

ON GENERALIZATIONS OF P_1 -RINGS AND *gsr*-RINGS

Dedicated to Professor Yukiyoji Kawada on his 60th birthday

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In his papers [5], [6] and [7], F. Szász considered several generalizations of strongly regular rings: A ring R is called a P_1 -ring (resp. *gsr*-ring) if $aR = aRa$ (resp. $aRa = a^2Ra^2$) for any $a \in R$. As was shown in [5, Theorem 6], [6, Theorem 9] and [7, Theorem 9], there holds the following:

(I) A ring R is strongly regular if and only if one of the following equivalent conditions is satisfied:

- 1) R is a semi-prime P_1 -ring.
- 2) R is semi-prime and $aR = a^2R$ for any $a \in R$.
- 3) R is a semi-prime *gsr*-ring.

More recently, S. Ligh and Y. Utumi [4, Theorems 1, 3] and K. Chiba and H. Tominaga [2, Theorem 2] have characterized a P_1 -ring R as follows:

(II) R is a P_1 -ring if and only if one of the following equivalent conditions is satisfied:

- 1) R is a direct sum of a strongly regular ring and a zero ring.
- 2) $aR \subseteq Ra^2$ for any $a \in R$.
- 3) $l(R) = r(R)$ and $R/l(R)$ is strongly regular.
- 4) $aR = a^2R$ and $Ra = Ra^2$ for any $a \in R$.
- 5) $R/l(R)$ and $R/r(R)$ are strongly regular.

Concerning *gsr*-rings, by the aid of Nagata-Higman theorem, F. Szász [7, Theorem 7] proved also the following:

(III) If a *gsr*-ring R with prime radical N is an algebra over a field whose characteristic is 0 or a prime $p \geq 5$, then $N^7 = 0$.

In this note, we shall introduce the notion of a P'_n -ring (resp. Q_n -ring) as a generalization of that of a P_1 -ring (resp. *gsr*-ring), and prove three theorems which contain (I), (II) and (III) respectively.

Throughout R will represent an associative ring (with or without 1), and n a positive integer. As for notations and terminologies used here, we follow [2].

If $aR^n = aR^n a$ (resp. $aR^n a = a^2R^n a^2$) for any $a \in R$, R is called a P'_n -ring

(resp. Q_n -ring). Obviously, the notion of a P'_1 -ring (resp. Q_1 -ring) coincides with that of a P_1 -ring (resp. gsr -ring). If R is a P'_n -ring or if $aR^n = a^2R^n$ for any $a \in R$, then it is easy to see that the set N of all nilpotent elements of R equals $l(R^n)$. While, if $aR^n \subseteq R^n a^2$ for any $a \in R$, then one can prove $N = l(R^{2n})$ (cf. the proof of [4, Theorem 3]). Finally, in case R is a Q_n -ring, for any $a \in N$ we have $aR^n a = 0$, so that $(aR)^{2n} \subseteq (aR^n)^2 = 0$, which means that N coincides with the prime radical of R .

Our first theorem is a generalization of (I).

Theorem 1. *The following are equivalent :*

- 1) R is strongly regular.
- 2) R is a semi-prime P'_n -ring.
- 3) R is semi-prime and $aR^n = a^2R^n$ for any $a \in R$.
- 4) R is semi-prime and $aR^n \subseteq R^n a^2$ for any $a \in R$.
- 5) R is a semi-prime Q_n -ring.

Proof. It is easy to see that 1) implies 2)–5). Suppose 2). In any rate, R is a reduced ring. Given $a \in R$, $a^{n+1} - ab_0 a = 0$ for some $b_0 \in R^n$. Since $(a^n - ab_0)^2 = 0$, it follows $a^n - ab_0 = 0$ and $a^n - ab_1 a = 0$ for some $b_1 \in R^n$. Repeating the same procedure, we obtain eventually $a - ab_n a = 0$ for some $b_n \in R^n$, proving 1). Next, noting that in a reduced ring $xy = 0$ if and only if $yx = 0$, the implications 3) \implies 1) and 4) \implies 1) will be shown quite similarly. Finally, suppose 5). If a is an arbitrary element of R then

$$a^{n+2} \in aR^n a = a^2 R^n a^2 = \dots = a^{2(n+2)} R^n a^{2(n+2)}.$$

Hence, by [1, Lemma 1] there exists an element $b \in R$ such that $a^{n+2} = a^{2(n+2)} b$ and $ab = ba$. Then it is easy to see that $(a - a^{n+3} b)^{n+2} = 0$. Since R is seen to be a reduced ring, we obtain $a = a^2(a^{n+1} b)$, which proves 1) (cf. also [1, Lemma 4]).

If e is an idempotent of R such that $ea = ae$ for any $a \in R$ with $a^2 = 0$ then e is central. To see this, it is enough to recall that $(ex - exe)^2 = 0 = (xe - exe)^2$ for any $x \in R$. Following A. Kertész and O. Steinfeld [3], R is called a Z -ring if every idempotent of R is central. For instance, any ring whose nilpotent elements are central is a Z -ring (cf. [4, Theorem 1]).

Lemma 1. (1) *If R is a P'_n -ring then it is a Z -ring.*

(2) *If $aR^n \subseteq R^n a^2$ for any $a \in R$ then R is a Z -ring.*

(3) *If $l(R^n) = r(R^n)$ and $R/l(R^n)$ is strongly regular, then R is a Z -ring.*

Proof. (1) Let $e=e^2$ and $a^2=0$ in R . Obviously, $ae=ae^n=0$ and $ea=e \cdot e^{n-1}a \in eR^n e$, whence it follows $ea=eae=0$. Hence, R is a Z -ring. Similarly, we can prove (2).

(3) Let $e=e^2$ in R . Since $R/l(R^n)$ is strongly regular, $ae-ea \in l(R^n)$ for any $a \in R$, so that $ae-eae=(ae-ea)e^n=0$, and similarly $ea-eae=0$. Hence $ae=eae=ea$.

Now, we are at the position to prove our principal theorem, which contains (II).

Theorem 2. *The following are equivalent :*

- 1) R is a direct sum of a strongly regular ring and a nilpotent ring of nilpotency index at most $n+1$.
- 2) R is a P'_n -ring.
- 3) $aR^n \subseteq R^n a^2$ for any $a \in R$.
- 4) R is a Z -ring and $R/l(R^n)$ is regular.
- 5) $l(R^n)=r(R^n)$ and $R/l(R^n)$ is strongly regular.
- 6) $aR^n=a^2R^n$ and $R^n a=R^n a^2$ for any $a \in R$.
- 7) $R/l(R^n)$ and $R/r(R^n)$ are strongly regular.
- 2')—4') *The left-right analogues of 2)—4).*

Proof. It is easy to see that 1) implies 2)—7).

2) \implies 4) By Lemma 1 (1) and Theorem 1.

6) \implies 7) \implies 5) \implies 4) By Theorem 1 and Lemma 1 (3).

4) \implies 1) By the regularity of $\bar{R}=R/l(R^n)$ we have $R=R^{n+1}+l(R^n)$. Now, let $x=\sum x_i^{(1)}x_i^{(2)}\dots x_i^{(n+1)}$ be an arbitrary element of $R^{n+1} \cap l(R^n)$. Then, by the regularity of \bar{R} and the nilpotency of $l(R^n)$ we can find a central idempotent e such that $x_i^{(1)}-x_i^{(1)}e \in l(R^n)$ for all i . Hence,

$$x = \sum (x_i^{(1)} - x_i^{(1)}e)x_i^{(2)}\dots x_i^{(n+1)} + \sum x_i^{(1)}ex_i^{(2)}\dots x_i^{(n+1)} = xe = xe^n = 0,$$

whence it follows $R=R^{n+1} \oplus l(R^n)$.

3) \implies 1) By Lemma 1 (2) and Theorem 1, R is a Z -ring such that $R/l(R^{2n})$ is strongly regular. Hence, the proof of 4) \implies 1) shows that $R=R^{2n+1} \oplus l(R^{2n})$. Since $R^{n+1}=R^{n+2}$, we readily see $R=R^{n+1} \oplus l(R^{2n})$. If $aR^{2n}=0$ then $aR^n \subseteq R^{n+1} \cap l(R^{2n})=0$, which means $l(R^{2n})=l(R^n)$. Thus $R=R^{n+1} \oplus l(R^n)$, proving 1).

Remark. By Theorem 2, any P'_n -ring is a Q_n -ring. However, the converse is not true in general : Let R be the subring of $(GF(2))_2$ consisting of $0, e_{11}, e_{12}$ and $e_{11}+e_{12}$, where e_{ij} 's are matrix units of $(GF(2))_2$. Then, $aR=a^2R$ and $aRa=a^2Ra^2$ for any $a \in R$, $Re_{12} \neq 0 = Re_{12}^2$, $l(R) \neq 0 = r(R)$,

and $R/I(R)$ is isomorphic to $\text{GF}(2)$.

Finally, we shall present a sharpening of (III).

Lemma 2. *Let R be a Q_n -ring. If $a_1, a_2, a_4, a_5 \in N$ and $a_3 \in R^n$ then $2a_1a_2a_3a_4a_5 = 0$.*

Proof. If a, b are in N then $aR^na = 0$, $bR^nb = 0$ and $(a+b)R^n(a+b) = 0$. Hence, one will easily see that $axb = -bxa$ for any $x \in R^n$. By making use of this relation, we obtain

$$a_1a_2a_3a_4a_5 = -a_4a_5a_3a_1a_2 = a_4a_2a_3a_1a_5 = -a_1a_2a_3a_4a_5,$$

namely, $2a_1a_2a_3a_4a_5 = 0$.

By Lemma 2, without making use of Nagata-Higman theorem, we readily obtain the following:

Theorem 3. *Let R be an algebra over a field of characteristic $\neq 2$. If R is a Q_n -ring then $N^{n+4} = 0$. Especially, if R is a gsr-ring then $N^5 = 0$.*

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