\bar{R}=\bar{I}\oplus\bar{R}_{r-1}$. Then $\bar{R}/\bar{I}_K\rightrightarrows\bar{R}_{r-1\,K}$, which is of finite length $r$, by Cor. to Lemma 1.

Now we are ready to prove the following

**Theorem 3.** *Let $I$ be a left $K$-, right $R$-submodule of $R$ such that ${}_KR/I$ is finitely generated, projective, and of constant rank $n$. Then $R=R_{n-1}\oplus I$.*

*Proof.* By assumption, ${}_KR/(pR+I)\rightrightarrows{}_K(K/p)^n$ ($n$ copies) for any maximal ideal $p$ of $K$. Then, by Cor. to Lemma 1 and Cor. to Lemma 2, $R = R_{n-1} + I + pR$ for any maximal ideal $p$ of $K$. Since ${}_KR/(R_{n-1} + I)$ is finitely generated, we have $R = R_{n-1} + I$, and the canonical homomorphism $R_{n-1} \to R/I$ is an epimorphism. Since the both sides are of rank $n$, this epimorphism is an isomorphism (cf. [2]). Thus $R = R_{n-1} \oplus I$.

In order to treat the case ${}_KR/I$ is not of constant rank, we need the following well known

**Lemma 4.** *Let ${}_KP(\neq 0)$ be a finitely generated, projective K-module. Then there exist uniquely pairwise orthogonal non-zero idempotents $e_i$ ($i = 1, \cdots, r$) with $\sum e_i = 1$ and non-negative integers $n_1 > \cdots > n_r$ such that $e_iP$ is a finitely generated, projective $e_iK$-module of constant rank $n_i$.*

**Theorem 5.** *Let $K$ be a commutative ring, and $R = K[R_1]$ a (*)-positively filtered ring over $K$. Let $I$ be a proper left $K$-, right $R$-submodule of $R$ such that ${}_KR/I$ is finitely generated and projective. Then there are pairwise orthogonal non-zero idempotents $e_i$ of $K$ with $\sum e_i = 1$ and non-negative integers $n_1 > \cdots > n_r$ such that $R = I(\oplus \oplus_{i=1}^{r} e_iR_{n_i-1})$.*

*Proof.* Put $R/I = P$ in the preceding lemma. Then $R = R_{n_i-1} + I + pR$ for all maximal ideal $p$ of $K$ with $e_i \notin p$ (cf. the proof of Th. 3), and so $e_iR = e_iR_{n_i-1} + e_iI + e_ipR$. Therefore $e_iR = e_iR_{n_i-1} + e_iI$. We note that $p(1 - e_i) = K(1 - e_i)$, and then, as in the proof of Th. 3, we can prove that $e_iR = e_iR_{n_i-1} \oplus e_iI$. Hence $R = I \oplus (\oplus_{i=1}^{r} e_iR_{n_i-1})$.

## REFERENCES

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