GALOIS OBJECTS AS MODULES OVER A HOPF ALGEBRA

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Let $\mathbf{C}$ be the category of coalgebras over a commutative ring R with identity and let H be a Hopf algebra with antipode which is a finitely generated projective R-module. In § 1, we shall define the notion of a Galois H-object in $\mathbf{C}$ as a generalization of that given in [2] and discuss the properties of such objects. In § 2, we shall state several results of Galois objects in the categery of R-algebras which are similar to those in $\mathbf{C}$. In § 3, a homomorphism from the group of Galois H^*-objects to Pic(H) for some Hopf algebra H will be considered, where $H^* = \mathrm{Hom}_R(H, R)$. This is a generalization of [3, Th. 2]. Finally we correct some errors in the previous paper [4].

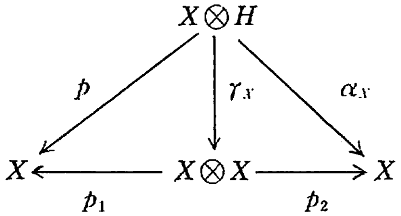
Throughout this paper, R will denote a commutative ring with identity and unadorned $\otimes$ will mean $\otimes_R$. Moreover we shall assume that every ring has an identity which is preserved by every homomorphism, every module is unital and every algebra is an R-algebra. Concerning coalgebras and Hopf algebras we shall use freely the notation and terminology in Sweedler [6]. Finally H will represent always a Hopf algebra with structure maps $(\mu, \eta, \Delta, \varepsilon)$.

1. Let $\mathbf{C}$ be the category of R-coalgebras. In this section we shall give several results which are similar to those stated in [2] for the category of cocommutative R-coalgebras. The tensor product $\otimes$ is the product in the category of cocommutative R-coalgebras (cf. [2]), but in general it is not so in $\mathbf{C}$. Therefore we are obliged to abandon the categorical argument employed in [2].

An H-object in $\mathbf{C}$ is defined to be a pair (X, α), where X is in $\mathbf{C}$ and $\alpha: X \otimes H \longrightarrow X$ in $\mathbf{C}$ such that the diagram below commute

$$\begin{array}{ccc} X\otimes H\otimes H & \xrightarrow{1\otimes\mu} & X\otimes H \\ {\scriptstyle \alpha\otimes 1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ X\otimes H & \xrightarrow[\alpha]{} & X \end{array} \quad \cdot \qquad \begin{array}{ccc} X\otimes R & \xrightarrow{1\otimes\eta} & X\otimes H \\ & \searrow \quad \swarrow{\scriptstyle \alpha} & \\ & X & \end{array}$$

the unlabeled map being the natural isomorphism. Occasionally we shall denote the pair (X, α) by X. If we need the explicit reference to α we shall write $\alpha=\alpha_X$. One may remark here that H itself with $\alpha_H=\mu$ is an H-object. Let $\mathbf{C}^H$ be the category of all H-objects. A map $f\colon X\longrightarrow Y$ in $\mathbf{C}^H$ is a $\mathbf{C}$-map such that $\alpha_Y(f\otimes 1_H)=f\alpha_X$ where 1_H is the identity map from H to H. For an H-object X, we define an R-module homomorphism γ_X: $X\otimes H\longrightarrow X\otimes X$ by $\gamma_X(x\otimes h)=\Delta_X(x)(1\otimes h)=(1_X\otimes\alpha)(\Delta_X\otimes 1_H)(x\otimes h)$ $(x\in X,\ h\in H)$. Then the following diagram commutes



where $p(x\otimes h)=\varepsilon(h)x$, $p_1(x\otimes y)=x\varepsilon_X(y)$ and $p_2(x\otimes y)=\varepsilon_X(x)y$ $(x, y\in X, h\in H)$.

Definition 1.1. Let H be a Hopf algebra. Then X in $\mathbf{C}^H$ will be called a *Galois H-object* in $\mathbf{C}$ if X is a finitely generated projective faithful R-module and the map $\gamma_X\colon X\otimes H\longrightarrow X\otimes X$ is an R-module isomorphism.

Let $\phi\colon G\longrightarrow H$ be a homomorphism of Hopf algebras, X in $\mathbf{C}^H$, and $\alpha_{\phi,X}\colon X\otimes G\longrightarrow X$ the composition

$$X\otimes G\xrightarrow{1\otimes\phi} X\otimes H\xrightarrow{\alpha_X} X.$$

Since $(X, \alpha_{\phi,X})$ is an object in $\mathbf{C}^G$, we can define a functor

$$\mathbf{C}^\phi\colon \mathbf{C}^H\longrightarrow\mathbf{C}^G$$

as follows: $\mathbf{C}^\phi(X)=(X, \alpha_{\phi,X})$ $((X, \alpha)\in\mathbf{C}^H)$, $\mathbf{C}^\phi(f)=f$ $(f\colon X\longrightarrow Y)$.

In the subsequent study of this section, we shall assume that Hopf algebras G and H are cocommutative Hopf algebras. First, we state the following lemma which is easy to be verified.

Lemma 1.2 (cf. [2, Remarks 4. 3 (d)]). *Let $\phi\colon G\longrightarrow H$ be a homomorphism of Hopf algebras, and X in $\mathbf{C}^G$. Viewing H as a left H-module via ϕ, $Y=X\otimes_G H$ is an H-object with the obvious right H-module structure and the coalgebra operations on Y satisfying the following formulae*

$$\Delta_Y(x\otimes h)=\textstyle\sum_{(x),(h)}(x_{(1)}\otimes h_{(1)})\otimes(x_{(2)}\otimes h_{(2)})$$
$$\varepsilon_Y(x\otimes h)=\varepsilon_X(x)\,\varepsilon(h)$$

where $\Delta_X(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$, *and* $\Delta(h)=\sum_{(h)} h_{(1)}\otimes h_{(2)}$ $(x\in X,\ h\in H)$. *Furthermore the diagram*

$$X\otimes G\otimes H \underset{\alpha_X\otimes 1_H}{\overset{\omega_{X,\phi}}{\rightrightarrows}} X\otimes H \xrightarrow{\pi_X} Y=X\otimes_G H$$

is a coequalizer diagram in **C**, *where* $\omega_{X,\phi}(x\otimes g\otimes h)=x\otimes\phi(g)h$ *and* π_X *is the canonical map.*

Theorem 1.3 (cf. [3, Th. 2. 9]). *Let* $\phi: G\longrightarrow H$ *be a homomorphism of cocommutative Hopf algebras, and let* $\tilde{\phi}(X)$ *be defined by the following coequalizer diagram*

$$X\otimes G\otimes H \underset{\alpha_X\otimes 1_H}{\overset{\omega_{X,\phi}}{\rightrightarrows}} X\otimes H \xrightarrow{\pi_X} X\otimes_G H=\tilde{\phi}(X)$$

where $\omega_{X,\phi}$ *and* π_X *are as in* Lemma 1. 2. *Then* $\tilde{\phi}: \mathbf{C}^G\longrightarrow\mathbf{C}^H$ *is the left adjoint of* $\mathbf{C}^\phi: \mathbf{C}^H\longrightarrow\mathbf{C}^G$. *In particular, if* $f: X\longrightarrow\mathbf{C}^\phi(Y)$ *is a* $\mathbf{C}^G$*-map* $(Y\in\mathbf{C}^H)$, *then the corresponding* $\mathbf{C}^H$*-map* $f: X\otimes_G H\longrightarrow Y$, *arising from adjointness, renders the diagram below commutative*

$$\begin{array}{ccc} X\otimes H & \xrightarrow{\pi_X} & X\otimes_G H \\ {\scriptstyle f\otimes 1_H}\downarrow & & \downarrow{\scriptstyle f} \\ Y\otimes H & \xrightarrow[\alpha_Y]{} & Y \end{array}.$$

Proof. Let X be in $\mathbf{C}^G$, and Y in $\mathbf{C}^H$. If $f: X\longrightarrow\mathbf{C}^\phi(Y)=Y$ is in $\mathbf{C}^G$, then we have $f\alpha_X=\alpha_Y(1_Y\otimes\phi)(f\otimes 1_G)$. It follows that $\alpha_Y(f\otimes 1_H)(\alpha_X\otimes 1_H)=\alpha_Y(\alpha_Y\otimes 1_H)(1_Y\otimes\phi\otimes 1_H)(f\otimes 1_G\otimes 1_H)=\alpha_Y(f\otimes 1_H)\,\omega_{X,\phi}$. Thus by Lemma 1. 2, there exists a unique $\mathbf{C}^H$-map $f: X\otimes_G H\longrightarrow Y$ such that $f\pi_X=\alpha_Y(f\otimes 1_H)$. As above we define

$$\Phi: \mathrm{Hom}_{\mathbf{C}^G}(X, \mathbf{C}^\phi(Y))\longrightarrow \mathrm{Hom}_{\mathbf{C}^H}(\tilde{\phi}(X), Y)$$

$$\Psi: \mathrm{Hom}_{\mathbf{C}^H}(\tilde{\phi}(X), Y)\longrightarrow \mathrm{Hom}_{\mathbf{C}^G}(X, \mathbf{C}^\phi(Y))$$

by $\Phi(f)=f$ and $\Psi(g)=g(1_X\otimes\eta_H): X\cong X\otimes R\longrightarrow X\otimes_G H\longrightarrow Y$. Then it is easy to see that $\Phi\Psi=1$ and $\Psi\Phi=1$.

If H is a Hopf algebra with antipode which is a finitely generated projective R-module, then H is called a *finite* Hopf algebra.

The following lemma will be easily shown.

Lemma 1.4 (cf. [2, Prop. 9. 1]). *If H is a finite Hopf algebra, then H is a Galois H-object in $\mathbf{C}$.*

Lemma 1.5. *Let H be a finite Hopf algebra. If X is a Galois H-object in $\mathbf{C}$, then X is a finitely generated projective H-module.*

Proof. Let X be a Galois H-object. Then $\varepsilon_X: X \longrightarrow R$ is an R-module epimorphism and thus R is an R-direct summad of X. Since the isomorphism $\gamma_X: X\otimes H \longrightarrow X\otimes X$ is a right H-module isomorphism, X is an H-module direct summand of $X\otimes X$. By the projectivity of $X\otimes X \cong X\otimes H$ as H-modules, X is a finitely generated projective H-module.

By Lemmas 1. 4 and 1. 5, we have the following

Theorem 1.6 (cf. [2, Th. 2. 20]). *Let $\phi: G \longrightarrow H$ be a homomorphism of finite cocommutative Hopf algebras. If X is a Galois G-object in $\mathbf{C}$, then $\tilde{\phi}(X) = X\otimes_G H$ is a Galois H-object in $\mathbf{C}$.*

Proof. Let X be a Galois G-object in $\mathbf{C}$, and define a map $\rho: X\otimes H \longrightarrow X\otimes X\otimes_G H$ by $\rho(x\otimes h) = \sum_{(x)} x_{(1)}\otimes x_{(2)}\otimes h$ $(x\in X,\ h\in H)$. Since $(1_X\otimes\alpha)(\Delta_X\otimes 1_G): X\otimes G \longrightarrow X\otimes X$ is an isomorphism of right G-modules, ρ is an isomorphism with the inverse $1_X\otimes\varepsilon_X\otimes 1_H$. We consider now the following diagram

$$\begin{array}{ccc} X\otimes H\otimes H & \xrightarrow{\ \delta_1\ } & X\otimes H\otimes H \\ {\scriptstyle \rho\otimes 1_H}\downarrow & & \downarrow{\scriptstyle \delta_2} \\ X\otimes X\otimes_G H\otimes H & \xrightarrow[\ \delta_3\ ]{} & X\otimes X\otimes_G H\otimes X\otimes_G H \end{array}$$

where

$\delta_1\ (x\otimes h\otimes k) = \sum_{(h)} x\otimes h_{(1)}\otimes h_{(2)}\, k$

$\delta_2\ (x\otimes h\otimes k) = \sum_{(x)} x_{(1)}\otimes x_{(2)}\otimes h\otimes x_{(3)}\otimes k$

$\delta_3\ (x\otimes y\otimes h\otimes k) = \sum_{(y),(h)} x\otimes y_{(1)}\otimes h_{(1)}\otimes y_{(2)}\otimes h_{(2)}\, k$

$(x,\ y\in X,\ h,\ k\in H)$. Then it is easy to see that the above diagram commutes, and so δ_1 is an isomorphism by Lemma 1. 4. Since $X\otimes X\otimes_G H\otimes X\otimes_G H$ is a submodule of $X\otimes X\otimes_G H\otimes X\otimes X\otimes_G H$ canonically, δ_2 has the inverse $1_X\otimes\varepsilon_X\otimes 1_H\otimes 1_X\otimes\varepsilon_X\otimes 1_H$, that is, δ_2 is an isomorphism. Therefore δ_3 is an isomorphism and $X\otimes_G H\otimes H \cong (X\otimes_G H)\otimes(X\otimes_G H)$. Clearly $X\otimes_G H$ being finitely generated projective R-module, $X\otimes_G H$ is a Galois H-object in $\mathbf{C}$.

Definition 1.7 (cf. [2, Def. and Remarks 2.22]). Let H be a finite Hopf algebra. We shall denote by $E_{\mathbf{C}}(H)$ the set of $\mathbf{C}^{G}$-isomorphism classes of Galois H-objects in $\mathbf{C}$.

If $\phi: G \longrightarrow H$ is a homomorphism of Hopf algebras, in virture of Th. 1.6, we can define a map $E(\phi): E_{\mathbf{C}}(G) \longrightarrow E_{\mathbf{C}}(H)$ by $E(\phi)((X)) = (\tilde{\phi}(X))$, where (X) means a $\mathbf{C}^{G}$-isomorphism class of X in $\mathbf{C}^{G}$. If $\phi: G \longrightarrow H$ and $\psi: H \longrightarrow J$ are homomorphisms of Hopf algebras, then we can check easily that $\mathbf{C}^{\psi\phi} \cong \mathbf{C}^{\phi}\mathbf{C}^{\psi}: \mathbf{C}^{J} \longrightarrow \mathbf{C}^{G}$. Moreover by the uniqueness of adjoints (or an easy direct computation), we obtain a natural equivalence of functors $\widetilde{\psi\phi} \cong \tilde{\psi}\tilde{\phi}: \mathbf{C}^{G} \longrightarrow \mathbf{C}^{J}$, which gives rise to the equality $E(\psi\phi) = E(\psi)E(\phi): E_{\mathbf{C}}(G) \longrightarrow E_{\mathbf{C}}(J)$.

We insert here the following which will be used in § 2.

Lemma 1.8. (a) *Let X be a Galois G-object in $\mathbf{C}$. Then $\tilde{\eta}_{H}\tilde{\varepsilon}_{G}(X) \cong H$ in $\mathbf{C}^{H}$.*

(b) *Let $\phi_i: G_i \longrightarrow H_i$ be homomorphisms of cocommutative Hopf algebras and let X_i be Galois G_i-object ($i=1, 2$). Then $\widetilde{\phi_1 \otimes \phi_2}(X_1 \otimes X_2) \cong \tilde{\phi}_1(X_1) \otimes \tilde{\phi}_2(X_2)$ in $\mathbf{C}^{H_1 \otimes H_2}$.*

Proof. It suffices to prove (a) only. Since R is the only Galois R-object over R, we have $\tilde{\varepsilon}_{G}(X) \cong R$ in $\mathbf{C}^{R}$. Thus $\tilde{\eta}_{H}\tilde{\varepsilon}_{G}(X) \cong \widetilde{\eta_{H}\varepsilon_{G}}(R) \cong \tilde{\eta}_{H}(R) \cong H$ in $\mathbf{C}^{H}$.

By Th. 1.6, we readily obtain the following

Theorem 1.9 (cf. [2, Th. 3.9 (a)]). *Let H be a finite, commutative, cocommutative Hopf algebra. Then $E_{\mathbf{C}}(H)$ is an abelian semi-group with the addition*

$$(X)+(Y) = (\tilde{\mu}(X \otimes Y)) = ((X \otimes Y) \otimes_{H \otimes H} H)$$

where H is a left $H \otimes H$-module by μ and $X \otimes Y$ is in $\mathbf{C}^{H \otimes H}$ by the natural way. (H) is the zero element in $E_{\mathbf{C}}(H)$.

2. In this section we shall give some statements which are duals of those in § 1.

Let H be a finite Hopf algebra. If (S, α) is a right H^*-comodule algebra, then S has a left H-module structure which is defined by

$$h(x) = \sum_{(x)} x_{(1)} \otimes \langle h, x_{(2)} \rangle \qquad (x \in S, \ h \in H)$$

where $\langle\ ,\ \rangle: H \otimes H^* \longrightarrow R$ denotes the duality pairing. Thus S is a left H-module algebra. Conversely, if S is a left H-module algebra, then we obtain a map $\alpha: S \longrightarrow S\otimes H^*$;

$$\alpha(S)=\sum_{i=1}^{h} h_i S\otimes h_i^* \qquad (s\in S,\ h_i\in H,\ h_i^*\in H^*)$$

where $\{h_i,\ h_i^*\}_{1\leq i\leq n}$ is an R-projective coordinate system of H. Since S is a left H-module algebra, S is a right H^*-comodule algebra with respect to α (cf. [4, p. 142]).

In the subsequent study, we shall assume, unless explicitly stated otherwise, that Hopf algebras are finite, commutative, cocommutative and every right H^*-comodule algebra (resp. left H-module algebra) will be regarded as a left H-module algebra (resp. right H^*-comodule algebra) in the above way.

The following definition is slightly different from [2, Def. 7. 3].

Definition 2.1. Let H be a Hopf algebra, and X a right H-comodule algebra. X will be called a *Galois H-object* if

(1) X is a faithfully flat R-module.

(2) γ_X: $X\otimes X \longrightarrow X\otimes H$ defined by $\gamma_X(x\otimes y)=(x\otimes 1)\ \alpha(y)$ is an R-module isomorphism.

The following theorem can be proved by the same method as in the proof of [2, Th. 9. 3].

Theorem 2.2. *Let H be a finite commutative Hopf algebra, and S a right H-comodule algebra. Then the following conditions are equivalent*:

(a) *S is a Galois H-object.*

(b) *S is a finitely generated faithful projective R-module and the mapping* f: $S\sharp H^* \longrightarrow \mathrm{End}_R(S)$ *defined by* $f(s\sharp h^*)(x)=sh^*(x)$ *is an R-module isomorphism* $(s,\ x\in S,\ h^*\in H^*)$.

Let $\mathbf{A}$ be the category of R-algebras and let $\mathbf{A}_0$ (resp. $\mathbf{C}_0$) be the full subcategory of $\mathbf{A}$ (resp. $\mathbf{C}$) whose objects are those of $\mathbf{A}$ (resp. $\mathbf{C}$) which are finitely generated projective R-modules. Then the functor $-^*$: $\mathbf{C}_0^{op} \longrightarrow \mathbf{A}_0$ is an isomorphism of categories, where $-^* = \mathrm{Hom}_R(-,\ R)$. From this fact, the following is immediate ([cf, 6, 1. 1 and 2, p. 34]).

Lemma 2.3. (a) *If* $(C,\ \Delta_C,\ \varepsilon_C)$ *is in* $\mathbf{C}_0$, *then* $(C^*,\ \Delta_C^*,\ \varepsilon_C^*)$ *is in* $\mathbf{A}_0$.

(b) *If* $(A,\ \mu_A,\ \eta_A)$ *is in* $\mathbf{A}_0$, *then* $(A^*,\ \mu_A^*,\ \eta_A^*)$ *is in* $\mathbf{C}_0$.

(c) *If* $(X,\ \alpha)$ *is in* $\mathbf{C}_0^H$, *then* $(X^*,\ \alpha^*)$ *is in* $\mathbf{A}_0^{H^*}$.

(d) *If* $(Y,\ \beta)$ *is in* $\mathbf{A}_0^H$, *then* $(Y^*,\ \beta^*)$ *is in* $\mathbf{C}_0^{H^*}$.

Here $\mathbf{A}_0^{(\ )}$ *is the category of ( )-comodule algebras in* $\mathbf{A}$.

By Th. 2. 2 and Lemma 2. 3, we have the following

Proposition 2.4. *Let H be a finite cocommutative Hopf algebra. Then X is a Galois H-object in $\mathbf{C}_0$ if and only if X^* is a Galois H^*-object in $\mathbf{A}_0$.*

Let H be finite cocommutative Hopf algebra and let $E_{\mathbf{A}_0}(H^*)$ be the set of isomorphism classes of Galois H^*-objects in $\mathbf{A}_0$. Then it is clear that the mapping $*: E_{\mathbf{C}_0}(H) \longrightarrow E_{\mathbf{A}_0}(H^*)$ defined by $*((X))=(X^*)$ gives the set of isomorphism. Since $E_{\mathbf{C}_0}(H)$ is an abelian semi-group by Th.1.9, the mapping $*$ defines the abelian semi-group structure on $E_{\mathbf{A}_0}(H^*)$. If $\mathbf{A}_{0,c}$ is the full subcategory of $\mathbf{A}_0$ whose objects are those of $\mathbf{A}_0$ which are commutative R-algebras, then by Th.1.4, Th.1.9 and the functor $-^*$, we can see that the semi-group structure on $E_{\mathbf{A}_c}(H^*)$ coinside with the group structure of [2, Chap. I].

Next we shall prove that $E_{\mathbf{A}_0}(H^*)$ has a group structure for finite, commutative, cocommutative Hopf algebra H. First, we have the following

Lemma 2.5. *Let ϕ: $G \longrightarrow H$ be a homomorphism of Hopf algebras and let X be a Galois G^*-object in $\mathbf{A}_0$. Then $\phi(X)=\mathrm{Hom}_G(H, X)$ is a Galois H^*-object in $\mathbf{A}_0$, where the multiplication on $\phi(X)$ is defined by the formula*

$$(f.g)(h)=\sum_{(h)} f(h_{(1)})g(h_{(2)})$$

($\Delta(h)=\sum_{(h)} h_{(1)}\otimes h_{(2)}$) and H acts on $\phi(X)$ via $(hf)(k)=f(kh)$ $(h, k\in H)$.

Proof. Let X be a Galois G^*-object in $\mathbf{A}_0$. Then by Prop.2.4, X^* is a Galois G-object in $\mathbf{C}_0$. Thus by Th.1.6, $X^*\otimes_G H$ is a Galois H-object in $\mathbf{C}_0$. Applying the functor $-^*$ to $X^*\otimes_G H$, we obtain

$$\mathrm{Hom}_R(X^*\otimes_G H, R)\cong \mathrm{Hom}_G(H, X)$$

as Galois H^*-objects in $\mathbf{A}_0$. The rest of the proof will be clear.

Lemma 2.6. *Let ϕ: $G\longrightarrow H$ be an epimorphism of Hopf algebras and let X be a Galois G^*-object in $\mathbf{A}_0$. Then*

$$\phi(X)=X^{\ker(\phi)}$$

in $\mathbf{A}_0^H$, where $\ker(\phi)=\{g\in G \mid (1\otimes\phi)\Delta_G(g)=g\otimes 1$ *in* $G\otimes H\}$, $X^{\ker(\phi)}=\{x\in X \mid gx=\varepsilon_G(g)x$ *for all* $g\in\ker(\phi)\}$, *and the action of $H=\phi(G)$ on $X^{\ker(\phi)}$ is given by $\phi(g)(x)=g(x)$.*

Proof. Let g be in $\ker(\phi)$. Then an easy computation show that $\phi(g)=\varepsilon_G(g)$. We define the maps j: $X^{\ker(\phi)}\longrightarrow\phi(X)$ and j': $\phi(X)\longrightarrow X^{\ker(\phi)}$ by $j(x)(\phi(g))=g(x)$ and $j'(f)=f(1)$, respectively. It is easy to verify that j, j' are morphisms in $\mathbf{A}_0^H$ and $jj'=1$, $j'j=1$.

By Lemmas 2. 5 and 2. 6, we have the following

Corollary 2.7. *Let ϕ : $G \longrightarrow H$ be an epimorphism of Hopf algebras and let X, Y be Galois G^*-objects in $\mathbf{A}_0$. Then $\phi(x)=(\widetilde{\phi}(X^*))^*$ in $\mathbf{A}_0^H$ and*

$$(X)+(Y)=(\mu(X\otimes Y))=((X\otimes Y)^{\ker(\mu)}) \text{ in } E_{\mathbf{A}_0}(G^*).$$

Lemma 2.8. *Let ϕ_i : $G_i \longrightarrow H_i$ be homomorphisms of finite, commutative, cocommutative Hopf algebras and let X_i be Galois G_i^*-objects ($i=1, 2$). Then $(\phi_1\otimes\phi_2)(X_1\otimes X_2)\cong\phi_1(X_1)\otimes\phi_2(X_2)$ in $\mathbf{A}_0^{H_1\otimes H_2}$.*

Proof. Note that $\widetilde{(\phi_1\otimes\phi_2)}(X_1^*\otimes X_2^*)\cong\widetilde{\phi}_1(X_1^*)\otimes\widetilde{\phi}_2(X_2^*)$ and $(\widetilde{\phi}_1(X_1^*))^*=\phi(X_1)$.

Recently, M. Takeuchi has pointed out the following lemma which is useful in our study, and he kindly permitted us to cite it here.

Lemma 2.9. *Let X be a Galois H^*-object. Then $X^H=R$.*

Proof. Let x be an arbitrary element in X^H. Then it is easy to see that $\alpha_X(x)=x\otimes 1$ in $X\otimes H^*$. Since $X\otimes X\cong X\otimes H^*$ as right H-modules (H acts on the second factor), we have $X\otimes X^H\cong X\otimes H^{*H}\cong X\otimes R$. Now the faithful flatness of X shows that $X^H=R$.

Theorem 2.10. *Let X be a Galois H^*-object in $\mathbf{A}_0$ and let H be a finite, commutative, cocommutative Hopf algebra. Then*

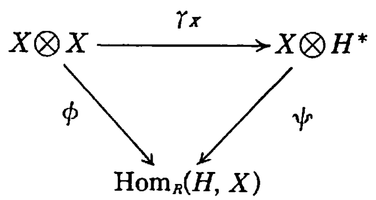
$$(\lambda(X))+(X)=(H^*)$$

where λ is the antipode of H. That is, $E_{\mathbf{A}_0}(H^*)$ is an abelian group.

Proof. By Cor. 2. 7, we have

$(\lambda(X))+(X)=(\mu(\lambda(X)\otimes X))=(\mu(\lambda\otimes 1)(X\otimes X))=((X\otimes X)^{\ker(\mu(\lambda\otimes 1))})$.

First, we assume that H is a free R-module with the basis $\{h_1=1, h_2, \cdots, h_m\}$, and consider the following commutative diagram



where $\phi(x\otimes y)(h)=x(hy)$ and $\psi(x\otimes h^*)(h)=h^*(h)x$ ($x, y\in X$, $h\in H$,

$h^*\in H^*$). Since X is a Galois H^*-object, there exist elements x_{ij} and y_{ij} in X such that

$$\gamma_X(\sum_{i=1}^n x_{ij}\otimes y_{ij})=1\otimes h_j^* \qquad (j=1, 2, \cdots, m)$$

and thus

$$\sum_{i=1}^n x_{ij}(h_k y_{ij})=\phi(\sum_{i=1}^n x_{ij}\otimes y_{ij})(h_k)=(\psi\gamma_X(\sum_{i=1}^n x_{ij}\otimes y_{ij}))(h_k)$$
$$=\psi(1\otimes h_j^*)(h_k)=h_j^*(h_k)=\delta_{j,k} \text{ (Kronecker's delta).}$$

Then it is easy to see that

$$\sum_{(h)}\sum_{i=1}^n h_{(1)}x_{ij}\otimes h_{(2)}y_{ij}=\varepsilon(h)\sum_{i=1}^n x_{ij}\otimes y_{ij}$$

for all $h\in H$. Noting that $\mathrm{im}(\Delta)=\ker(\mu(\lambda\otimes 1))$, we have

$$\sum_{i=1}^n x_{ij}\otimes y_{ij} \text{ in } (X\otimes X)^{\ker(\mu(\lambda\otimes 1))}.$$

By Lemma 2. 9, we can define a map

$$\tau: (X\otimes X)^{\ker(\mu(\lambda\otimes 1))} \longrightarrow \mathrm{Hom}_R(H, R)$$

by $\tau=\phi$. Then τ is an H-module and algebra homomorphism. Moreover $\tau(\sum_{i=1}^n x_{ij}\otimes y_{ij})=h_j^*$, τ is an epimorphism. A counting of ranks then yields that τ is an isomorphism. In general, using the localization argument, we have $(X\otimes X)^{\ker(\mu(\lambda\otimes 1))}\cong \mathrm{Hom}_R(H, R)$ in $\mathbf{A}_0^{H^*}$.

The next theorem is a generalization of [3, Prop. 2].

Theorem 2.11. *Let H be a finite, commutative, cocommutative Hopf algebra. Then the direct sum $E_{\mathbf{A}_0}(H^*)\oplus E_{\mathbf{A}_0}(H^*)$ is a direct summand of $E_{\mathbf{A}_0}(H^*\otimes H^*)$.*

Proof. Let i, j : $H\otimes H\longrightarrow H$ be homomorphisms defined by $i(h\otimes k)=(1\otimes\varepsilon)(h\otimes k)$, $j(h\otimes k)=(\varepsilon\otimes 1)(h\otimes k)$, respectively. Let f: $E_{\mathbf{C}_0}(H)\oplus E_{\mathbf{C}_0}(H)\longrightarrow E_{\mathbf{C}_0}(H\otimes H)$ and g: $E_{\mathbf{C}_0}(H\otimes H)\longrightarrow E_{\mathbf{C}_0}(H)\oplus E_{\mathbf{C}_0}(H)$ be homomorphisms defined by

$f((X), (Y))=(X\otimes Y)$ and $g((Z))=((\tilde{i}(Z)), (\tilde{j}(Z)))$, respectively. Then by Lemma 1. 8 (a), we have $\tilde{i}(X\otimes Y)\cong\widetilde{(1\otimes\varepsilon)}(X\otimes Y)\cong X\otimes\tilde{\varepsilon}(Y)\cong X$ and $\tilde{j}(X\otimes Y)\cong Y$ in $\mathbf{C}_0^H$. Thus $gf=1$. The rest of the proof will be clear.

3. Throughout this section we assume again that H is a finite Hopf algebra.

Theorem 3.1 (cf. [2, Th. 9. 6]). *Every Galois H^*-object X is a projective H-module.*

Proof. Noting that H^* is a left H-module via $(hf)(k)=f(kh)$ $(h, k\in H, f\in H^*)$, $\gamma_X: X\otimes X \longrightarrow X\otimes H^*$ is a left H-module isomorphism, where the left H-module structure of $X\otimes X$ is given by $h(x\otimes y)=x\otimes hy$. Since X is a projective R-module, we can apply [5, Lemma 2 and Prop. 3] to obtain that $X\otimes X$ is a projective left H-module. Also, by Th. 2. 2, X is a direct summand of $X\otimes X$ as left H-modules, and therefore X is a projective H-module.

Now let H be a commutative Hopf algebra, and S an H-module algebra. Then for the smash product $S\sharp H$, we consider the following condition:

($\sharp$) If $\sum_{i=1}^n s_i\sharp hh_i=(1\sharp h)(\sum_{i=1}^n s_i\sharp h_i)$ for all $h\in H$, then every s_i is in R. If H is a such a Hopf algebra as in [4, Remark 1. 6 (1) or (3)] and if S is an H-module algebra, then the smash product $S\sharp H$ satisfies the condition ($\sharp$).

For such a Hopf algebra, we have the following

Theorem 3.2. *Let H be a finite, commutative, cocommutative Hopf algebra, and X a Galois H^*-object. If $X\sharp H$ satisfies the condition* ($\sharp$), *then X is a rank* 1 *projective H-module.*

Proof. By Th. 2. 2, f: $X\sharp H \longrightarrow \mathrm{End}_R(X)$ is an isomorphism and $f(H)$ is in $\mathrm{Hom}_H(X, X)$. Let g be an arbitrary element in $\mathrm{Hom}_H(X, X)$. Since f is an isomorphism, there exists an element $\sum_{i=1}^n x_i\sharp h_i$ in $X\sharp H$ such that $f(\sum_{i=1}^n x_i\sharp h_i)=g$. Then an easy computation shows that $(1\sharp h)(\sum_{i=1}^n x_i\sharp h_i)=\sum_{i=1}^n x_i\sharp hh_i$ for all $h\in H$. Therefore by the condition ($\sharp$), we have $x_i\in R$ for all i. Hence $H=\mathrm{Hom}_H(X, X)$. Since X is finitely generated projective H-module by Th. 3. 1, we have $X\otimes_H X^*\cong \mathrm{Hom}_H(X, X)\cong H$.

In case H is a commutative Hopf algebra, we can consider the abelian group $\mathrm{Pic}(H)$ of isomorphism classes of projective H-modules of rank 1, where $\mathrm{cl}(P)+\mathrm{cl}(Q)$ is defined to be $\mathrm{cl}(P\otimes_H Q)$ $(\mathrm{cl}(P), \mathrm{cl}(Q)\in \mathrm{Pic}(H))$. The inverse element $-\mathrm{cl}(P)$ is $\mathrm{cl}(\mathrm{Hom}_H(P, H))$([1, §5, no. 4]).

Theorem 3.3. *Let H be a finite, commutative, cocommutative Hopf algebra such that $H\cong H^*$ as H-module. Assume that for any Galois H^*-object X, $X\sharp H$ satisfies the condition* ($\sharp$). *Then the map θ: $E_{A_0}(H^*)\longrightarrow \mathrm{Pic}(H)$ defined by $\theta((X))=\mathrm{cl}(X)$ is a homomorphism of abelian groups and*

$$0 \longrightarrow \text{Harr-}H^2(R, H) \longrightarrow E_{A_0}(H^*) \longrightarrow \text{Pic}(H)$$

is an exact sequence of abelian groups, where Harr-$H^2(R, H)$ *is the generalized* Harrison *cohomology group of second order defined in* [4].

Proof. By. Th. 3. 2, θ is well defined. If (X), (Y) are in $E_{A_0}(H^*)$, then we can define f: $(X\otimes Y)\otimes_{H\otimes H} H \longrightarrow X\otimes_H Y$ by $f(x\otimes y\otimes h)=xh\otimes y(=x\otimes hy)$. Then f is well defined and the map $g(x\otimes y)=x\otimes y\otimes 1$ is the inverse map of f. Thus f is an isomorphism. Hence θ is a homomorphism of abelian groups. The rest of the proof will be clear by [4, Th. 2. 8].

4. The statement of [4, Lemma 2. 4] is incorrect, and should be read as follows:

Lemma. *Let H be a finite Hopf algebra, and S a faithfully flat R-module. Then S is a Galois H^*-object in the sense of* Def. 2. 4 (*in this paper*) *if and only if S is an H-module algebra such that the mapping ϕ defined in* [4, p. 142] *is a $\otimes^2 H$-module isomorphism.*

By the way, we claim that as was mentioned in [4, p. 138, lines 12–14] Th. 2. 6, and Prop. 2. 7 of [4] were stated under the hypotheses that H is a finite, commutative, cocommutative Hopf algebra which is isomorphic to H^* as H-modules.

Finally, if we replace the definition of Galois objects in [4] by Def. 2. 4 in this paper, then the results in [4, § 2] are still valid.

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