COBORDISM THEORY WITH REALITY

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Introduction

M. F. Atiyah [1] has defined the KR-theory of complex vector bundles with reality which relates KO-, KSC- and K-theories to each other. According to A. L. Edelson [8] the classifying bundle in the KR-theory is the complex universal bundle γ_n with the reality given by the conjugation for each n.

P. S. Landweber [10] discussed the equivariant homotopy groups of the equivariant spectrum of the Thom spaces $T(\gamma_n)$ with reality ($n=1, 2, \cdots$). In this paper we discuss the cobordism theory denoted by $MR^{*,*}(\ )$ based on this spectrum. In this theory, there hold the Thom isomorphism theorem and the splitting principle for real vector bundles in the sense of Atiyah. Consequently we can consider the Chern classes for real vector bundles with values in our cobordism $MR^{*,*}(\cdot)$.

The layout of this paper is as follows. In § 1 we recall some basic properties on real vector bundles. We introduce the Thom spectrum (2. 4) with reality in § 2, construct the equivariant cohomology theories (Theorems 3. 8 and 3. 9) with respect to the Thom spectrum by the fashion of G. W. Whitehead [12] in § 3, and prove the Thom isomorphism theorem (Theorem 4. 7) along the line of T. tom Dieck [5, 6] in § 4. In § 5 we give an exact sequence (Theorem 5. 8) which shows a relation between our cobordism theory and the usual complex coboridism theory. The exact sequence is a generalization of the exact sequence of Landweber ([10, (2. 1)]). § 6 is devoted to the splitting principle (Theorems 6. 6, 6. 8 and 6. 9) and the Chern classes (Theorem 6. 11) for the real vector bundles.

1. Real vector bundles

In this section we summarize some basic properties of real vector bundles which owe to M. F. Atiyah [1] and A. L. Edelson [8].

A *real space* is a Hausdorff space X together with an involution $\tau=\tau_X: X \longrightarrow X$, a *real pair* is a topological pair (X, A) together with an involution $\tau: (X, A) \longrightarrow (X, A)$ and a *real map* is a continuous map f: $(X, A) \longrightarrow (Y, B)$ which commutes with the involutions. A *real open set* is an open set U such that $\tau(U)=U$.

A *real vector bundle* ξ over a real space X is a complex vector bundle over X such that the total space E is also a real space and

(i) the projection $p: E \longrightarrow X$ is a real map,

(ii) the map $\tau | E_x: E_x \longrightarrow E_{\tau(x)}$ is conjugate linear, that is, $\tau(ae) = \bar{a}\tau(e)$ for any complex number a and $e \in E_x = p^{-1}(x)$.

We consider the n-dimensional complex space C^n as a real space with the standard conjugation c as the involution. If $V_k(C^n) = U(n)/U(n-k)$ denotes the Stiefel manifold of orthonormal k-frames in C^n, the conjugation in $U(n)$ defines the involution of $V_k(C^n)$ which sends $(v_1, \cdots, v_k)$ to $(\bar{v}_1, \cdots, \bar{v}_k)$. This involution induces the involution τ on the Grassmann manifold $G_k(C^n)$, that is, if $V \in G_k(C^n)$ has basis $\{v_1, \cdots, v_k\}$, $\tau(V) = \overline{V}$ is the subspace of C^n having basis $\{\bar{v}_1, \cdots, \bar{v}_k\}$. The classifying bundle

$$E(\gamma_k^n) = \{(V, x) \in G_k(C^n) \times C^n \mid x \in V\}$$

admits the involution τ defined by $\tau(V, x) = (\overline{V}, \bar{x})$. Thus we have a k-dimensional real vector bundle γ_k^n over the real space $G_k(C^n)$. We have the real inclusions

$$G_k(C^n) \subset G_k(C^{n+1}) \subset \cdots\cdots \subset \bigcup_{k \leqslant m} G_k(C^m) = G_k(C^\infty) = BU(k),$$
$$E(\gamma_k^n) \subset E(\gamma_k^{n+1}) \subset \cdots\cdots \subset \bigcup_{k \leqslant m} E(\gamma_k^m) = E(\gamma_k),$$

and $\gamma_k = (E(\gamma_k), p, BU(k))$ is a k-dimensional real vector bundle over the real space $BU(k)$.

Two real vector bundles E and E' over a real space X are *real isomorphic*, if there exists an isomorphism $h: E \longrightarrow E'$ of complex vector bundles such that h is also a real map. In the category of real vector bundles, an n-dimensional *trivial* bundle over a real space X is a real vector bundle which is real isomorphic to the product bundle $X \times C^n$ with the involution $\tau \times c$. Then

Proposition 1.1 (cf. [1, p. 374]). *Any real vector bundle is locally trivial in the category of real vector bundles.*

Let $\{U_i\}$ be a finite covering of a real compact space X by real open sets and $\{f_i'\}$ be a partition of unity with respect to the covering $\{U_i\}$. Then

Lemma 1.2. *The family $\{f_i\}$, defined by $f_i(x) = (f_i'(x) + f_i'\tau(x))/2$, is a partition of unity with respect to the covering $\{U_i\}$ and satisfies*

$$f_i\tau(x) = f_i(x) \quad \textit{for any} \quad x \in X.$$

Let $\xi = (E, p, X)$ be an n-dimensional real vector bundle over a real compact space X. Let $\{U_i \mid 1 \leqslant i \leqslant m\}$ be a finite covering of X by real open sets giving the local triviality of ξ. Then we have

Lemma 1.3. *Consider the composition*

$$g_i = p_2 \cdot h_i : p^{-1}(U_i) \xrightarrow{h_i} U_i \times C^n \xrightarrow{p_2} C^n \quad (1 \leqslant i \leqslant m),$$

where h_i is the real isomorphism and p_2 is the projection on the second factor. Then g_i is a real map which is a complex isomorphism on each fibre.

By a *real Gauss map* of ξ we mean a real map $g: E \longrightarrow C^m$ which is a complex linear monomorphism on each fibre. By using f_i and g_i in the above lemmas, we obtain a real Gauss map

$$g: E \longrightarrow C^n \oplus \cdots \oplus C^n = C^{mn}$$

by $g(e) = (f_1(p(e))\, g_1(e), \cdots, f_m(p(e))\, g_m(e))$. Thus we have

Proposition 1.4. *For any n-dimensional real vector bundle $\xi = (E, p, X)$ over a real compact space X, there exists a real Gauss map $g: E \longrightarrow C^{mn}$ for sufficiently large m.*

The real Gauss map g defines the real bundle map

$$\hat{g}: E \longrightarrow E(\gamma_n^{mn}), \quad \bar{g} = g(\xi): X \longrightarrow G_n(C^{mn}),$$

by $\hat{g}(e) = (g(E_{p(e)}), g(e))$, $\bar{g}(x) = g(E_x)$ for $e \in E$, $x \in X$, and we have

Proposition 1.5 (cf. [8, Prop. II. 1]). *By corresponding the real map $g(\xi)$ to a real vector bundle ξ, we have a bijective correspondence*

$$R\mathrm{Vect}_n(X) \approx [X;\ BU(n)]_R$$

of the set $R\mathrm{Vect}_n(X)$ of real isomorphism classes of n-dimensional real vector bundles over the real compact space X onto the set $[X;\ BU(n)]_R$ of real homotopy classes of real maps $X \longrightarrow BU(n)$.

By this result, $\gamma_n = (E(\gamma_n),\ p,\ BU(n))$ will be called the n-dimensional *universal real vector bundle.*

By a *real Hermitian metric* on ξ we mean a real map $\beta: E(\xi \oplus \xi) \longrightarrow C$ such that $\beta | E_x \times E_x$ is an inner product on E_x for each $x \in X$. By using the real Gauss map g of Proposition 1. 4 and the inner product $\langle\ ,\ \rangle$ on C^{mn}, we obtain a real Hermitian metric β on ξ by $\beta(e, e') = \langle g(e),\ g(e')\rangle$. Thus we have

Proposition 1.6. *There exists a real Hermitian metric on any n-dimensional real vector bundle over a real compact space.*

Furthermore, we have

Proposition 1.7. *Let ξ_0 be a real subbundle of an n-dimensional real vector bundle ξ over a real compact space. Then one can define the real subbundle $\xi_0^\perp$ of ξ by*

$$E(\xi_0^\perp)=\{v\in E(\xi) \mid \beta(v, e)=0 \text{ for any } e\in E(\xi_0)_{p(v)}\}$$

and it holds the following isomorphism as real vector bundles:

$$\xi \cong \xi_0 \oplus \xi_0^\perp.$$

2. The real Thom spectrum

Let $\mathscr{U}$ be the category whose objects are real compact Hausdorff spaces and whose maps are real maps, and $\mathscr{U}^2$ the one of pairs in $\mathscr{U}$. Let $\mathscr{U}_0$ be the category whose objects are real spaces in $\mathscr{U}$ with base points preserved by the involutions and whose maps are real maps preserving base points.

For $(X, A)\in \mathscr{U}^2$, X/A is an object of $\mathscr{U}_0$ which is obtained from X by collapsing A to the base point of X/A. For $X_1, \cdots, X_n\in \mathscr{U}_0$, the reduced join $X_1\wedge\cdots\wedge X_n$ is also an object of $\mathscr{U}_0$.

Let I be the interval $[-1, 1]$ and let $S=S^1=I/\dot{I}$, and consider the n-fold reduced join $S^n=S\wedge\cdots\wedge S$. Then we obtain the real space

(2. 1) $$\Sigma^{p,q}=(\Sigma^{p,q}, T_{p,q})\in \mathscr{U}_0$$

for any integers $p, q\geqslant 0$, where $\Sigma^{p,q}=S^q\wedge S^p$ and the involution $T_{p,q}$: $\Sigma^{p,q} \longrightarrow \Sigma^{p,q}$ is given by $T_{p,q}(a, b)=(a, -b)$ for $a=(a_1, \cdots, a_q)\in S^q$, $b=(b_1, \cdots, b_p)\in S^p$.

The *real Thom space* of a real vector bundle ξ over a space X in $\mathscr{U}$, denoted by $T(\xi)$, is the real space with base point which is the one-point compactification of $E(\xi)$. Let $D(\xi)$ and $S(\xi)$ be the associated unit disk and sphere bundles of ξ, respectively, for some real Hermitian metric on ξ. Then, the real Thom space $T(\xi)$ is real homeomorphic to the real space $D(\xi)/S(\xi)$.

Let ξ and η be real vector bundles over spaces X and Y in $\mathscr{U}$, respectively. Then the real Thom space $T(\xi\times\eta)$ and the real space $T(\xi)\wedge T(\eta)$ are real homeomorphic. Especially, the real Thom space $T(\theta^n\oplus\xi)$ is real homeomorphic to the real space $\Sigma^{n,n}\wedge T(\xi)$, where θ^n is the n-dimensional trivial real vector bundle over X.

Since a real bundle map f: $\xi \longrightarrow \eta$ induces a real map $T(f)$: $T(\xi) \longrightarrow T(\eta)$ of real Thom spaces, we have the real inclusions

$$T(\gamma_k^n)\subset T(\gamma_k^{n+1})\subset\cdots\cdots\subset \bigcup_{k\leqslant m} T(\gamma_k^m)=MU(k).$$

The real space $MU(k)$ will be called the *real Thom space* of the universal real vector bundle γ_k.

Let $od: C^n \longrightarrow C^{2(n+m)}$ and $ev: C^n \longrightarrow C^{2(n+m)}$ be the real inclusions given by

$$od(x_1, \cdots, x_n) = (x_1, 0, \cdots, x_n, 0, 0, \cdots, 0)$$
$$ev(x_1, \cdots, x_n) = (0, x_1, \cdots, 0, x_n, 0, \cdots, 0).$$

Then the real bundle map

$$a(n, m): \gamma_k^n \times \gamma_l^m \longrightarrow \gamma_{k+l}^{2n+2m}$$

can be defined by

$$a(n, m)((V, x), (W, y)) = od_*(V, x) \oplus ev_*(W, y),$$

and this induces the real bundle map

$$a_{k,l}: \gamma_k \times \gamma_l \longrightarrow \gamma_{k+l}$$

and the base point preserving real map

(2.2) $$\mu_{k,l}: MU(k) \wedge MU(l) \longrightarrow MU(k+l).$$

Similarly to the complex case, we have the following

Proposition 2.3. *The diagrams*

$$\begin{array}{ccc} MU(p) \wedge MU(q) \wedge MU(r) & \xrightarrow{1 \wedge \mu_{q,r}} & MU(p) \wedge MU(q+r) \\ \downarrow{\scriptstyle \mu_{p,q} \wedge 1} & & \downarrow{\scriptstyle \mu_{p,q+r}} \\ MU(p+q) \wedge MU(r) & \xrightarrow{\mu_{p+q,r}} & MU(p+q+r) \end{array}$$

and

$$\begin{array}{ccc} MU(p) \wedge MU(q) & \xrightarrow{T} & MU(q) \wedge MU(p) \\ & {\scriptstyle \mu_{p,q}} \searrow \quad \swarrow {\scriptstyle \mu_{q,p}} & \\ & MU(p+q) & \end{array}$$

are homotopy commutative as base point preserving real maps, where $T(x \wedge y) = y \wedge x$.

Now, we consider the base point preserving real map

$$\varepsilon_k = \mu_{1,k} \cdot (i \wedge 1): \Sigma^{1,1} \wedge MU(k) \xrightarrow{i \wedge 1} MU(1) \wedge MU(k) \xrightarrow{\mu_{1,k}} MU(k+1),$$

where $i: \Sigma^{1,1} = T(\gamma_1^1) \longrightarrow MU(1)$ is the real inclusion. The sequence

(2.4) $$\{MU(k), \varepsilon_k \mid k \in N\}$$

of the real spaces and the real maps will be called the *real Thom spctrum.*

3. The cohomology theory

For any real spaces X and Y with base points, let $[X; Y]_R$ denote the set of all real homotopy classes of real maps $X\longrightarrow Y$ preserving base points. We denote by $[f]_R$ the real homotopy class of $f: X\longrightarrow Y$, especially by $[y_0]_R=0$ the one of the constant map y_0 to the base point $y_0\in Y$. As in the usual cases, the set $[\Sigma^{p,q}\wedge X;\ Y]_R$ has the natural group structure with the identity element 0 for $q\geqq 1$ by the multiplication defined along the first coordinate axis, and this is abelian for $q\geqq 2$, where $\Sigma^{p,q}$ is the real space of (2. 1).

For any $X\in \mathscr{U}_0$, the real Thom spectrum (2. 4) defines the homomorphism

$$\nu_k^\sharp=\varepsilon_{k\sharp}\cdot\Sigma^{1,1}: [\Sigma^{k-p,k-q}\wedge X; MU(k)]_R\xrightarrow{\Sigma^{1,1}}[\Sigma^{k+1-p,k+1-q}\wedge X; \Sigma^{1,1}\wedge MU(k)]_R$$
$$\xrightarrow{\varepsilon_{k\sharp}} [\Sigma^{k+1-p,k+1-q}\wedge X; MU(k+1)]_R.$$

Then $\{[\Sigma^{k-p,k-q}\wedge X;\ MU(k)]_R,\ \nu_k^\sharp\}$ forms a direct system, and we can consider the abelian group

(3. 1) $$\widetilde{MR}^{p,q}(X)=\operatorname{Dir}\lim_k\ [\Sigma^{k-p,k-q}\wedge X;\ MU(k)]_R.$$

Remind the definition of the complex cobordism theory. Then, by forgetting the reality structure, we obtain the natural homomorphism

(3. 2) $$\rho:\ \widetilde{MR}^{p,q}(X)\longrightarrow \widetilde{MU}^{p+q}(X),$$

where $\widetilde{MU}^*(\ \ )$ is the reduced complex cobordism theory.

For any map $f: X\longrightarrow Y$ in $\mathscr{U}_0$, the homomorphisms

$$(1\wedge f)^\sharp:\ [\Sigma^{k-p,k-q}\wedge Y;\ MU(k)]_R\longrightarrow [\Sigma^{k-p,k-q}\wedge X;\ MU(k)]_R$$

commute with $\nu_k^\sharp$, and induce the homomorphism

$$f^*=\widetilde{MR}^{p,q}(f):\ \widetilde{MR}^{p,q}(Y)\longrightarrow \widetilde{MR}^{p,q}(X).$$

Similarly to the usual complex cobordism theory, we have

Proposition 3.3. (i) *If $f_0,\ f_1\in\mathscr{U}_0$ are real homotopic maps, then $f_0^*=f_1^*$.*

(ii) *For $X\in\mathscr{U}_0$ and any non-negative integers $r,\ s$ we have the suspension isomorphism*

$$\sigma^{r,s}:\ \widetilde{MR}^{p,q}(X)\cong\widetilde{MR}^{p+r,q+s}(\Sigma^{r,s}\wedge X).$$

Let the interval $I=[-1,\ 1]$ be a trivial real space (i. e. the involution is the identity) with base point -1. For a space X and a map $f: X\longrightarrow Y$ of $\mathscr{U}_0$, the *cone* CX over X, the *reduced suspension* $\Sigma^{0,1}X=\Sigma^{0,1}\wedge X$ of X and the *reduced mapping cone* $C(f)=CX\cup_f Y$ of f are also spaces of $\mathscr{U}_0$. Let $a(f): Y \longrightarrow C(f)$ be the canonical inclusion and $b(f)$: $C(f) \longrightarrow C(f)/Y = \Sigma^{0,1}X$ the projection, which are also real. Then we have the following lemma (cf. [3]).

Lemma 3.4. *For any real space V with base point the following sequence is exact:*

$$\cdots \longrightarrow [\Sigma^{0,1}Y; V]_R \xrightarrow{\Sigma^{0,1}f^\sharp} [\Sigma^{0,1}X; V]_R \xrightarrow{b(f)^\sharp} [C(f); V]_R$$
$$\xrightarrow{a(f)^\sharp} [Y; V]_R \xrightarrow{f^\sharp} [X; V]_R.$$

Now, by applying Lemma 3. 4 to $V = MU(k)$ and $1 \wedge f: \Sigma^{k-p,k-q} \wedge X \longrightarrow \Sigma^{k-p,k-q} \wedge Y$ and by taking the direct limit of (3. 1), we have

Proposition 3.5. *The following sequence is exact:*

$$\cdots \longrightarrow \widetilde{MR}^{p,q-1}(Y) \xrightarrow{\Sigma^{0,1}f^*} \widetilde{MR}^{p,q-1}(X) \xrightarrow{b(f)^*\sigma^{0,1}} \widetilde{MR}^{p,q}(C(f))$$
$$\xrightarrow{a(f)^*} \widetilde{MR}^{p,q}(Y) \xrightarrow{f^*} \cdots,$$

where $\sigma^{0,1}$ is the suspension isomorphism of Proposition 3. 3 (ii).

By a *real complex* we mean a CW-complex X together with a cellular involution τ whose fixed poit set is a subcomplex of X (cf. Bredon [3]). Let $\mathscr{W}$, $\mathscr{W}^2$ and $\mathscr{W}_0$ be the subcategories of $\mathscr{U}$, $\mathscr{U}^2$ and $\mathscr{U}_0$, respectively, consisting of real spaces having the real homotopy types of real finite complexes.

Let X be a real complex, and A a subcomplex invariant under the involution. Then, for any real space V, X has the real (equivariant) homotopy extension property with respect to A (cf. [3]). Therefore we have

Lemma 3.6. *If (X, A) is a pair in $\mathscr{W}_0$ and $i: A \subset X$, then the real space X/A is of the same real homotopy type as $C(i)$ and the projection $p: C(i) \longrightarrow C(i)/CA = X/A$ is a real homotopy equivalence.*

Proposition 3. 5 and Lemma 3. 6 show the following

Proposition 3.7. *For any pair (X, A) in $\mathscr{W}_0$, we have the exact sequence*

$$\cdots \longrightarrow \widetilde{MR}^{p,q-1}(A) \xrightarrow{\delta} \widetilde{MR}^{p,q}(X/A) \xrightarrow{j^*} \widetilde{MR}^{p,q}(X) \xrightarrow{i^*} \widetilde{MR}^{p,q}(A) \longrightarrow \cdots.$$

Now, in virtue of Propositions 3. 3 and 3. 7, we obtain

Theorem 3.8. *For any integer p, $\widetilde{MR}^{p,*}(\ )$ of (3. 1) is a generalized cohomology theory on $\mathscr{W}_0$.*

By the standard argument of the generalized cohomology theory (cf., e. g., [7]), we have the following theorems.

Theorem 3.9. *For $(X, A)\in \mathscr{W}^2$, we define*

$$MR^{p,q}(X, A)=\widetilde{MR}^{p,q}(X/A).$$

Then, for any integer p, $MR^{p,*}(\ ,\ )$ is a generalized cohomology theory on $\mathscr{W}$.

Theorem 3.10. *For any triad $(X; A, B)$ in $\mathscr{W}$ (i. e. (X, A), $(X, B)\in \mathscr{W}^2$), there exists the Mayer-Vietoris exact sequence*

$$\cdots \longrightarrow MR^{p,q-1}(A\cap B) \xrightarrow{\Delta} MR^{p,q}(A\cup B) \xrightarrow{\alpha} MR^{p,q}(A)\oplus MR^{p,q}(B) \xrightarrow{\beta} MR^{p,q}(A\cap B)\longrightarrow \cdots .$$

For any base point preserving real maps $f\colon \Sigma^{k-p,k-q}\wedge X\longrightarrow MU(k)$ and $g\colon \Sigma^{l-r,l-s}\wedge Y\longrightarrow MU(l)$, we consider the composition

$$\Sigma^{k+l-p-r,k+l-q-s}\wedge X\wedge Y\xrightarrow{e\wedge 1\wedge 1}\Sigma^{k-p,k-q}\wedge\Sigma^{l-r,l-s}\wedge X\wedge Y$$
$$\xrightarrow{1\wedge T\wedge 1}\Sigma^{k-p,k-q}\wedge X\wedge\Sigma^{l-r,l-s}\wedge Y\xrightarrow{f\wedge g}MU(k)\wedge MU(l)\xrightarrow{\mu_{k,l}}MU(k+l),$$

where e is the natural real homeomorphism and $\mu_{k,l}$ is the real map of (2. 2), and define the cross product

$$\wedge : [\Sigma^{k-p,k-q}\wedge X; MU(k)]_R\times[\Sigma^{l-r,l-s}\wedge Y; MU(l)]_R$$
$$\longrightarrow [\Sigma^{k+l-p-r,k+l-q-s}\wedge X\wedge Y; MU(k+l)]_R$$

by $[f]_R\wedge[g]_R=[\mu_{k,l}\cdot(f\wedge g)\cdot(1\wedge T\wedge 1)\cdot(e\wedge 1\wedge 1)]_R$. In virtue of Proposition 2. 3, this cross product commutes with $\nu_k^{\sharp}$ and induces the cross product

$$\wedge : \widetilde{MR}^{p,q}(X)\otimes\widetilde{MR}^{r,s}(Y)\longrightarrow\widetilde{MR}^{p+r,q+s}(X\wedge Y). \tag{3.11}$$

Hence, the cohomology theory $\widetilde{MR}^{*,*}$ is the multiplicative theory.

Finally, we notice the dual situation of the group of (3. 1). For any $X\in\mathscr{W}_0$, we have also the composition

$$\nu_{k\sharp}=(\varepsilon_k\wedge 1)_\sharp\cdot\Sigma^{1,1}\colon [\Sigma^{k+p,k+q}; MU(k)\wedge X]_R\xrightarrow{\Sigma^{1,1}}$$
$$[\Sigma^{k+1+p,k+1+q}; \Sigma^{1,1}\wedge MU(k)\wedge X]_R\xrightarrow{(\varepsilon_k\wedge 1)_\sharp}[\Sigma^{k+1+p,k+1+q}; MU(k+1)\wedge X]_R$$

and the direct system $\{[\Sigma^{k+p,k+q};\ MU(k)\wedge X]_R,\ \iota_{k\sharp}\}$. Therefore, we can define the abelian group

(3. 12) $$\widetilde{MR}_{p,q}(X)=\operatorname{Dir}\operatorname*{Lim}_k\ [\Sigma^{k+p,k+q};\ MU(k)\wedge X]_R.$$

4. The Thom isomorphism theorem

Denote by

(4. 1) $$t_k\in\widetilde{MR}^{k,k}(\Sigma^{k,k})$$

the element represented by the real inclusion $\Sigma^{k,k}=T(\gamma_k^k)\subset MU(k)$. Then the definition of the cross product (3. 11) shows the following

Proposition 4.2. *The suspension isomorphism*

$$\sigma^{k,k}:\ \widetilde{MR}^{p,q}(X)\longrightarrow\widetilde{MR}^{p+k,q+k}(\Sigma^{k,k}\wedge X)$$

is given by $\sigma^{k,k}(x)=t_k\wedge x$ *for any* $x\in\widetilde{MR}^{p,q}(X)$.

Let ξ be an n-dimensional real vector bundle over a space X in $\mathscr{U}$. In virtue of Proposition 1. 5, there is a real bundle map $f: E(\xi)\longrightarrow E(\gamma_n)$, and f induces the base point preserving real map $T(f):\ T(\xi)\longrightarrow MU(n)$ of the Thom spaces. The element

(4. 3) $$t(\xi)\in\widetilde{MR}^{n,n}(T(\xi))$$

represented by $T(f)$ is called the *Thom class* of ξ. Let $s:\ X^+(=X/\phi)\longrightarrow T(\xi)$ be the base point preserving real map induced by the 0-section $X\longrightarrow E(\xi)$. Then the element

$$e(\xi)=s^*t(\xi)\in MR^{n,n}(X)$$

is called the *Euler class* of ξ.

Similarly to the usual case, as for the Thom classes of the real vector bundles we have

Proposition 4.4. (i) *If* $h:\ \eta\longrightarrow\xi$ *is a real bundle map, then*

$$t(\eta)=T(h)^*(t(\xi)).$$

(ii) *Under the identification* $T(\xi\times\xi')=T(\xi)\wedge T(\xi')$ *we have*

$$t(\xi\times\xi')=t(\xi)\wedge t(\xi').$$

(iii) *The element* t_k *of* (4. 1) *is the Thom class of the trivial real vector bundle* C^k *over a point.*

Let ξ and η be real vector bundles over a space X in $\mathscr{U}$. Let $d: X\longrightarrow X\times X$ be the diagonal map. Since $\xi\oplus\eta=d^*(\xi\times\eta)$, we have the base point preserving real map

$$\Delta:\ T(\xi\oplus\eta)\longrightarrow T(\xi\times\eta)=T(\xi)\wedge T(\eta),$$

which satisfies $\Delta^* t(\xi \times \eta) = t(\xi \oplus \eta)$.

Now, we can define the *Thom homomorphism*

$$(4.5) \qquad \Psi(\xi): \widetilde{MR}^{p,q}(T(\eta)) \longrightarrow \widetilde{MR}^{p+n,q+n}(T(\xi \oplus \eta))$$

(ξ is n-dimensional) by

$$\Psi(\xi)(x) = \Delta^*(t(\xi) \wedge x) \quad \text{for} \quad x \in \widetilde{MR}^{p,q}(T(\eta)).$$

Especially, if η is a 0-dimensional real vector bundle over X, then we obtain the Thom homomorphism

$$\Psi(\xi): MR^{p,q}(X) \longrightarrow \widetilde{MR}^{p+n,q+n}(T(\xi)).$$

By a parallel argument to T. tom Dieck [5], [6] we obtain the following

Proposition 4.6. *Let ξ, η, ζ be real vector bundles over a space X in $\mathscr{U}$, and let ξ', η' be ones over a space X' in $\mathscr{U}$.*

(i) *If* $\dim \xi = \dim \xi' = n$ *and real bundle maps $f: \xi \longrightarrow \xi'$ and $g: \eta \longrightarrow \eta'$ induce the same real map $\tilde{f} = \tilde{g}: X \longrightarrow X'$ of base spaces, then the following diagram is commutative:*

$$\begin{array}{ccc} \widetilde{MR}^{p,q}(T(\eta')) & \xrightarrow{\Psi(\xi')} & \widetilde{MR}^{p+n,q+n}(T(\xi' \oplus \eta')) \\ \downarrow T(g)^* & & \downarrow T(f \oplus g)^* \\ \widetilde{MR}^{p,q}(T(\eta)) & \xrightarrow{\Psi(\xi)} & \widetilde{MR}^{p+n,q+n}(T(\xi \oplus \eta)) . \end{array}$$

(ii) *If* $\dim \xi = n$ *and* $\dim \zeta = m$, *then the following diagram is commutative:*

$$\begin{array}{ccc} \widetilde{MR}^{p,q}(T(\eta)) & \xrightarrow{\Psi(\xi \oplus \zeta)} & \widetilde{MR}^{p+m+n,q+m+n}(T(\xi \oplus \zeta \oplus \eta)) \\ & \searrow \Psi(\zeta) \qquad \Psi(\xi) \nearrow & \\ & \widetilde{MR}^{p+m,q+m}(T(\zeta \oplus \eta)) . & \end{array}$$

(iii) *For the n-dimensional trivial real vector bundle θ^n over X, we have*

$$\Psi(\theta^n) = \sigma^{n,n}: \widetilde{MR}^{p,q}(T(\eta)) \cong \widetilde{MR}^{p+n,q+n}(\Sigma^{n,n} \wedge T(\eta)).$$

Theorem 4.7. *For any real vector bundles ξ and η over a space X in $\mathscr{U}$, the Thom homomorphism $\Psi(\xi)$ of* (4.5) *is an isomorphism.*

By this theorem and the standard arqument we have

Theorem 4.8. *Let $\xi=(E, p, X)$ be an n-dimensional real vector bundle over a space X in $\mathscr{U}$, and S(E) its associated sphere bundle. Then there exists the Thom-Gysin exact sequence*:

$$\cdots \longrightarrow MR^{p,q}(X) \xrightarrow{e(\xi)\cdot} MR^{p+n,q+n}(X) \longrightarrow MR^{p+n,q+n}(S(E)) \longrightarrow MR^{p,q+1}(X) \longrightarrow \cdots,$$

where $e(\xi)$ is the Euler class of ξ and $\cdot$ is the internal product induced from the cross product.

5. Exact sequences

In this section, we show that there is an exact sequence containing the homomorphism ρ of (3. 2).

For any real spaces X and Y with base points, let $RF(X; Y)$ denote the set of all real maps $X \longrightarrow Y$ preserving base points. Consider $S=I/\dot{I}$ $(I=[-1, 1])$ and the subsets

$$D_{+}^{p,q}=\{(a, t, b)\in S^q\wedge S\wedge S^p \mid t\geqslant 0\}$$
$$D_{-}^{p,q}=\{(a, t, b)\in S^q\wedge S\wedge S^p \mid t\leqslant 0\}$$

of $S^q\wedge S\wedge S^p$ which is the real space $\Sigma^{p+1,q}$ or $\Sigma^{p,q+1}$ of (2. 1). We identify the real set $\Sigma_0^{p,q}=D_{+}^{p,q}\cap D_{-}^{p,q}$ with $\Sigma^{p,q}$ by means of the real map e: $\Sigma^{p,q} \longrightarrow \Sigma^{p+1,q}$ or $\Sigma^{p,q+1}$ defined by $e(a, b)=(a, 0, b)$.

In the sequel we use the arguments of J. Levine [11] in a slightly generalized form.

Lemma 5.1 (cf. [11, Lemma 3. 4]). *If $f: D_{+}^{p,q}\wedge X \longrightarrow Y$ satisfies $f\mid \Sigma^{p,q}\wedge X\in RF(\Sigma^{p,q}\wedge X;\ Y)$, then there exists uniquely a map h in $RF(\Sigma^{p+1,q}\wedge X;\ Y)$ which is an extension of f.*

Proof. The desired map h is defined by

$$h\mid D_{+}^{p,q}\wedge X=f, \quad h\mid D_{-}^{p,q}\wedge X=\tau_Y\cdot f\cdot(T_{p+1,q}\wedge \tau_X),$$

where τ_X and τ_Y are the involutions of X and Y, respectively. q. e. d.

Lemma 5.2 (cf. [11, Lemma 3. 7]). *Let X be a CW-complex. If f_0, f_1: $\Sigma^{p+1,q}\wedge X \longrightarrow Y$ are homotopic and g_s: $D_{-}^{p,q}\wedge X \longrightarrow Y$ $(0\leqslant s\leqslant 1)$ is a homotopy of $g_0=f_0\mid D_{-}^{p,q}\wedge X$ to $g_1=f_1\mid D_{-}^{p,q}\wedge X$, then $\{g_s\}$ is extendible to a homotopy of f_0 to f_1.*

Now, for any spaces X and Y with base points, we denote by $F(X; Y)$ the set of all maps $X \longrightarrow Y$ preserving base points, and by

$[X; Y]$ the set of all homotopy classes of maps in $F(X; Y)$.

Let $\alpha \in [S^{p+q+1} \wedge X; Y]$ and $\beta \in [\Sigma^{p+1,q} \wedge X; Y]_R$ be given classes. If $g \in RF(\Sigma^{p+1,q} \wedge X; Y)$ is a representative of β, we can choose a representative $f \in F(S^{p+q+1} \wedge X; Y)$ of α snch that

$$f | D_-^{p,q} \wedge X = g | D_-^{p,q} \wedge X.$$

Since $f | \Sigma^{p,q} \wedge X = g | \Sigma^{p,q} \wedge X \in RF(\Sigma^{p,q} \wedge X; Y)$, Lemma 5.1 shows that there is a map

$$h \in RF(\Sigma^{p+1,q} \wedge X; Y)$$

snch that $h | D_+^{p,q} \wedge X = f | D_+^{p,q} \wedge X$.

Lemma 5.3 (cf. [11, 3.8]). *The class* $[h]_R \in [\Sigma^{p+1,q} \wedge X; Y]_R$ *depends only on* α *and* β.

By the above lemma we can denote

(5.4) $$\alpha \cdot \beta = [h]_R \in [\Sigma^{p+1,q} \wedge X; Y]_R$$

for $\alpha \in [S^{p+q+1} \wedge X; Y]$, $\beta \in [\Sigma^{p+1,q} \wedge X; Y]_R$. Then, it is easy to check that the operation $(\alpha, \beta) \longrightarrow \alpha \cdot \beta$ satisfies the following properties.

Proposition 5.5. (i) *If* $f: X' \longrightarrow X$ *is a map in* $\mathscr{W}_0$ *and* g: $Y \longrightarrow Y'$ *is a real map preserving base points, then*

$$(f^\sharp \alpha) \cdot (f^\sharp \beta) = f^\sharp(\alpha \cdot \beta), \quad (g_\sharp \alpha) \cdot (g_\sharp \beta) = g_\sharp(\alpha \cdot \beta).$$

(ii) *The following diagram is commutative*:

$$\begin{array}{ccc} [S^{p+q} \wedge X; Y] \times [\Sigma^{p,q} \wedge X; Y]_R & \xrightarrow{\ \cdot\ } & [\Sigma^{p,q} \wedge X; Y]_R \\ \downarrow S^2 \times \Sigma^{1,1} & & \downarrow \Sigma^{1,1} \\ [S^{p+q+2} \wedge X; S^2 \wedge Y] \times [\Sigma^{p+1,q+1} \wedge X; \Sigma^{1,1} \wedge Y]_R & \xrightarrow{\ \cdot\ } & [\Sigma^{p+1,q+1} \wedge X; \Sigma^{1,1} \wedge Y]_R. \end{array}$$

Proposition 5.6 (cf. [11, 4.2]). *If* $\alpha_1, \alpha_2 \in [S^{p+q+1} \wedge X; Y]$ *and* $\beta, \beta_1, \beta_2 \in [\Sigma^{p+1,q} \wedge X; Y]_R$, *then*

$$\alpha_1 \cdot (\alpha_2 \cdot \beta) = (\alpha_1 \alpha_2) \cdot \beta, \quad (\alpha_1 \alpha_2) \cdot (\beta_1 \beta_2) = (\alpha_1 \cdot \beta_1)(\alpha_2 \cdot \beta_2),$$

where $\alpha_1 \alpha_2$ *and* $\beta_1 \beta_2$ *mean the multiplications in the groups* $[S^{p+q+1} \wedge X; Y]$ *and* $[\Sigma^{p+1,q} \wedge X; Y]_R$, *respectively.*

Now, we prove the next proposition, which is essential to show the main result of this section.

Proposition 5.7 (cf. [11, Theorem 4.3]). *For* $X \in \mathscr{W}_0$ *and any real space* Y *with base point, the sequence*

$$\cdots \longrightarrow [\Sigma^{p,q+1}\wedge X;\, Y]_R \xrightarrow{\rho} [S^{p+q+1}\wedge X;\, Y] \xrightarrow{\phi} [\Sigma^{p+1,q}\wedge X;\, Y]_R \xrightarrow{\psi} [\Sigma^{p,q}\wedge X;\, Y]_R \longrightarrow \cdots$$

is exact, where the homomorphisms ρ, ϕ *and* ψ *are given as follows*:

$\rho([f]_R)=[f]$ (*forgetting the reality*) *for* $[f]_R\in[\Sigma^{p,q+1}\wedge X;\, Y]_R$,
$\phi(\alpha)=\alpha\cdot 0$ (*given by* (5.4)) *for* $\alpha\in[S^{p+q+1}\wedge X;\, Y]$,
$\psi([g]_R)=[g\mid \Sigma^{p,q}\wedge X]_R$ *for* $[g]_R\in[\Sigma^{p+1,q}\wedge X;\, Y]_R$.

Proof. (a) Im $\rho=$ Ker ϕ. We will show that $\phi\rho([f']_R)=0$ for any $f'\in RF(\Sigma^{p,q-1}\wedge X;\, Y)$. Let $\omega:(I,\dot I)\longrightarrow(I,\dot I)$ $(I=[-1,1])$ be a map such that $\omega([-1,0])=-1$, $\omega(1)=1$ and ω is homotopic to the identity relative $\dot I$. Then $[f']_R$ is represented by the map $f\in RF(\Sigma^{p,q+1}\wedge X;\, Y)$ given by $f(a,t,b,x)=f'(a,\omega(t),b,x)$. Since $f(D^{p,q}_{-}\wedge X)=y_0$, the definition of (5.4) shows that $\phi\rho([f]_R)=[f]\cdot 0$ is represented by the extension $h\in RF(\Sigma^{p+1,q}\wedge X;\, Y)$ of $f\mid D^{p,q}_{+}\wedge X$. Define a homotopy $h_s: D^{p,q}_{+}\wedge X\longrightarrow Y$ $(0\leqslant s\leqslant 1)$ by

$$h_s(a,t,b,x)=f(a,\ s+(1-s)t,b,x)\quad (0\leqslant t\leqslant 1).$$

Then, this satisfies

$$h_0=f\mid D^{p,q}_{+}\wedge X=h\mid D^{p,q}_{-}\wedge X,\quad h_1=y_0,\quad h_s\mid\Sigma^{p,q}\wedge X\in RF(\Sigma^{p,q}\wedge X;\, Y).$$

Therefore $[h]_R=0$ by Lemma 5.1, as desired.

Next, we will show that, if $\alpha\in[S^{p+q+1}\wedge X;\, Y]$ and $\phi(\alpha)=0$, then $\alpha\in\text{Im}\,\rho$. Let α be represented by $f\in F(S^{p+q+1}\wedge X;\, Y)$ such that $f\mid D^{p,q}_{-}\wedge X=y_0$. Since $\phi(\alpha)=0$, there is a homotopy $h_s: D^{p,q}_{-}\wedge X\longrightarrow Y$ $(0\leqslant s\leqslant 1)$ such that

$$h_0=f\mid D^{p,q}_{+}\wedge X,\quad h_1=y_0,\quad h_s\mid\Sigma^{p,q}\wedge X\in RF(\Sigma^{p,q}\wedge X;\, Y).$$

Now, define $f'\in RF(\Sigma^{p,q+1}\wedge X;\, Y)$ by

$$f'(a,t,b,x)=\begin{cases} h_t(a,0,b,x) & \text{if}\quad 0\leqslant t\leqslant 1,\\ y_0 & \text{if}\quad -1\leqslant t\leqslant 0.\end{cases}$$

Then we have easily $\rho([f']_R)=[f]$, as desired.

(b) Im $\phi=$ Ker ψ. It is clear that $\psi\phi(\alpha)=0$ for any $\alpha\in[S^{p+q+1}\wedge X;\, Y]$.

Suppose $\psi(\beta)=0$ for $\beta=[g]_R\in[\Sigma^{p+1,q}\wedge X;\, Y]_R$. Then $g\mid\Sigma^{p,q}\wedge X$ is real homotopic to y_0 in $RF(\Sigma^{p,q}\wedge X;\, Y)$. By the homotopy extension theorem and Lemma 5.1, there exists a real map $g'\in RF(\Sigma^{p+1,q}\wedge X;\, Y)$ which is real homotopic to y_0 in $RF(\Sigma^{p+1,q}\wedge X;\, Y)$ and $g'\mid\Sigma^{p,q}\wedge X=g\mid\Sigma^{p,q}\wedge X$. Define f: $S^{p+q+1}\wedge X\longrightarrow Y$ by

$$f\mid D^{p,q}_{+}\wedge X=g\mid D^{p,q}_{+}\wedge X,\quad f\mid D^{p,q}_{-}\wedge X=g'\mid D^{p,q}_{-}\wedge X.$$

Then we have $[f]\cdot 0=[g]_R=\beta$ by the definition of (5. 4).

(c) Im $\psi=$Ker ρ. Let $[f]_R\in[\Sigma^{p,q}\wedge X;\ Y]_R$. Then, $[f]_R\in$Ker ρ if and only if f is extendible over $D_+^{p,q}\wedge X$. By Lemma 5. 1, this is equivalent to $[f]_R\in$Im ψ.

These complete the proof of the proposition. q. e. d.

Consider the case of $Y=MU(k)$ in the above proposition. In virtue of Proposition 5. 5, the homomorphisms ρ, ϕ and ψ commute with the homomorphisms $\nu_k^\sharp$ of the direct systems $\{[\Sigma^{k-p,k-q}\wedge X;\ MU(k)]_R,\ \nu_k^\sharp\}$ of (3. 1) and $\{[S^{2k-p-q}\wedge X; MU(k)],\ \nu_k^\sharp\}$ with respect to the usual complex Thom spectrum, and hence we have the following theorem which is a generalization of the exact sequence of Landweber [10, (2. 1)].

Theorem 5.8. *For any* $X\in\mathscr{W}_0$ *we have the exact sequence*

$$\cdots\longrightarrow\widetilde{MR}^{p,q-1}(X)\xrightarrow{\rho}\widetilde{MU}^{p+q-1}(X)\xrightarrow{\phi}\widetilde{MR}^{p-1,q}(X)\xrightarrow{\psi}\widetilde{MR}^{p,q}(X)\longrightarrow\cdots,$$

where ρ *is the homomorphism of* (3. 2) *forgetting the reality.*

Similarly, consider the case of $Y=MU(k)\wedge X$ in Proposition 5. 7. Then

Theorem 5.9. *For any* $X\in\mathscr{U}_0$ *we have the exact sequence*

$$\cdots\longrightarrow\widetilde{MR}_{p,q+1}(X)\xrightarrow{\rho}\widetilde{MU}_{p+q+1}(X)\xrightarrow{\phi}\widetilde{MR}_{p+1,q}(X)\xrightarrow{\psi}\widetilde{MR}_{p,q}(X)\longrightarrow\cdots.$$

As for the coefficient ring $MR^{*,*}$, we have

$$MR^{-p,-q}=\widetilde{MR}^{-p,-q}(\Sigma^{0,0})=\widetilde{MR}_{p,q}(\Sigma^{0,0})=MR_{p,q}.$$

In Theorem 5. 8, put $X=\Sigma^{0,0}$. The obtained exact sequence is the one of Landweber. $MR^{*,*}$ has been partially computed by Landweber [10].

6. The splitting principle and the Chern classes

Let $s_n\in\widetilde{MU}^{2n}(S^{2n})$ be the element represented by the natural inclusion $S^{2n}\subset MU(n)$. Then $\widetilde{MU}^{2n}(S^{2n})$ is a free abelian group generated by s_n (cf. [4]).

Lemma 6.1. *For any integer* $n\geqslant 0$, *the homomorphism*

$$\rho:\widetilde{MR}^{n,n}(\Sigma^{n,n})\longrightarrow MU^{2n}(S^{2n})$$

of (3. 2) *is an isomorphism and* $\rho(t_n)=s_n$, *where* t_n *is the element of* (4. 1).

Proof. It is clear that $\rho(t_n)=s_n$, and hence ρ is surjective and we have the exact sequence

$$\widetilde{MR}^{n-1,n}(\Sigma^{n,n}) \longrightarrow \widetilde{MR}^{n,n}(\Sigma^{n,n}) \xrightarrow{\rho} \widetilde{MU}^{2n}(S^{2n}) \longrightarrow 0$$

by Theorem 5. 8. The first term is isomorphic to $\widetilde{MR}^{-1,0}(\Sigma^{0,0})$ by the suspension isomorphism of Proposition 3. 3 (ii), and $\widetilde{MR}^{-1,0}(\Sigma^{0,0})=MR_{1,0}=0$ by [10, (3. 5)]. Thus we obtain the lemma. q. e. d.

We consider the n-dimensional complex projective space CP_n as a real complex with the involution induced by the conjugation c in C^{n+1}, which will be called the *standard projective n-space.* By the natural manner the canonical line bundle η_n over CP_n is a real line bundle over the real complex CP_n, which is also called the canonical line bundle. We can check that

$$T(\eta_n)=CP_{n+1} \quad \text{and} \quad CP(\infty)=MU(1)$$

as real speces with base points. Furthermore, the inclusion $CP_n\subset MU(1)$ is a real map preserving base points, and this represents the element $x_n\in MR^{1,1}(CP_n)$.

Theorem 6.2. *The $MR^{*,*}$-module $MR^{*,*}(CP_n)$ of the standard projective n-space CP_n is a free $MR^{*,*}$-module with basis* 1, x_n, $\cdots$, $(x_n)^n$, *with the relation* $(x_n)^{n+1}=0$. *In other words,*

$$MR^{*,*}(CP_n)=MR^{*,*}[x_n]/((x_n)^{n+1}).$$

Proof. The theorem is obvious for $n=0$. Suppose inductively that it is true for $n-1$, and consider the exact sequence

$$\cdots \longrightarrow \widetilde{MR}^{*,*}(\Sigma^{n,n}) \xrightarrow{j^*} MR^{*,*}(CP_n) \xrightarrow{i^*} MR^{*,*}(CP_{n-1}) \longrightarrow \cdots$$

of the pair (CP_n, CP_{n-1}) of the real complexes. By the inductive assumption, i^* is an epimorphism, hence we get the commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \widetilde{MR}^{*,*}(\Sigma^{n,n}) & \xrightarrow{j^*} & MR^{*,*}(CP_n) & \xrightarrow{i^*} & MR^{*,*}(CP_{n-1}) & \longrightarrow 0 \\
& \downarrow \rho & & \downarrow \rho & & \downarrow \rho & \\
0 \longrightarrow & \widetilde{MU}^{*}(S^{2n}) & \xrightarrow{j^*} & MU^{*}(CP_n) & \xrightarrow{i^*} & MU^{*}(CP_{n-1}) & \longrightarrow 0
\end{array}$$

of the exact sequences, and $i^*(x_n)^n=0$. Therefore, there exists an element $a\in\widetilde{MR}^{n,n}(\Sigma^{n,n})$ such that $j^*a=(x_n)^n$. Let $y^n\in\widetilde{MU}^2(CP_n)$ be the canonical generator of $MU^*(CP_n)$. Since $j^*\rho a=(y_n)^n=j^*s_n$, we have $\rho a=s_n$ and so $a=t_n$ by Lemma 6.1. Furthermore, $(x_n)^{n+1}=j^*(b\wedge t_n)$ for some $b\in MR^{1,1}$. Therefore $(x_n)^{n+1}=0$, because $MR^{1,1}=MR_{-1,-1}=0$ by [10, p. 272]. These show the theorem by the induction on n. q. e. d.

Proposition 6.3. *Let X be a space in $\mathscr{W}$ and let $\bar{x}_n=p^*x_n\in MR^{1,1}(X\times CP_n)$, where $p: X\times CP_n\longrightarrow CP_n$ is the projection. Then $MR^{*,*}(X\times CP_n)$ is a free $MR^{*,*}(X)$-module with basis 1, $\bar{x}_n, \cdots, (\bar{x}_n)^n$, and $(\bar{x}_n)^{n+1}=0$. In other words,*

$$MR^{*,*}(X\times CP_n)=MR^{*,*}(X)\,[\bar{x}_n]/((\bar{x}_n)^{n+1}).$$

Proof. Suppose inductively that $MR^{*,*}(X\times CP_{n-1})$ is as stated (this is trivial for $n=1$). Then, similarly to the above proof, we get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{MR}^{*,*}(\Sigma^{n,n}) & \xrightarrow{j^*} & MR^{*,*}(CP_n) & \xrightarrow{i^*} & MR^{*,*}(CP_{n-1}) & \longrightarrow & 0\\
 & & \downarrow{\bar{p}^*} & & \downarrow{p^*} & & \downarrow{p^*} & & \\
0 & \longrightarrow & \widetilde{MR}^{*,*}(X^+\wedge\Sigma^{n,n}) & \xrightarrow{j^*} & MR^{*,*}(X\times CP_n) & \xrightarrow{i^*} & MR^{*,*}(X\times CP_{n-1}) & \longrightarrow & 0
\end{array}$$

of the exact sequences. By Proposition 4.2, $\widetilde{MR}^{*,*}(X^+\wedge\Sigma^{n,n})$ is a free $MR^{*,*}(X)$-module generated by the Thom class t_n, and $\bar{p}^*t_n=t_n$ by Proposition 4.4. Therefore, $MR^{*,*}(X\times CP_n)$ is freely generated by 1, $\bar{x}_n, \cdots, (\bar{x}_n)^n$, and $(\bar{x}_n)^{n+1}=p^*(x_n)^{n+1}=0$. q. e. d.

Now, let $\xi=(E, p, X)$ be an n-dimensional real vector bundle over a space X in $\mathscr{W}$, and consider its associated projective bundle

(6.4) $$P\xi=(PE, \pi, X).$$

PE is a real space with the induced involution and π is a real map. Especially, if X is a finite real complex, then PE is a finite real complex. Furthermore, let

$$\gamma=(LE, \pi_L, PE)$$

be the canonical line bundle over PE. Then this is also a real line bundle, and its classifying map $PE\longrightarrow MU(1)$ represents the element

(6.5) $$x\in MR^{1,1}(PE).$$

Applying the standard Mayer-Vietoris argument, we obtain

Theorem 6.6. *Let ξ be an n-dimensional real vector bundle over a* *space* X *in* $\mathscr{W}$. *Then* $MR^{*,*}(PE)$ *of the projective bundle* PE *of* (6.4) *is a free* $MR^{*,*}(X)$*-module with basis* $1, x, \cdots, x^{n-1}$, *where* x *is the element of* (6.5).

Corollary 6.7. *The induced homomorphism*

$$\pi^* : MR^{*,*}(X) \longrightarrow MR^{*,*}(PE)$$

of the projection π *of* (6.4) *is a monomorphism.*

Similarly to the usual case, by the standard argument we have the following theorems.

Theorem 6.8. *For any* n*-dimensional real vector bundle* ξ *over a space* X *in* $\mathscr{W}$, *there exist a space* F *and a map* $\pi : F \longrightarrow X$ *in* $\mathscr{W}$ *satisfying the following conditions:*

1) $\pi^* : MR^{*,*}(X) \longrightarrow MR^{*,*}(F)$ *is a monomorphism.*
2) $\pi^*\xi$ *is real isomorphic to the sum of* n *real line bundles over* F.

Theorem 6.9. *For any two* n *and* m*-dimensional real vector bundles* ξ *and* η *over a space* X *in* $\mathscr{W}$ *respectively, there exist a space* F *and a map* $\pi : F \longrightarrow X$ *in* $\mathscr{W}$ *satisfying the following conditions:*

1) $\pi^* : MR^{*,*}(X) \longrightarrow MR^{*,*}(F)$ *is a monomorphism.*
2) $\pi^*(\xi)$ *and* $\pi^*(\eta)$ *are real isomorphic to the sums of* n *and* m *real line bundles over* F, *respectively.*

Finally, we can define an analogue of the Chern classes of complex vector bundles for real vector bundles.

Let $\xi = (E, p, X)$ be an n-dimensional real vector bundle over a space X in $\mathscr{W}$. Let $x \in MR^{1,1}(PE)$ be the element of (6.5). Then, by Theorem 6.6 x^n can be expressed as a linear combination of $1, x, \cdots, x^{n-1}$ over $MR^{*,*}(X)$ uniquely. Put

$$x^n - c_1(\xi)x^{n-1} + c_2(\xi)x^{n-2} - \cdots + (-1)^n c_n(\xi) = 0. \tag{6.10}$$

Then we obtain $MR^{*,*}$*-Chern classes*

$$c_k(\xi) \in MR^{k,k}(X), \quad 0 \leqslant k \leqslant n \quad (c_0(\xi) = 1).$$

The *total Chern class* is defined by

$$c(\xi) = 1 + c_1(\xi) + \cdots + c_n(\xi)$$

as usual.

By a parallel argument to the usual case we obtain

Theorem 6.11. *The total Chern classes satisfy the following properties*:

1) *If a real bundle map* $f: \xi \longrightarrow \eta$ *covers a real map* $f: X \longrightarrow Y$ *of base spaces, then* $f^*c(\eta) = c(\xi)$.

2) *If* ξ *and* η *are real vector bundles over* X, *then*

$$c(\xi \oplus \eta) = c(\xi)c(\eta).$$

3) *If* η_n *is the canonical line bundle over the standard projective* n-*space* CP_n, *then* $c(\eta_n) = 1 + x_n$ *where* x_n *is the element in Theorem* 6.2.

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(*Received March 13, 1976*)