ON FINITE UNIONS OF SUBRINGS

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In this paper we shall give a sufficient condition for rings not to be covered by a finite number of proper subrings. This enables us to obtain a necessary and sufficient condition for Artinian simple rings to have finite coverings by proper subrings, wherefrom the results of Venkatachaliengar-Soundararajan [7] and Kishimoto-Motose [1] are deduced. Moreover we shall give some related results.

Throughout this paper, A will represent a ring with 1, and A-modules will be unital. Unless otherwise specified, subrings of A will mean those which contain 1. If no left (resp. right) A-module can be covered by a finite number of proper A-submodules, or equivalently, if no finitely generated left (resp. right) A-module can be covered by a finite number of proper A-submodules, A is called a *left* (resp. *right*) *um-ring* (cf. [4]). For any ring R (with or without 1), $J(R)$ denotes the Jacobson radical of R and R^+ the additive group of R.

We begin with the following restatement of [4, Proposition 1. 7] for non-commutative rings.

Lemma 1. *If there exists an infinite subset S of A such that $x-y$ is a unit of A for all $x \neq y$ in S, then A is a left um-ring.*

Proof. Suppose there exists a left A-module M such that $M = \bigcup_{j=1}^{m} M_j$ with proper A-submodules M_j. We may assume that $M_i \not\subset \bigcup_{j\neq i} M_j$ for each i. Choose $u_1 \in M_1 \setminus \bigcup_{j\neq 1} M_j$ and $u_2 \in M_2 \setminus \bigcup_{j\neq 2} M_j$, and consider the infinite set $\{u_1 + su_2 \mid s \in S\}$, which is contained in $M \setminus M_2$. Then there exist some $x \neq y$ in S such that both $u_1 + xu_2$ and $u_1 + yu_2$ are contained in the same M_k $(k \neq 2)$. But, this yields a contradiction $u_2 \in M_k$.

The latter half of the next is a partial extension of [4, Theorem 2. 2].

Theorem 1. (1) *A is a left um-ring if and only if so is $\bar{A} = A/J(A)$.*

(2) *If A is semilocal ($A/J(A)$ is Artinian), then the following are equivalent:*

1) *A is a left um-ring.*

2) *A is a right um-ring.*

3) *A/I is infinite for every maximal ideal I of A.*

4) *There exists an infinite subset S of A such that $x-y$ is a unit for all $x \neq y$ in S.*

Proof. (1) It suffices to prove the if part. Suppose a finitely generated left A-module M is covered by a finite number of A-submodules: $M=\cup_{i=1}^{n} M_i$. Since $_{\bar{A}}(M/J(A)M)$ can not be covered by any finite number of proper $\bar{A}$-submodules, we have $M=M_k+J(A)M$ for some k. By Nakayama's lemma, it follows then $M=M_k$. Hence, A is a left *um*-ring.

(2) By Lemma 1, it suffices to show that 1) implies 3). Suppose there exists an ideal I such that $\bar{A}=A/I$ is a finite simple ring, and consider the left A-module $M=\bar{A}\oplus\bar{A}$. Evidently, every Au $(u\in M)$ is a proper A-submodule of M and $M=\cup_{u\in M} Au$, which is a contradiction.

Lemma 2. *Let B be a subring of A.*

(1) *If B contains an infinite subset S such that $x-y$ is a unit of B for all $x\neq y$ in S, then A can not be covered by any finite number of proper subrings containing B.*

(2) *If $(A^+:B^+)<\infty$, then any element $b\in B$ with inverse in A is a unit of B.*

Proof. (1) Since B is a left *um*-ring by Lemma 1, our assertion is almost evident.

(2) There exist some positive integers $s>t$ such that $(b^{-1})^s-(b^{-1})^t\in B$. Hence, $b^{-1}=b^{s-t-1}+b^{s-1}b_0$ with some $b_0\in B$.

Now we get the following key result.

Theorem 2. *If A contains a subring B such that $\bar{B}=B/J(B)$ is an infinite Artinian simple ring, then A can not be covered by any finite number of proper subrings.*

Proof. Let $\bar{B}=\sum_{i,j=1}^{n}\bar{D}\bar{e}_{ij}$, where $\bar{D}=D/J(B)$ is an infinite division subring of $\bar{B}$ and $\bar{e}_{ij}$'s are matrix units. Since $J(D)=J(B)$, from the beginning we may assume that $\bar{B}$ is a division ring. Suppose $A=\cup_{j=1}^{m} A_j$ with proper subrings A_j, where we may assume that $A_i\not\subset\cup_{j\neq i} A_j$ for each i. Let $A_0=\cap_{j=1}^{m} A_j$, and $B_0=A_0\cap B$. Then, by [6, Lemma 1] or [3, (4. 4)] we have $\infty>(A^+:A_0^+)\geqq(B^+:B_0^+)\geqq(B^+:(B_0+J(B))^+)$. Combining this with the infinity of $\bar{B}$, it follows that $B_0+J(B)/J(B)$ is infinite. Now, let S be a complete representative system of $B_0+J(B)/J(B)$ contained in B_0. Obviously, S is infinite and $x-y$ is a unit in B for all $x\neq y$ in S. Since $(B^+:B_0^+)<\infty$, $x-y$ is then a unit in B_0 by Lemma 2 (2). But this contradicts Lemma 2 (1).

Remark. It should be noted that in the preceding theorem the subrings A_j need not contain 1. This is obvious since in Lemma 2 (2), the assumption that the subring B contains 1 is superfluous. The same remark applies to the next corollary.

Corollary 1. *Let A be an Artinian simple ring. Then, A can not be covered by any finite number of proper subrings if and only if A is either a finite field or an infinite ring.*

Proof. In case A is infinite, by Theorem 2 A can not be covered by any finite number of proper subrings. Next, if A is a finite field then A can not be so, for the multiplicative group of A is cyclic. Finally, if A is a finite simple ring which is not a field, then each element of A generates a (commutative) proper subring.

Evidently, Corollary 1 contains the result in [7]. Moreover, Corollary 1 enables us to prove a slight generalization of the result of K. Kishimoto and K. Motose [1].

Corollary 2. *If A is Artinian semisimple then A can not be covered by any finite number of prime proper subrings.*

Proof. If A is not simple, it is easy to see that A can not be covered by prime proper subrings. In what follows, we assume that A is simple : $A=\sum_{i,j=1}^{n} De_{ij}$ where D is a division ring and e_{ij}'s are matrix units. By Corollary 1, it suffices to consider the case that A is finite of characteristic p and $n>1$. Let c be a generating element of the multiplicative group of D, and $a=\sum_{i=2}^{n} e_{i,i-1}$. To be easily seen, $(c+a)^{p^k}=c$ for some k. Accordingly, if $c+a$ is contained in a prime (and hence simple) subring B then c and a are in B. Hence, B is of capacity n and contains D, whence it follows $B=A$.

Remark. (1) It is an easy consequence of Corollary 1 that if a semilocal ring A can not be covered by any finite number of proper subrings then, for every maximal ideal I of A, A/I is either a finite field or an infinite ring. However, the converse need not be true.

(2) It will be almost evident that an Artinian simple ring can not be covered by any finite number of prime proper subrings, not necessarily containing 1.

If the set of subrings of A is totally ordered by inclusion then A can not be covered by any finite number of proper subrings. We shall prove finally the following

Theorem 3. *If A is an infinite simple ring then the following are equivalent:*

1) *The set of commutative subrings of A is totally ordered by inclusion.*

2) *Every proper subring of A is finite.*

3) *A is the direct limit of an infinite tower of fields $K_0 \subset K_1 \subset K_2 \subset \cdots$ where K_i is the field of order p^{q^i}, p and q are primes.*

Proof. Obviously 3) implies 1) and 2).

1)$\Longrightarrow$3) It is easy to see that A is a (commutative) field of characteristic $p>0$ and is algebraic over its prime field. Hence, by [5, Corollary to Proposition 6] it follows 3).

2)$\Longrightarrow$3) Suppose there exists an infinite proper subring I of A which does not contain 1. Then, $\boldsymbol{Z}\cdot 1+I$ must be A, and so I is a proper ideal of A. This contradiction shows that every proper subring of A (with or without 1) is finite. Hence, 3) is a consequence of [2, Theorem 1].

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