NOTE ON THE MAXIMAL QUOTIENT RING OF A GALOIS SUBRING

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Let A be a ring with identity, G a finite group of automorphisms of A, and A^G the subring of A consisting of all elements of A left fixed by all elements of G. When A has a classical left quotient ring $Q_{cl}(A)$ and the extension of G to $Q_{cl}(A)$ is identified with G, A^G has $Q_{cl}(A)^G$ as its classical left quotient ring under suitable hypotheses (cf. [2], [3], [4], [8] and [9]). In stead of classical left quotient rings, we shall consider here maximal left quotient rings in the sense of Utumi-Lambek. As was shown by Utumi [10], a ring A always has its maximal left quotient ring $Q_{max}(A)$ determined uniquely up to isomorphism over A and every ring automorphism of A can be extended uniquely to that of $Q_{max}(A)$. We shall now identify the unique extension of G to $Q_{max}(A)$ with G. As was noted in [2], in general it is not true that $Q_{max}(A)^G = Q_{max}(A^G)$. The purpose of this note is to prove the last equality under the hypothesis that A is a G-Galois extension of A^G, namely, there exist $x_1, \cdots\cdots, x_n; y_1, \cdots\cdots, y_n \in A$ such that $\sum_i x_i \sigma(y_i) = \delta_{1,\sigma}$ for all $\sigma \in G$ (cf. [7]).

Throughout the present note, it is always assumed that every ring has an identity, every subring of a ring contains the same identity and that every module as well as every ring homomorphism is unital. Furthermore, A will represent a ring, and G a finite group of automorphisms of A, which will be identified with the unique extension of G to the maximal left quotient ring $Q_{max}(A)$ of A.

1. Lemmas. We shall recall here several terminologies which will be used in the sequel. Let ${}_RM \subset {}_RN$ be left R-modules. If M has nonzero intersection with every nonzero R-submodule of N, then M is an *essential submodule* of N (or N is an *essential extension* of M). If, for each x, $0 \neq y \in N$ there exists $a \in R$ such that $ax \in M$ and $ay \neq 0$, then N is a *rational extension* of M (or M is a *dense submodule* of N). If a ring extension S of R is a rational extension of R as a left R-module, then S is called a *left quotient ring* of R. For the notion and information about maximal left quotient rings see [10] or [6, § 4. 3].

The next lemma is well known. However, for the sake of completeness, we shall give here the proof.

Lemma 1. *Let $_RM$ and $_RN$ be left R-modules, and let $_R\hat{N}$ be the injective hull of $_RN$. Then the following statements are equivalent*:

1) $\mathrm{Hom}_R(M, \hat{N}) = 0$.

2) *For each* $x \in M$, $0 \neq y \in N$, *there exists* $a \in R$ *such that* $ax = 0$ *and* $ay \neq 0$.

Proof. 1) $\Longrightarrow$ 2): Let $x \in M$, $0 \neq y \in N$. We may assume $x \neq 0$. Let I be the left annihilator of x in R. Then the right multiplication map of x from R to Rx induces an R-isomorphism of R/I to Rx. If $Iy = 0$, then the right multiplication map of y induces a nonzero R-homomorphism of R/I to N, and so, $_R\hat{N}$ being injective, $\mathrm{Hom}_R(M, \hat{N}) \neq 0$, contradicting 1).

2) $\Longrightarrow$ 1): If there exists an R-homomorphism f of M to $\hat{N}$ such that $f(x) \neq 0$ for some $x \in M$, then, N being essential in $\hat{N}$, there exists $a \in R$ with $0 \neq af(x) \in N$, and so we have $a' \in R$ such that $a'(ax) = 0$ and $a'(af(x)) \neq 0$. This is a contradiction.

Lemma 2. *Let S/R be a ring extension, $\hat{S}$ the injective hull of $_SS$, and $\hat{R}$ that of $_RR$. Let α: $\mathrm{Hom}_R(S, \hat{R}) \to \hat{S}$ be an S-isomorphism. Then, for an arbitrary left S-module $_SX$, the map*

$$\alpha'(X): \mathrm{Hom}_R(X, \hat{R}) \to \mathrm{Hom}_S(X, \hat{S})$$

defined by

$$[\alpha'(X)(g)](x) = \alpha(g\cdot\rho_x) \quad (g \in \mathrm{Hom}_R(X, \hat{R}),\ x \in X)$$

is bijective, where $\rho_x : S \to X$ *is defined by* $(\rho_x)(s) = sx\ (s \in S)$.

Proof. To be easily seen, $\alpha'(X)$ is the composite of the following isomorphisms:

$$\mathrm{Hom}_R(X, \hat{R}) \cong \mathrm{Hom}_R(S \otimes {}_SX, \hat{R}) \cong \mathrm{Hom}_S(X, \mathrm{Hom}_R(S, \hat{R})) \cong \mathrm{Hom}_S(X, \hat{S}).$$

Following F. Kasch [5], a ring extension S/R is called a *Frobenius extension* if $_RS$ is finitely generated projective and $_SS_R \cong {}_S\mathrm{Hom}(_RS, {}_RR)_R$.

Let $\Delta = \Delta(A; G)$ be the trivial crossed product of A with G, that is, $\Delta = \bigoplus_{\sigma\in G} Au_\sigma$; $\{u_\sigma\}_{\sigma\in G}$ is a free generator for Δ over A, $au_\sigma \cdot bu_\tau = a\sigma(b)u_{\sigma\tau}$ $(a, b \in A;\ \sigma, \tau \in G)$. Then the map

$$h : \Delta \to A, \quad h(\textstyle\sum_{\sigma\in G} a_\sigma u_\sigma) = a_1 \quad (a_\sigma \in A)$$

induces a left Δ-, right A-bimodule isomorphism

$$\Phi : \Delta \to \mathrm{Hom}(_A\Delta, {}_AA), \quad (\Phi(d))(x) = h(xd) \quad (d, x \in \Delta)$$

whose inverse is given by

$$\Phi^{-1}(f) = \sum_{\sigma \in G} \sigma(f(u_{\sigma^{-1}}))u_\sigma \quad (f \in \mathrm{Hom}({}_A\Delta, {}_AA)).$$

Therefore, Δ/A is a Frobenius extension.

Lemma 3. *Let $\hat{A}$ and $\hat{\Delta}$ be the injective hulls of ${}_AA$ and ${}_\Delta\Delta$, respectively. Then there exists a left Δ-module isomorphism* $\mathrm{Hom}_A(\Delta, \hat{A}) \cong \hat{\Delta}$.

Proof. At first, we shall show that $\Delta \otimes_A \hat{A}$ is an essential extension of $\Delta (\cong \Delta \otimes_A A)$ as left Δ-modules. To see this, let $x = \sum_{\sigma \in G} u_\sigma \otimes x_\sigma (\{x_\sigma\}_{\sigma \in G} \subset \hat{A})$ be an arbitrary nonzero element of $\Delta \otimes_A \hat{A}$. We have then $x_\sigma \neq 0$ for some σ. However, ${}_A\hat{A}$ is an essential extension of ${}_AA$, and so there exists some $a_\sigma \in A$ such that $0 \neq \sigma^{-1}(a_\sigma)x_\sigma \in A$. Since

$$a_\sigma x = \sum_{\tau \in G} u_\tau \cdot \tau^{-1}(a_\sigma) \otimes x_\tau = u_\sigma \otimes \sigma^{-1}(a_\sigma)\, x_\sigma + y$$

with $y = \sum_{\tau \neq \sigma} u_\tau \otimes \tau^{-1}(a_\sigma)\, x_\tau$, if y is nonzero then we can choose similarly some $a_\tau \in A$ ($\tau \neq \sigma$) with $0 \neq \tau^{-1}(a_\tau)\, \tau^{-1}(a_\sigma)x_\tau \in A$. Repeating the same argument, we have eventually $a \in A$ such that $\sigma^{-1}(a)x_\sigma \in A$ for all $\sigma \in G$ and $\sigma^{-1}(a)x_\sigma \neq 0$ for some $\sigma \in G$. Since $\{u_\sigma\}_{\sigma \in G}$ is a free generator for Δ over A, we have then $0 \neq ax \in \Delta$, and so $\Delta \otimes_A \hat{A}$ is an essential extension of Δ as left A- and hence as left Δ-modules. Next, Δ/A being a Frobenius extension, we have $\Delta \otimes_A \hat{A} \cong \mathrm{Hom}_A(\Delta, \hat{A})$ as left Δ-modules by [5, (II), p. 15]. The latter is clearly an injective Δ-module. Hence, noting the mention cited above, the uniqueness of the injective hull up to isomorhism yields the conclusion.

Now, we shall denote by t the trace map

$$t : A \longrightarrow A^G, \ t(x) = \sum_{\sigma \in G} \sigma(x) \qquad (x \in A),$$

and say that t is *left nondegenerate* if $t(Aa) \neq 0$ for all nonzero $a \in A$, or equivalently, if $t(I) \neq 0$ for all nonzero left ideals I of A. The *right nondegeneracy* of t is defined symmetrically.

Lemma 4. *Assume that the trace map t is left nondegenerate.*

1) *If I is a dense left ideal of A, then $t(I)$ and $I \cap A^G$ are both dense left ideals of A^G.*

2) *$Q_{\max}(A)^G$ is a left quotient ring of A^G.*

Furthermore, assume that for every dense left ideal D of A^G the left ideal AD of A is dense. Then

3) *$Q_{\max}(A)^G$ is the maximal left quotient ring of A^G.*

Proof. 1): Let I be a dense left ideal of A. Let x, $0 \neq y$ be elements of A^G. Then, there exists $a \in A$ such that $ax \in I$ and $ay \neq 0$.

But, t being left nondegenerate, there exists $a'\in A$ such that $0\neq t(a'ay) = t(a'a)y\in A^G$. It follows therefore that $t(I)$ is a dense left ideal of A^G. Noting that the intersection of a finite number of dense left ideals is a dense left ideal and $\sigma(I)$ is dense in A for each $\sigma\in G$, we see that $I_0=\cap_{\sigma\in G}\sigma(I)$ is dense in A, and so $t(I_0)$ is dense by the above. Therefore $I\cap A^G$ is dense by $t(I_0)\subset I_0\subset I$.

2): Let x, $0\neq y$ be elements of $Q_{\max}(A)^G$. Then there exists $a\in A$ such that ax, $ay\in A$ with $ay\neq 0$. Then, in the same way as in 1), we can find an element $a'\in A$ such that $t(a'a)x\in A^G$ and $t(a'a)y\neq 0$, which yields 2).

3): In this proof, we shall use freely [6, Corollary to Prop. 8, p. 99] and write left module homomorphisms on the right side. Let $f: D\longrightarrow A^G$ be an arbitrary left A^G-module homomorphism of a dense left ideal D of A^G to A^G. Then the map

$$\bar{f}: AD\longrightarrow A$$

defined by

$$(\textstyle\sum_k a_kd_k)\bar{f}=\sum_k a_k\cdot(d_k)f \qquad (a_k\in A,\ d_k\in D)$$

is well-defined. Indeed, let assume $\sum_k a_kd_k=0$ $(a_k\in A,\ d_k\in D)$. Since $t(a\sum_k a_k\cdot(d_k)f)=\sum_k t(aa_k)(d_k)f=(\sum_k t(aa_k)d_k(f=(t(a\sum_k a_kd_k))f$ for all $a\in A$, the left nondegeneracy of t yields $\sum_k a_k\cdot(d_k)f=0$ as desired. Now AD is dense in A by the assumption, and so there exists $q\in Q_{\max}(A)$ such that $(x)\bar{f}=xq$ for all $x\in AD$. Especially, we have $(d)f=dq$ for all $d\in D$. It remains to prove $q\in Q_{\max}(A)^G$. Since $d(q-\sigma(q))=(d)f-\sigma((d)f)=0$ $(d\in D,\ \sigma\in G)$, this follows from the density of AD.

Lemma 5. *If A is a G-Galois extension of A^G, then AD is a dense left ideal of A whenever D is a dense left ideal of A^G.*

Proof. Let us set $B=A^G$, and $C=\mathrm{End}(A_B)$. There exist $x_1,\cdots,x_n$; $y_1,\cdots y_n\in A$ such that $\sum_i x_i\sigma(y_i)=\delta_{\sigma,1}$ for all $\sigma\in G$. Then the map

$$j: \Delta=\Delta(A;\ G)\longrightarrow C$$

defined by

$$j(\textstyle\sum_\sigma a_\sigma u_\sigma)(x)=\sum_\sigma a_\sigma\sigma(x) \qquad (x\in A)$$

is a ring isomorphism whose inverse is given by

$$j^{-1}(c)=\textstyle\sum_\sigma(\sum_i c(x_i)\sigma(y_i))u_\sigma \qquad (c\in C).$$

Moreover, if $i_1: A\rightarrow\Delta$ is the natural injection and $i_2: A\rightarrow C$ is the left multiplication map then $ji_1=i_2$. Therefore, we may and shall identify

C with J via j. Since $x = \sum_i t(xx_i)y_i = \sum_i x_i t(y_i x)$ for all $x \in A$, t is left and right nondegenerate. Let D be a dense left ideal of B. We shall show that $\mathrm{Hom}_A(A/AD, \hat{A}) = 0$, which will complete the proof by Lemma 1. Using Lemmas 2 and 3, it is sufficient to show $\mathrm{Hom}_C(A/AD, \hat{C}) = 0$. Let $x \in A$ and $0 \neq c \in C$. We have then $c(x') \neq 0$ for some $x' \in A$. Since t is left nondegenerate, there exists $a \in A$ such that $t(ac(x')) \neq 0$. Further, D being dense in B, there exists $b \in B$ such that $bt(ac(x')) \neq 0$ and $bt(ax) \in D \subset AD$. Then $c' = i_2(b) \cdot t \cdot i_2(a)$ is an element of C such that $c' \cdot x \in AD$ and $c' \cdot c \neq 0$, and so $\mathrm{Hom}_C(A/AD, \hat{C}) = 0$ by Lemma 1.

2. **Main theorem.** We are now ready for proving our main theorem.

Theorem. *Let A be a G-Galois extension of A^G. Then $Q_{\max}(A)^G = Q_{\max}(A^G)$, and moreover $Q_{\max}(A) = A$ if and only if $Q_{\max}(A^G) = A^G$.*

Proof. Put $Q = Q_{\max}(A)$. There exist $x_1, \cdots, x_n$; $y_1, \cdots, y_n \in A$ such that $\sum_i x_i \sigma(y_i) = \delta_{\sigma,1}$ for all $\sigma \in G$. In the proof of Lemma 5 we have seen that the trace map t is nondegenerate. Therefore by Lemmas 4 and 5 we have $Q^G = Q_{\max}(A^G)$. It is easy to see that $x = \sum_i x_i t(y_i x) = \sum_i t(xx_i)y_i$ for all $x \in Q$, where t is the trace map of Q to Q^G. It follows then that $Q = A \cdot Q^G = Q^G \cdot A = A \cdot Q_{\max}(A^G) = Q_{\max}(A^G) \cdot A$, and so $Q = A$ if and only if $Q_{\max}(A^G) = A^G$.

Obviously the maximal left quotient ring of a ring has no proper left quotient rings (see [6, Corollary to Prop. 2, p. 95]). Hence the following is an easy combination of our theorem and Lemma 4.

Proposition. *If $Q = Q_{\max}(A)$ is a G-Galois extension of Q^G such that the trace map $t : A \to A^G$ is left nondegenerate, then Q^G is the maximal left quotient ring of A^G.*

Remark 1. If A is a semiprime ring without $|G|$-torsion, then the trace map t is left and right nondegenerate. If in addition the left singular ideal of A is zero, then $Q_{\max}(A)^G = Q_{\max}(A^G)$. In fact, $I = \{a \in A \mid t(Aa) = 0\}$ is clearly a G-invariant left ideal of A such that $t(I) = 0$. Thus I is nilpotent by [1, Proposition 2. 3]. However, A is semiprime, and so $I = 0$. Hence, t is left nondegenerate. Similarly, t is right nondegenerate. Since the left singular ideal of A is zero, $Q = Q_{\max}(A)$ is a regular, left self-injective ring. Hence, Q is injective as a left A-module. Moreover, the left quotient ring Q of A has no $|G|$-torsion. Thus we can apply the above argument to see that the trace map $t : Q \to Q^G$ is left and right nondegenerate. Now, let D, f and $\bar{f}$ be same as in the proof of Lemma 4 3). The injectivity of $_AQ$ implies the existence of $q \in Q$ such that $(x)\bar{f} = xq$ for all $x \in AD$, and so the proof enables us to see that $d(q - \sigma(q)) = 0$ for all $d \in D$, $\sigma \in G$. However, $Q^G \cdot D$ is a dense left ideal of Q^G by Lemma 4 2). Hence, the right nondegeneracy of $t : Q \to Q^G$ implies that the right annihilator of $Q^G \cdot D$ in Q is zero, which yields $q \in Q^G$. It follows therefore $Q^G = Q_{max}(A^G)$.

Remark 2. If A is commutative and the trace map t is nondegenerate then $Q_{max}(A)^G = Q_{max}(A^G)$. In fact, the nondegeneracy of t implies that if J is an ideal of A^G whose annihilator in A^G is zero then the annihilator of J in A is zero. However, in a commutative ring, a dense ideal is nothing but an ideal whose annihilator is zero. Now, the assertion is a consequence of Lemma 4.

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