ALGEBRAS WITH ONLY FINITELY MANY SUBALGEBRAS

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By the proof of [1], one will readily see that if a ring R contains finitely many commutative subrings then R is finite. In this note, corresponding to this result, we shall prove

Theorem. *Let R be an algebra over a field K. Then the following are equivalent:*

1) *R contains finitely many subalgebras.*

2) *R contains finitely many commutative subalgebras.*

3) a) *R is finite, or* b) *R is finite dimensional over K and every subalgebra of R has a single generating element.*

Proof. It remains to prove 2) $\Longrightarrow$ 3) $\Longrightarrow$ 1).

2) $\Longrightarrow$ 3) First, we claim that if K is infinite then every subalgebra S has a single generating element. Suppose $S \neq [s]$ (the subalgebra generated by s) for any $s \in S$. Since $S = \cup_{s \in S} [s]$ and the set of subalgebras $[s]$ is finite, by [2, (4.4)] or [3, Lemma 1] there exists a commutative subalgebra S_0 such that $1 < (S^+ : S_0^+) < \infty$, where S^+ denotes the additive group of the ring S. But this contradicts the infinity of K. Now, we shall prove that R is finite dimensional over K. In any rate, by the above argument, we can find a commutative subalgebra R_0 of R with $(R^+ : R_0^+) < \infty$ and a finite chain of subalgebras $R_0 \supset R_1 \supset \cdots \supset R_r = 0$ such that R_i is a maximal ideal of R_{i-1} $(i = 1, \cdots, r)$. Hence, we may restrict our attention to the case that R is a commutative simple algebra. In case $R^2 = 0$, our assertion is trivial. If $R^2 \neq 0$ then R is a field. Since R must be a finitely generated algebraic extension field over K, R is obviously finite dimensional over K.

3) $\Longrightarrow$ 1) It suffices to prove that b) implies 1). We shall proceed by the induction concerning $[R : K]$. If the commutative algebra R contains no non-zero nilpotent ideals, then R is the direct sum of fields R_i. Since every subalgebra of R has a single generating element and especially each R_i contains only finitely many subalgebras, one will easily see that R itself contains finitely many subalgebras. In what follows, we assume that R contains a non-zero ideal $M = [m]$ with $M^2 = 0$. By the induction hypothesis, the set Σ_0 of subalgeberas of R containing M is finite: $\Sigma_0 = \{S_1, \cdots, S_t\}$. Let Σ be the set of subalgebras S of R with $S \cap M = 0$. If S has a non-zero

ideal $N=[n]$ with $N^2=0$, then it is easy to see that $nm=0$, whence it follows $(N+M)^2=0$ and $[N+M:K]=1$, a contradiction. We have therefore seen that each S in Σ is semi-simple, and hence possesses the identity e_S. We claim here that if $S+M=S'+M=S_j\in\Sigma_0$ $(S, S'\in\Sigma)$ then $e_S=e_{S'}$, which may be denoted as e_j. In fact, noting that $e_{S'}=e_S+km$ with some $k\in K$, $e_{S'}^3=e_{S'}^2=e_{S'}$ implies $e_S+3kme_S=e_S+2kme_S=e_S+km$, and so $0=kme_S=km$. Now, we set $\Sigma_1=\{S\in\Sigma \mid \mathrm{Ann}(M)\supseteq S\}$ and $\Sigma_2=\{S\in\Sigma \mid \mathrm{Ann}(M)\not\supseteq S\}$. If $S, S'\in\Sigma_1$ and $S+M=S'+M=S_j\in\Sigma_0$ then, by the above we have $S=e_jS\subseteq e_jS'+e_jM=S'$, namely, $S=S'$. This means that Σ_1 is finite. Finally, we shall prove that Σ_2 is finite, which will complete the proof. Let S be in Σ_2, and $S+M=S_j\in\Sigma_0$. Obviously, $e_jm=m$ and $S/S\cap\mathrm{Ann}(M)\cong M$. Hence, $S=Ke_j+(S\cap\mathrm{Ann}(M))$. Since $\mathrm{Ann}(M)\subset R$ (if Σ_2 is non-empty), by the induction hypothesis $\mathrm{Ann}(M)$ contains finitely many subalgebras. Thus we have seen Σ_2 is finite.

Remark. Let R be the three dimensional algebra over an infinite field K generated by x with $x^4=0$. Then, to be easily seen, the subalgebra generated by x^2 and x^3 can not be generated by a single element and R contains infinitely many subalegbras.

REFERENCES

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Added in proof. If R is an infinite dimensional algebra over a field K such that all commutative proper subalgebras of R are finite dimensional then R is the direct limit of an infinite tower of finite dimensional extension fields over K. This fact can be used to prove that an algebra over a field is finite dimensional if and only if the both chain conditions for subalgebras are satisfied. We wish to thank Prof. T. J. Laffey for informing that the last result has been carried over to alternative algebras in his forthcomming paper [Commutative R-subalgebras of R-infinite R-algebras and the Schmidt problem for R-algebras (to appear in Advances in Mathematics)].