NOTE ON EXCHANGE PROPERTY

Dedicated to Professor Kiiti Morita on his 60th birthday

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Every module concerned in the present note will be a unital right module over a ring R with 1. Let η be a cardinal number. Then, a module M is said to have the *η-exchange property*, if for any direct sum $A=\bigoplus_{B\in\mathfrak{B}} B$ such that $\eta > |\mathfrak{B}|$ (the cardinal of the set $\mathfrak{B}$) and $A=M\oplus M'$ there exist submodules $B'\subseteq B$ such that $A=M\oplus(\bigoplus_{B\in\mathfrak{B}}B')$. We say that M has the *finite exchange property* if M has the η-exchange property for any finite η. These properties were defined and investigated by Crawley and Jónnson [1], and also studied by Harada [3] in connection with Krull-Schmidt-Azumaya theorem. Finally, M is said to have the *η-exchange property w. r. t. completely indecomposable modules*, if for any direct sum A of completely indecomposable modules $B\in\mathfrak{B}$ such that $\eta\geqslant|\mathfrak{B}|$ and $A=M\oplus M'$ there exist some $B'\subseteq B$ such that $A=M\oplus(\bigoplus_{B\in\mathfrak{B}}B')$.

In our subsequent study, $\mathfrak{N}$ will represent always a set of completely indecomposable modules. We consider a sequence $\{N_i\}_1^\infty$ of modules N_i in $\mathfrak{N}$ and non-isomorphisms $f_i: N_i \longrightarrow N_{i+1}$ $(i=1, 2, \cdots)$. The sequence $\{f_i\}_1^\infty$ will be called a *sequence of non-isomorphisms in* $\mathfrak{N}$. Such a sequence is said to be *proper* if $N_i \neq N_j$ for $i\neq j$. If for any $m\in N_1$ there exists a natural number n (depending on m) such that $f_n\cdots f_1(m)=0$, then the sequence $\{f_i\}_1^\infty$ is said to be *locally T-nilpotent*. If every sequence (resp. proper sequence) of non-isomorphisms in $\mathfrak{N}$ is locally T-nilpotent then $\mathfrak{N}$ is defined to be *locally T-nilpotent* (resp. *locally semi-T-nilpotent*). Let $\mathfrak{N}=\bigcup_{\rho\in P}\mathfrak{N}(\rho)$ be the partition of $\mathfrak{N}$ into the isomorphism classes. Moreover, let $\mathfrak{N}'=\bigcup_{\rho\in P'}\mathfrak{N}(\rho)$ and $\mathfrak{N}''=\bigcup_{\rho\in P''}\mathfrak{N}(\rho)$ where $P'=\{\rho\in P\mid|\mathfrak{N}(\rho)|<\aleph_0\}$ and $P''=\{\rho\in P\mid|\mathfrak{N}(\rho)|\geqslant\aleph_0\}$. In their paper [4] Harada and Ishii proved that if $\mathfrak{N}'$ is locally T-nilpotent then $\bigoplus_{N\in\mathfrak{N}'}N$ has the $\aleph_0$-exchange property. The purpose of this note is to prove the following theorems.

Theorem 1. *If $\mathfrak{N}'$ is locally semi-T-nilpotent then $\bigoplus_{N\in\mathfrak{N}'}N$ has the $\aleph_0$-exchange property.*

Theorem 2. *If $\mathfrak{N}$ is locally semi-T-nilpotent and every N in $\mathfrak{N}$ is finitely generated, then $\bigoplus_{N\in\mathfrak{N}}N$ has the $\aleph_0$-exchange property.*

In advance of proving our theorems, we claim that the statement of

[4, Lemma 8] is obscure and should be read as follows:

Lemma 1. *Assume that a module M has the finite exchange property. If $M \oplus M' = \oplus_{i=1}^{\infty} A_i$ and $K_i = \oplus_{\iota=i}^{\infty} A_\iota$, then there exist respective direct summands A_i' and K_{i+1}' of A_i* and *K_{i+1} such that $M \oplus M' = M \oplus (\oplus_{i=1}^{n} A_i') \oplus K_{n+1}'$ for any n.*

Proof. For the sake of completeness, we shall give here the proof. Assume that we have found A_i' and K_{i+1}' ($i=1, 2, \cdots, n$) such that $M \oplus M' = M \oplus (\oplus_{i=1}^{m} A_i') \oplus K_{m+1}'$ for all $m \leqslant n$. We may set $K_{n+1} = K_{n+1}' \oplus K_{n+1}''$ with $K_{n+1}'' \subseteq M \oplus (\oplus_{i=1}^{n} A_i')$. Then, $M \oplus (\oplus_{i=1}^{n} A_i') = K_{n+1}'' \oplus L$ with some L. Let $A_i = A_i' \oplus A_i''$. Since $M \approx (\oplus_{i=1}^{n} A_i'') \oplus K_{n+1}''$, K_{n+1}'' has the finite exchange property by [1, Lemma 3.10]. Then, by $K_{n+1} = A_{n+1} \oplus K_{n+2}$ there exist respective direct summands A_{n+1}' and K_{n+2}' of A_{n+1} and K_{n+2} such that $K_{n+1} = K_{n+1}'' \oplus A_{n+1}' \oplus K_{n+2}'$. Thus, $M \oplus M' = M \oplus (\oplus_{i=1}^{n} A_i') \oplus K_{n+1}' = K_{n+1}'' \oplus L \oplus K_{n+1}' = K_{n+1}'' \oplus A_{n+1}' \oplus K_{n+2}' \oplus L = M \oplus (\oplus_{i=1}^{n+1} A_i') \oplus K_{n+2}'$.

Proof of Theorem 1. Let $M = \oplus_{N \in \mathfrak{N}'} N = \oplus_{\rho \in P'} (\oplus_{N \in \mathfrak{N}(\rho)} N)$. Then M has the finite exchange property by [6, Prop. 1.7]. Assume $A = \oplus_{i=1}^{\infty} A_i = M \oplus M'$, and put $K_i = \oplus_{\iota=i}^{\infty} A_\iota$. By Lemma 1, there exist some A_i' and K_{i+1}' such that $A_i = A_i' \oplus A_i''$, $K_{i+1} = K_{i+1}' \oplus K_{i+1}''$ and $A = M \oplus (\oplus_{i=1}^{n} A_i') \oplus K_{n+1}'$ for any n. Since $(\oplus_{i=1}^{n} A_i'') \oplus K_{n+1}'' \approx M$, by Kanbra theorem and Krull-Schmidt-Azumaya theorem we have

$$(*) \qquad A_i'' = \oplus_{\rho \in P'} (\oplus_{B \in \mathfrak{B}(i,\rho)} B)$$

where every $B \in \mathfrak{B}(i, \rho)$ is isomorphic to $N \in \mathfrak{N}(\rho)$ and $\sum_i |\mathfrak{B}(i, \rho)| \leqslant |\mathfrak{N}(\rho)| < \aleph_0$. Suppose $M^* = M \oplus (\oplus_{i=1}^{\infty} A_i') \neq A$. Then, noting that $A = (\oplus_{i=1}^{\infty} A_i') \oplus (\oplus_{i=1}^{\infty} A_i'')$, we can find i_1 and ρ_1 such that $\mathfrak{B}(i_1, \rho_1)$ contains some B_1 which is not contained in M^*. Now, let $a_1 \in B_1 \backslash M^*$, and consider the projection $p: A \longrightarrow K_{m+1}'$ with respect to the decompostion $A = M \oplus (\oplus_{i=1}^{m} A_i') \oplus K_{m+1}'$ where $m = i_1$. Then $x_1 = p(a_1) \notin M^*$. Writing x_1 in $K_{m+1} = (\oplus_{i=m+1}^{\infty} A_i') \oplus (\oplus_{i=m+1}^{\infty} A_i'')$, we have $x_1 = \sum_i x_i' + \sum_i x_i''$. There exists then some $i_2 > i_1$ such that $x_{i_2}'' \notin M^*$. From (*) we can find some $\mathfrak{B}(i_2, \rho_2)$ which contains B_2 such that $q(x_1) \in B_2 \backslash M^*$, where $q: A \longrightarrow B_2$ is the projection with respect to the decomposition $A = (\oplus_{i=1}^{\infty} A_i') \oplus (\oplus_{\rho \in P'} (\oplus_{B \in \mathfrak{N}(i,\rho)} B)$. Putting $g_1 = qp | B_1 : B_1 \longrightarrow B_2$, we have $a_2 = g_1(a_1) \in B_2 \backslash M^*$. Repeating the same argument to a_2 and so on, we can find $g_k: B_k \longrightarrow B_{k+1}$ ($B_k \in \mathfrak{B}(i_k, \rho_k)$, $i_k < i_{k+1}$) such that $g_k \cdots g_1(a_1) \notin M^*$. Since $\sum_i |\mathfrak{B}(i, \rho_1)| < \aleph_0$, there exists a natural number k_1 such that $f_1 = g_{k_1} \cdots g_1 : B_1 \longrightarrow B_{k_1+1}$ is a non-isomorphism. Repeating the same procedure, we can find natural numbers $k_1 < k_2 < \cdots$ such that $f_n = g_{k_n} \cdots g_{k_{n-1}+1}$ are non-isomorphisms. Obviously, $\{f_n\}_1^\infty$ may be regarded as a sequence of non-isomorphisms in $\mathfrak{N}$ and $f_n \cdots f_1(a_1) \neq 0$ for any n. This contradiction shows that $M \oplus (\oplus_{i=1}^\infty A_i') = A$, namely, M has the $\aleph_0$-exchange property.

Corollary 1. *If $\mathfrak{N}$ is locally semi-T-nilpotent and $|P''| < \aleph_0$, then $M = \oplus_{N \in \mathfrak{N}} N$ has the $\aleph_0$-exchange property.*

Proof. For every $\rho \ni P''$, $\mathfrak{N}(\rho)$ is locally (semi-) T-nilpotent, and so $M_\rho = \oplus_{N \in \mathfrak{N}(\rho)} N$ has the $\aleph_0$-exchange property by [4, Lemma 4]. Hence, by [1, Lemma 3. 10], $\oplus_{\rho \in P''} M_\rho$ has the same property. Combining this with Theorem 1, we see that $M = (\oplus_{N \in \mathfrak{N}'} N) \oplus (\oplus_{\rho \in P''} M_\rho)$ has the $\aleph_0$-exchange property again by [1, Lemma 3. 10].

Proof of Theorem 2. Let $A = \oplus_{i=1}^\infty A_i = M \oplus M'$. As in the proof of Theorem 1, we can find A_i' and K_{i+1}' such that $A_i = A_i' \oplus A_i''$, $\oplus_{t=i+1}^\infty A_t = K_{i+1} = K_{i+1}' \oplus K_{i+1}''$ and $A = M \oplus (\oplus_{i=1}^n A_i') \oplus K_{n+1}'$ for any n. Let $p: A \longrightarrow \oplus_{i=1}^\infty A_i''$ be the projection with respect to the decomposition $A = (\oplus_{i=1}^\infty A_i') \oplus (\oplus_{i=1}^\infty A_i'')$. First we claim that if $\mathfrak{F}$ is an arbitrary finite subset of $\mathfrak{N}$ then $p(\oplus_{N \in \mathfrak{F}} N)$ is a direct summand of $\oplus_{i=1}^\infty A_i'$. Since $M^* = \oplus_{N \in \mathfrak{F}} N$ is finitely generated, $M^* \subseteq (\oplus_{i=1}^m A_i') \oplus (\oplus_{i=1}^m A_i'')$ for some m. Accordingly, if $q: A \longrightarrow (\oplus_{i=1}^m A_i'') \oplus K_{i+1}''$ is the projection with respect to the decomposition $A = (\oplus_{i=1}^m A_i') \oplus K_{m+1}' \oplus (\oplus_{i=1}^m A_i'') \oplus K_{m+1}''$ then $p(M^*) = q(M^*)$. Since $q|M$ is a monomorphism, $q(M) = q(M^*) \oplus L = p(M^*) \oplus L$ with some L, and so $p(M^*)$ is a direct summand of $(\oplus_{i=1}^m A_i''$ and hence) $\oplus_{i=1}^\infty A_i''$. Next, as in the proof of Theorem 1, we have

$$(*) \qquad A_i'' = \oplus_{\rho \in P} (\oplus_{B \in \mathfrak{B}(i,\rho)} B)$$

where every $B \in \mathfrak{B}(i, \rho)$ is isomorphic to $N \in \mathfrak{N}(\rho)$. Since $\mathfrak{N}$ is locally semi-T-nilpotent, $p(M)$ itself is a direct summand of $\oplus_{i=1}^\infty A_i''$ by [3, Th. 3. 2. 5]. Combining this with the fact that $p(M)$ has the $\aleph_0$-exchange property w. r. t. completely indecomposable modules (cf. [2, Cor. 2 of Prop. 1]), it follows $\oplus_{i=1}^\infty A_i'' = p(M) \oplus (\oplus_{i=1}^\infty A_i''')$ with some $A_i''' \subseteq A_i''$, and so $A = (\oplus_{i=1}^\infty A_i') \oplus p(M) \oplus (\oplus_{i=1}^\infty A_i''')$. Noting that $p|M$ is a monomorphism, we obtain eventually $A = M \oplus (\oplus_{i=1}^\infty (A_i' \oplus A_i'''))$.

Corollary 2. *Let S be a direct sum of semi-perfect modules. Then, S is semi-perfect if and only if S has the $\aleph_0$-exchange property.*

Proof. By [3, Cor. 5. 1. 13], S is the direct sum of indecomposable semi-perfect modules: $S = \oplus_{T \in \mathfrak{T}} T$. If S is semi-perfect then every T is cyclic (cf. for instance [3, Th. 5.2.4′]) and $\mathfrak{T}$ is locally semi-T-nilpotent by [3, Cor. 5.1.13]. Hence, S has the $\aleph_0$-exchange property by Theorem 2. Conversely, if S has the $\aleph_0$-exchange property then $\mathfrak{T}$ is locally semi-T-nilpotent by [8, Th. 1], and so S is semi-perfect by [3, Cor. 2.2.2].

Corollary 3 (cf. [7, Th. 8]). *Assume that R is a direct sum of indecomposable right ideals. Then a projective module S is semi-perfect if and only if S has the $\aleph_0$-exchange property.*

Proof. By the validity of Corollary 2, it remains only to prove the "if" part. To our end, it suffices to prove that S is a direct sum of semi-perfect modules. By Kaplansky theorem, S is a direct sum of countably generated projective modules. In what follows, we may assume therefore that S is a direct summand of a countably generated free module $F=\bigoplus_{i=1}^{\infty} F_i$, where $F_{iR}\approx R_R$. Then $F=S\oplus(\bigoplus_{i=1}^{\infty} F_i')$, where $F_i=F_i'\oplus F_i''$. Since $S\approx\bigoplus_{i=1}^{\infty} F_i''$, each F_i'' has the $\aleph_0$-exchange property by [1, Lemma 3.10], and then one will easily see that F_i'' is a direct sum of indecomposable modules. Hence, $S=\bigoplus_{j\in\mathfrak{J}} S_j$ with indecomposable S_j. Again by [1, Lemma 3.10], each S_j has the $\aleph_0$-exchange property, and so S_j is completely indecomposable by [5, Prop. 1]. Therefore, S_j is semi-perfect by [3, Th. 5.2.4′].

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