ELEMENTARY PROOFS OF SOME THEOREMS ON SPECIAL FOURIER SERIES

TAKESHI KANO

1. Introduction. A real sequence $\{a_n\}$ is said to be quasi-convex if

$$\sum_{n=1}^{\infty} (n+1)|\Delta^2 a_n| < \infty$$

where $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$, $\Delta a_n = a_n - a_{n+1}$.

It is known that a bounded convex sequence is quasi-convex and a bounded quasi-convex sequence is of bounded variation, viz

$$\sum_{n=1}^{\infty} |\Delta a_n| < \infty.$$

We denote by $(Ta)_n$, the n-th arithmetic mean of $\{a_n\}$,

$$(Ta)_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n).$$

Hardy [4] proved that if

$$\sum_{n=1}^{\infty} a_n \sin nx \tag{1}$$

is the Fourier series of a function $f(x) \in L^p(0, 2\pi)$ $(p \geq 1)$, then

$$\sum_{n=1}^{\infty} (Ta)_n \sin nx \tag{2}$$

is the Fourier series of a function $\varphi(x) \in L^p(0, 2\pi)$.

G. and S. Goes [2] obtained for a special sequence $\{a_n\}$ the following

Theorem A. *Let $\{a_n\}$ be a real null-sequence of bounded variation. Then (2) is the Fourier series of an L^1-function if and only if*

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty. \tag{3}$$

Hence Theorem A, when combined with the above result of Hardy, has the following interesting corollary.

Corollary 1.[1] *If $\{a_n\}$ is a real null-sequence of bounded variation, then (3) is necessary for (1) being a Fourier series.*

[1] Cf. [2; Proof of Theorem 5.3]

If $\{a_n\}$ is a quasi-convex null-sequence, the following important theorem due to Teljakovskiĭ [7] is known.

Theorem B. *Let $\{a_n\}$ be a quasi-convex real null-sequence. Then* (1) *is the Fourier series of an L^1-function if and only if* (3) *holds.*

Thus we have

Corollary 2. *When $\{a_n\}$ is a quasi-convex null-sequence,* (1) *is a Fourier series if and only if* (2) *is a Fourier series.*

For the proof of Theorem A, the following theorem [2; Theorem 6. 1] is required.

Theorem C. *A bounded sequence $\{a_n\}$ is of bounded variation if and only if $\{(Ta)_n\}$ is a quasi-convex sequence.*

Proofs of Theorems A and C due to G. and S. Goes are by a theory of the so-called BK-space, and so they are not elementary. In the next section we shall first give an elementary proof of Theorem C and then prove Theorem A in an elementary way depending upon Theorems B and C.

2. We need the following lemmas to prove Theorem C.

Lemma 1. *Put $S_n=\sum_{k=1}^{n} a_k$, $t_n=(Ta)_n=\frac{S_n}{n}$. Then*

$$\sum_{k=1}^{n} |\Delta t_k| \leq \sum_{k=1}^{n} |\Delta a_k|.$$

Proof. Writing $d_k=\Delta a_k=a_k-a_{k+1}$, we have

$$\begin{aligned}\Delta t_k=t_k-t_{k+1}&=\frac{S_k}{k}-\frac{S_{k-1}}{k+1}\\&=\left(\frac{1}{k}-\frac{1}{k+1}\right)(S_k-ka_{k+1})\\&=\left(\frac{1}{k}-\frac{1}{k+1}\right)(d_1+2d_2+\cdots+kd_k).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{k=1}^{n} |\Delta t_k|&\leq\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)(|d_1|+2|d_2|+\cdots+k|d_k|)\\&\leq|d_1|\left(1-\frac{1}{n+1}\right)+2|d_2|\left(\frac{1}{2}-\frac{1}{n+1}\right)+\cdots+n|d_n|\left(\frac{1}{n}-\frac{1}{n+1}\right)\\&\leq|d_1|+|d_2|+\cdots+|d_n|\\&=\sum_{k=1}^{n}|\Delta a_k|.\end{aligned}$$

Lemma 2. *Assume that $\{a_n\}$ is of bounded variation. Then $\{t_n\}$ is quasi-convex if and only if it is of bounded variation.*

Proof. It will suffice to prove that $\{t_n\}$ is quasi-convex when it is of bounded variation. A simple calculation shows that

$$\Delta t_n = \frac{S_n}{n} - \frac{S_{n+1}}{n+1} = \frac{1}{n(n+1)}(S_n - na_{n+1}),$$

$$\Delta^2 t_n = \Delta(\Delta t_n) = \frac{S_n}{n(n+1)} - \frac{S_{n+1}}{(n+1)(n+2)} + \frac{a_{n+2}}{n+2} - \frac{a_{n+1}}{n+1}$$

$$= \frac{2(S_n - na_{n+1})}{n(n+1)(n+2)} + \frac{1}{n+2}(a_{n+2} - a_{n+1}).$$

Hence

$$(n+2)\Delta^2 t_n = 2\Delta t_n + \Delta a_{n+1}. \tag{4}$$

Thus we have by Lemma 1

$$\sum_{n=1}^{\infty}(n+2)|\Delta^2 t_n| \leq 2\sum_{n=1}^{\infty}|\Delta t_n| + \sum_{n=1}^{\infty}|\Delta a_n| < \infty,$$

which shows that $\{t_n\}$ is quasi-convex.

Proof of Theorem C. If a bounded sequence $\{a_n\}$ is of bounded variation, then $(Ta)_n$ is also of bounded variation by Lemma 1, so it is quasi-convex by Lemma 2. Conversely, if $(Ta)_n$ is quasi-convex ($(Ta)_n$ is bounded since a_n is bounded), then it is necessarily of bounded variation. Therefore, from (4) we obtain

$$\sum_{n=2}^{\infty}|\Delta a_n| \leq \sum_{n=1}^{\infty}(n+2)|\Delta^2 t_n| + 2\sum_{n=1}^{\infty}|\Delta t_n| < \infty,$$

which proves that a_n is of bounded variation.

Proof of Theorem A. If $\{a_n\}$ is a null-sequence of bounded variation, then $t_n = (Ta)_n$ is quasi-convex according to Theorem C. Thus, by Theorem B, (2) is the Fourier series of an L^1-function if and only if

$$\sum_{n=1}^{\infty}\frac{|t_n|}{n} < \infty. \tag{5}$$

On the other hand, since

$$\frac{a_{n+1}}{n+1} = t_{n+1} - t_n + \frac{t_n}{n+1},$$

we have

$$\left| \sum_{n=2}^{\infty} \frac{|a_n|}{n} - \sum_{n=1}^{\infty} \frac{|t_n|}{n+1} \right| \leq \sum_{n=1}^{\infty} |\Delta t_n|,$$

which implies that (5) holds if and only if (3) holds, whenever $\{a_n\}$ is a null-sequence of bounded variation. Thus our proof is complete.

3. It has been stated without proof by Szidon [6] that if $\{a_n \log n\}$ is a real sequence of bounded variation, i. e.

$$\sum_{n=1}^{\infty} |\Delta(a_n \log n)| < \infty, \tag{6}$$

then

$$\sum_{n=1}^{\infty} a_n \cos nx \tag{7}$$

is the Fourier series of an L^1-function. A proof of this fact seems to have been first published by T. Kano [5; Theorem C], and an independent one by G. Goes [3; Theorem 5. 1]. Their proofs are of a different character.

On the other hand, it is elementary to prove that if

$$\sum_{n=1}^{\infty} |\Delta a_n| \log n < \infty \text{ and } a_n \to 0, \tag{8}$$

then both of (1) and (7) converge in the metric of L^1, and hence they are Fourier series (cf. [1; p. 26]). Note that, in the case of sine series (1), (6) ceases to be a sufficient condition for (1) being a Fourier series, as is easily seen from the example $a_n = 1/\log(n+1)$. That condition (6) is weaker than condition (8) has been proved by G. Goes [3; Theorem 4. 3] in the following form.

Theorem D. *Condition* (8) *holds if and only if both of* (3) *and* (6) *hold.*

Goes applied a theory of *BK*-space to prove this theorem, however, we shall give an entirely simple and elementary proof of this theorem.

Since

$$\Delta(a_n \log n) = \Delta a_n \cdot \log n - a_{n+1} \log\left(1 + \frac{1}{n}\right),$$

we have inequalities

$$|\Delta a_n| \log n \leq |\Delta(a_n \log n)| + \frac{|a_{n+1}|}{n}, \tag{9}$$

$$|\Delta(a_n \log n)| \leq |\Delta a_n| \log n + \frac{|a_{n+1}|}{n}. \tag{10}$$

Therefore, if $\{a_n \log n\}$ is of bounded variation and in addition (3) holds, then (8) follows from (9). Conversely, if (8) holds, then

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n} = \sum_{n=1}^{\infty} \left\{ \frac{1}{n} \left| \sum_{k=n}^{\infty} \Delta a_k \right| \right\} \leq \sum_{n=1}^{\infty} \left\{ \frac{1}{n} \sum_{k=n}^{\infty} | \Delta a_k | \right\}$$

$$= \sum_{N=1}^{\infty} \left\{ | \Delta a_N | \sum_{n=1}^{N} \frac{1}{n} \right\} \ll \sum_{N=1}^{\infty} | \Delta a_N | \log N < \infty,$$

i. e. (3) holds. Thus we conclude from (10) that $\{a_n \log n\}$ is of bounded variation. This completes our proof of Theorem D.

Acknowledgement : The author wishes to thank Professor G. Goes for some helpful advice made during the preparation of this paper.

REFERENCES

[1] R. P. BOAS, JR.: Integrability theorems for trigonometric transforms, Springer, 1967.

[2] G. and S. GOES: Sequences of bounded variation and sequences of Fourier coefficients. I, Math. Z. **118** (1970), 93—102.

[3] G. GOES: Sequences of bounded variation and sequences of Fourier coefficients. II, J. Math. Analys. and Appl. **39** (1972), 476—494.

[4] G. H. HARDY: Note on some points in the integral calculus, LXVI, The arithmetic mean of a Fourier constant, Messenger of Math. **58** (1928), 50—52.

[5] T. KANO: Coefficients of some trigonometric series, J. Fac. Sci. Shinshu Univ. **3** (1968), 153—162.

[6] S. SZIDON: Reihentheoretisches Sätze und ihre Anwendungen in der Theorie der Fourierreihen, Math. Z. **10** (1921), 121—127.

[7] S. A. TELJAKOVSKIĬ: Some estimates for trigonometric series with quasi-convex coefficients (in Russian), Mat. Sbornik **63** (105) (1964) 426—444. = A. M. S. Translations (2) **86** (1970), 109—131.

DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY

(Received October 21, 1976)