NOTE ON FULLY IDEMPOTENT IDEALS AND s-UNITAL IDEALS

To Professor R. Takekuma on his sixteenth birthday
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Throughout the present note R will represent a ring (with or without identity). R is said to be *fully left idempotent* (or *left weakly regular*) if every left ideal of R is idempotent or equivalently if $a\in(Ra)^2$ for any $a\in R$ (cf. [4]). R is said to be *fully idempotent* if every ideal of R is idempotent or equivalently if $a\in(RaR)^2$ for any $a\in R$ (cf. [2]). Finally, R is said to be *left* (resp. *right*) *s-unital* (or *D-regular*) if $a\in Ra$ (resp. $a\in aR$) for any $a\in R$ (cf. [5]). If an ideal I of R is a fully left idempotent ring (resp. fully idempotent ring), then I is called a *fully left idempotent ideal* (resp. *fully idempotent ideal*). Similarly, I is called an *s-unital ideal* if I is a left and right s-unital ring. We shall denote by $W(R)$, $W'(R)$ and $W^*(R)$ the sum of all fully left idempotent ideals of R, the sum of all fully idempotent ideals of R and the sum of all s-unital ideals of R, respectively.

In this note we shall prove the following:

Theorem 1. (1) *$W(R)$ is the unique maximal fully left idempotent ideal of R.*

(2) $W(R/W(R))=0$.

(3) *Let $(R)_n$ be the $n\times n$ matrix ring over R. Then* $W((R)_n)=(W(R))_n$.

(4) *If I is an ideal of R then* $W(I)=W(R)\cap I$.

Theorem 2. (1) *$W'(R)$ is the unique maximal fully idempotent ideal of R.*

(2) $W'(R/W'(R))=0$.

(3) $W'((R)_n)=(W'(R))_n$.

(4) *If I is an ideal of R then* $W'(I)=W'(R)\cap I$.

Theorem 3. (1) *$W^*(R)$ is the unique maximal s-unital ideal of R.*

(2) $W^*(R/W^*(R))=0$.

(3) $W^*((R)_n)=(W^*(R))_n$.

Borrowing the idea from B. Brown and N. H. McCoy [1], V. Gupta

[3] has proved Theorem 1 for rings with identity. Moreover, the W'-radical class and the W'-radical class are considered in [4] and [2], respectively. In case R is an integral domain, it is easy to see that R is left s-unital if and only if R contains 1. Accordingly, if R is an integral domain with 1 and I is an ideal of R with $R\supset I\supset 0$ then $W^*(I)=0\neq I=W^*(R)\cap I$, which shows that the W^*-radical class is not hereditary. Moreover, one may remark that if R is the subring of $(\mathrm{GF}(2))_2$ consisting of 0, $e=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $\begin{pmatrix}0&0\\1&0\end{pmatrix}$ and $\begin{pmatrix}1&0\\1&0\end{pmatrix}$ then $eW(R)e=eW'(R)e=eW^*(R)e=0\neq eRe=W(eRe)=W'(eRe)=W^*(eRe)$.

In advance of proving our theorems we shall state three lemmas.

Lemma 1. *Let I be an ideal of R.*

(1) *I is a fully left idempotent ideal if and only if $a\in(Ra)^2$ for any $a\in I$.*

(2) *I is a fully idempotent ideal if and only if $a\in(RaR)^2$ for any $a\in I$.*

Proof. (1) *If $a\in I$* and $a\in(Ra)^2$ then $(Ra)^2=(Ra)^4=(Ia)^2$. The converse is trivial.

(2) If $a\in I$ and $a\in(RaR)^2$ then $(RaR)^2=(RaR)^6=(IaI)^2$. The converse is trivial.

Lemma 2 (cf. [3, Lemma 2]). (1) *If $x\in(Ra)^2$ and $a-x\in(R(a-x))^2$ then $a\in(Ra)^2$.*

(2) *If $x\in(RaR)^2$ and $a-x\in(R(a-x)R)^2$ then $a\in(RaR)^2$.*

Proof. We shall prove only (1). Since $x\in(Ra)^2$, it follows $R(a-x)\subseteq Ra$. Hence, $a-x\in(Ra)^2$, which together with $x\in(Ra)^2$ implies $a\in(Ra)^2$.

The next is given in [5, Proposition 5(1)], [2, Theorem 2.5] and [5, Corollary 1].

Lemma 3. (1) *R is fully left idempotent if and only if so is $(R)_n$.*

(2) *R is fully idempotent if and only if so is $(R)_n$.*

(3) *R is left s-unital if and only if so is $(R)_n$.*

Proof of Theorem 1. (1) By Lemma 1 (1), it suffices to show that if the principal ideals (u) and (v) are fully left idempotent then $u+v\in(R(u+v))^2$. Let $a=u+v$ and $x=\sum_i x_i a y_i a$, where $u=\sum_i x_i u y_i u$ $(x_i, y_i\in R)$. Then $a-x=v-\sum_i x_i u y_i v-\sum_i x_i v y_i a\in(v)$, whence it follows $a-x\in$

$(R(a-x))^2$. Hence $a\in(Ra)^2$ by Lemma 2 (1).

(2) If $x\in(Ra)^2$ and $a-x\in W(R)$ then $a\in(Ra)^2$ by (1) and Lemma 2 (1). Now, (2) is obvious by Lemma 1 (1).

(3) Since $(W(R))_n\subseteq W((R)_n)$ by (1) and Lemma 3 (1), it remains only to prove the converse inclusion. Given $x\in R$, $E_{\lambda\mu}(x)$ will denote the element of $(R)_n$ with x in the (λ, μ)-position and zeros elsewhere. Let $A=(a_{ij})$ be an arbitrary element of $W((R)_n)$. Since $A\in((R)_nA)^2$, there exist some $X_k=(x_{ij}^{(k)})$ and $Y_k=(y_{ij}^{(k)})$ such that $A=\sum_k X_kAY_k$. Then by a brief computation we have

$$W((R)_n)\ni\sum_{k,\lambda,\mu} E_{1\lambda}(x_{p\lambda}^{(k)})AE_{\mu 1}(y_{\mu q}^{(k)})=E_{11}(a_{pq}).$$

Hence $W((R)_n)=(I)_n$, where I is the ideal of R consisting of all the elements x which appear in the (1, 1)-position of some elements in $W((R)_n)$. Recalling that $E_{11}(a_{pq})\in((R)_nE_{11}(a_{pq}))^2$, one will easily see that $a_{pq}\in(Ra_{pq})^2$, namely, $I\subseteq W(R)$ (Lemma 1 (1)).

(4) By (1) and Lemma 1 (1), $W(R)\cap I\subseteq W(I)$. Conversely, if $a\in W(I)$ and $x\in R$ then $ax\in(Ia)^2x\subseteq IaI\subseteq W(I)$ and similarly $xa\in W(I)$. Hence, $W(I)$ is an ideal of R and $W(I)\subseteq W(R)\cap I$ by (1).

Proof of Theorem 2. Although (1) and (2) are given in [2, Theorem 4. 4], the proof is quite similar to that of Theorem 1 and may be left to readers.

Proof of Theorem 3. (1) Let I and J be s-unital ideals of R. If $a\in I$ and $b\in J$ then there exist some $f\in I$ and $g\in J$ such that $fa=a$ and $g(b-fb)=b-fb$. Obviously, $e=f+g-gf\in I+J$ and $e(a+b)=a+b$, and similarly $(a+b)e'=a+b$ for some $e'\in I+J$. Hence, $W^*(R)$ is s-unital.

(2) Let $I\supseteq W^*(R)$ be an ideal of R such that for any $a\in I$ there exist $e, e'\in I$ with $a-ea$, $a-ae'\in W^*(R)$. Then by (1) there exist $f, f'\in W^*(R)$ such that $f(a-ea)=a-ea$ and $(a-ae')f'=a-ae'$. Since $a=(e+f-fe)a=a(e'+f'-e'f')$ and $e+f-fe$, $e'+f'-e'f'\in I$, it follows $I\subseteq W^*(R)$.

(3) By (1) and Lemma 3 (3), $(W^*(R))_n\subseteq W^*((R)_n)$. If $A=(a_{ij})\in W^*((R)_n)$ then there exist some $X=(x_{ij})$ and $Y=(y_{ij})$ such that $A=XA=AY=XAY$. It is easy to see that

$$W^*((R)_n)\ni\sum_{\lambda,\mu} E_{1\lambda}(x_{p\lambda})AE_{\mu 1}(y_{\mu q})=E_{11}(a_{pq}).$$

Hence $W^*((R)_n)=(I)_n$, where I is the ideal of R consisting of all the elements x which appear in the (1, 1)-position of some elements in $W^*((R)_n)$. Recalling that there exist $Z, Z'\in W^*((R)_n)$ such that $E_{11}(a_{pq})=ZE_{11}(a_{pq})=E_{11}(a_{pq})Z'$, one will easily see $a_{pq}=za_{pq}=a_{pq}z'$ for some $z, z'\in I$, and hence $I\subseteq W^*(R)$.

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