

ON ANNIHILATOR IDEALS

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In [5], it was shown that a ring without non-zero nilpotent elements is von Neumann regular if and only if every principal left ideal is the left annihilator of an element of A . This together with our result mentioned in [1] will be extended in the first theorem of this note. In [3] and [4], rings whose maximal left ideals are left annihilators are studied. A characterization of an arbitrary regular ring is here given in terms of finitely generated maximal and projective left annihilators (Theorem 2). Several characteristic properties of a semi-simple Artinian ring are obtained in terms of annihilators (Theorem 3).

Throughout, A denotes an associative ring with identity and Z is the left singular ideal of A . We recall that (1) A is called *reduced* if A contains no non-zero nilpotent elements; (2) A left ideal I is *closed* in A if I has no proper essential extensions in A ; (3) A left (right) A -module M is *p -injective* if, for any principal left (right) ideal P of A and any left (right) A -homomorphism $g: P \rightarrow M$, there exists $c \in M$ such that $g(b) = bc$ ($g(b) = cb$) for all $b \in P$. Then, A is strongly regular if and only if A is regular reduced, and it is easy to see that any p -injective left (right) ideal of A is idempotent.

The next lemma is proved in [6] (see [4, Lemma 3] for the proof).

Lemma 1. *The following conditions are equivalent :*

- 1) $Z = 0$ and every closed left ideal of A is two-sided ;
- 2) A is a reduced ring and $I + l(I)$ ($= I \oplus l(I)$) is essential in A for every left ideal I of A ;
- 3) A is a reduced ring and every closed left ideal of A is a left annihilator.

The characterization of a strongly regular ring given in [5] and [6] (see [1]) is improved in the next theorem.

Theorem 1. *The following conditions are equivalent :*

- 1) A is strongly regular ;
- 2) A is a reduced ring and every principal left ideal of A is a left annihilator ;
- 3) A is a reduced ring and every maximal left ideal of A is *p -injective* ;

4) A is a reduced ring and every maximal left ideal of A is either p -injective or a left annihilator.

5) $Z = 0$ and every finitely generated left ideal of A is the left annihilator of a left ideal;

6) $Z = 0$ and every principal left ideal of A is closed in A and two-sided.

Proof. Obviously, 3) implies 4) and 1) implies 5) and 6). 1) implies 3) by [5, Lemma 2] (cf. also [2]), and each of 5) and 6) does 2) by Lemma 1.

2) \implies 1) Let $b \in A$. In the reduced ring A , as is well-known, $ab^2 = 0$ if and only if $ab = 0$, namely, $l(b) = l(b^2)$. Since Ab is a left annihilator, $Ab = l(r(Ab)) = l(r(b)) = l(l(b)) = l(l(b^2)) = Ab^2$, which proves 1).

4) \implies 1) Let $b \in A$. We claim first $Ab + l(b) = A$. If not, there exists a maximal left ideal L containing $Ab + l(b)$. In case L is p -injective, considering the canonical injection $i: Ab \rightarrow L$, we can find $c \in L$ with $b = bc$. Then, $1 - c \in r(b) = l(b) \subseteq L$, whence it follows a contradiction $1 \in L$. On the other hand, in case $L = l(t)$ with some $0 \neq t \in A$, we have $t \in r(Ab + l(b)) \subseteq r(b) \subseteq L$. Then, $t^2 = 0$, a contradiction. Now, let $ab + d = 1$, $a \in A$, $d \in l(b)$. Then $ab^2 = b$, which proves 1).

A left ideal I of A is called a *maximal left annihilator* if $I = l(S)$ for some non-empty subset $S \neq 0$ of A and for any left annihilator J with $I \subseteq J$, either $J = I$ or $J = A$. In that case, $I = l(s)$ for any $0 \neq s \in S$.

Theorem 2. *The following conditions are equivalent:*

1) A is regular;

2) A is a semi-prime ring such that every finitely generated left ideal is the projective left annihilator of an element of A .

3) A is a semi-prime ring such that every finitely generated left ideal is either a maximal left annihilator or the projective left annihilator of an element of A .

Proof. If A is regular, every finitely generated left ideal of A is obviously the projective left annihilator of an element. Hence, 1) \implies 2) \implies 3). Assume 3). At any rate, every finitely generated left ideal of A is the left annihilator of an element. And so, it suffices to prove that any principal left ideal is projective. First, we claim $Z = 0$. Suppose there exists $0 \neq z \in Z$. Since Az can not be projective, Az is a maximal left annihilator. Accordingly, for any $w \notin Az$ we have $(Az \subset) Az + Aw = A$, which means that Az is a maximal left ideal and $Az = Z$. Moreover,

recalling that Z contains no non-zero idempotents, we have $Az = l(t)$ with some $0 \neq t \in Z$. But then $(At)^2 \subseteq ZAt \subseteq Zt = Azt = 0$, which contradicts the semi-primeness of A . Thus $Z = 0$. Now, assume that a principal left ideal I is a maximal left annihilator. Since $Z = 0$, there exists $0 \neq b \in A$ with $I \cap Ab = 0$. Recalling that $I \oplus Ab (\supseteq I)$ is projective, we see that I is projective, completing the proof.

The next result is motivated by [3, Corollary 2].

Theorem 3. *The following conditions are equivalent :*

- 1) *A is semi-simple, Artinian.*
- 2) *Every left ideal of A is the left annihilator of an idempotent right ideal ;*
- 3) *The right annihilator of any maximal left ideal is a non-zero p -injective right ideal ;*
- 4) *The right annihilator of any maximal left ideal contains a non-zero idempotent right ideal.*

Proof. Obviously $1) \implies 2) \implies 4)$, and $1) \implies 3) \implies 4)$ (cf. [5, Lemma 2] or [2]). In order to see that 4) implies 1), it suffices to prove that every maximal left ideal I of A is a direct summand. Let J be a nonzero idempotent right ideal contained in $r(I)$. Let $s, t \in J$ with $st \neq 0$. Then $I = l(J) = l(s) = l(t) = l(st)$. Suppose I is essential in A . Then there exists $b \in A$ such that $0 \neq bs \in I$, which implies $b \in l(st) = I$. This contradicts $bs \neq 0$.

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