ALMOST COMMUTATIVE RINGS AND THEIR INTEGRAL EXTENSIONS

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Recently reduced rings (rings without nonzero nilpotent elements), especially reduced regular rings, draw considerable attention. In many ways, this type of rings behaves very much like commutative rings. It is the purpose of this paper to investigate some general properties of such class of rings.

Throughout, R will represent a ring with identity element 1, and C the center of R.

Definition 1. R is said to be *almost commutative* (*AC-ring*), if for any prime ideal $P(\neq R)$ of R and $a\notin P$, there exists x such that $ax\in C-P$.

For a prime ideal P, $C-P$ is multiplicatively closed. Thus if $ax\in C-P$ and $xy\in C-P$ then $(x^2y)a=xa(xy)=(ax)(xy)\in C-P$. The notion of AC-ring is right-left symmetric.

Commutative rings and reduced regular rings are AC-rings. Finite direct sum of AC-rings is an AC-ring and a homomorphic image of an AC-ring is again an AC-ring.

Theorem 1. *In an AC-ring R, every prime (semi-prime) ideal is completely prime (completely semi-prime).*

Proof. Let P be a prime ideal in R. If $bc\in P$ and $c\notin P$, then let $z\in R$, $cz\in C-P$. Since $bczR=bRcz\subset P$, we have $b\in P$. Since a semi-prime ideal N is the intersection of all prime ideals that contain N, if $a\notin N$ then $a\notin P$ for some prime ideal P containing N and $a^n\notin P$ for all n. Hence, $a^n\notin N$ and N is completely semi-prime.

For an ideal A in an AC-ring R, its prime radical $\boldsymbol{P}(A)=\{x\in R \mid x^n \in A \text{ for some } n\}$, the same as in the commutative case. Consequently, the prime radical $\boldsymbol{P}$ of the ring R is just the set of nilpotent elements. An AC-ring is reduced if and only if it is semi-prime; a domain if and only if it is prime; and a division ring if and only if it is simple.

Theorem 2. *Let R be an AC-ring and I be a submodule of R, considering R as a left or right R-module. The following statements are*

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equivalent:

1) *I is a maximal right ideal.*

1') *I is a maximal left ideal.*

2) *I is a maximal ideal.*

3) *I is a right primitive ideal.*

3') *I is a left primitive ideal.*

Proof. By the symmetry of an AC-ring, it suffices to prove the equivalence of 1), 2), and 3). Obviously, 2) implies 3). Assume 1), and let $A=(I:R)=\{x\in R \mid Rx\subset I\}$, which is the largest ideal of R contained in I. If there exists a maximal ideal M containing A properly then there exists $a\in I-M$ and $x\in R$ such that $ax\in(C\cap I)-M$. But this contradicts $ax\in A\subset M$. Hence, R/A is a division ring and $I=A$, proving 2). Finally, assume 3). Then there exists a maximal right ideal I' of R containing I such that $(I'/I: R/I)=0$. As it was shown just above, $I'/I=(I'/I:R/I)=0$. $I=I'$.

Thus as in the commutative case, the Jacobson radical J of an AC-ring is the intersection of all maximal ideals.

Corollary 1. *A semi-simple AC-ring is a subdirect product of division rings.*

Theorem 3. *The following statements are equivalent*:

1) *R is a direct sum of finite number of division rings.*

2) *R is a semisimple AC-ring with minimum condition on ideals.*

3) *R is a reduced AC-ring with minimum condition on ideals.*

Proof. If R is an AC-ring with minimum condition on ideals then every prime ideal P is maximal for R/P is a division ring. Using the same technique as in commutative case, R has a finite number of maximal ideals only. By the Chinese Remainder Theorem, R is a direct sum of division rings if either R is reduced or semisimple.

Theorem 4. *If R is an AC-ring then the following statements are equivalent*:

1) *R is a regular ring.*

2) *Every ideal in R is semi-prime.*

3) *R is reduced and R/P is a regular ring (in fact a division ring) for every prime ideal P of R.*

Proof. Every ideal in a regular ring is semi-prime. Thus an AC-regular ring must be reduced. This shows 1) implies 2) and 3). If 2) then R/P is a division ring for every prime ideal P. Hence P is a maximal right ideal. If 3) then R/P is also a division ring, since R/P is a domain. Both cases imply R is a regular ring by Theorem 3 [4].

As we expect it, we can use the same definitions for primary ring, primary ideal, and local ring as in the commutative case for AC-rings. Recall a ring is *primary* if every zero divisor is nilpotent and is *local* if it has exactly one maximal ideal.

Theorem 5. *The following statements are equivalent for an AC-ring R:*

1) *R is primary.*
2) *Every right zero divisor of R is nilpotent.*
2′) *Every left zero divisor of R is nilpotent.*
3) *R has a minimal prime ideal P which contains all zero divisors.*

Proof. First, assume 2), and let $xy=0$, $y\neq 0$. If $x\notin \boldsymbol{P}$ then $x\notin P$ for some prime ideal P. Choose $u\in R$ such that $ux\in C-P$. By the hypothesis, $0=uxy=yux$ implies $ux\in \boldsymbol{P}$. It contradicts $ux\notin P$. This proves 2) implies 1). Similarly, 2′) implies 1). If R is primary then $\boldsymbol{P}$ consists of all zero divisors of R and is the unique minimal prime ideal. Finally assume 3), and let P be a minimal prime ideal containing all zero divisors. Let $x\in R$ be not nilpotent. The theorem follows if we can show $x\notin P$. Let $T=\{x^k s \mid k\geqslant 0, s\notin P\}$. Obviously, 1 and x are in T. But $0\notin T$, otherwise $x^k s=0$ for some k which implies $s\in P$. If $x^m s$ and $x^n s'$ in T, let $u\in R$ where $su\in C-P$. Then $(x^m s)\, u\, (x^n s')=x^{m+n}(sus')$ and $sus'\notin P$. Hence, T is an m-system and there exists a prime ideal P' with $P'\subset R-T$. Since P is minimal and $R-T\subset P$, we have $x\notin P'=P$.

An ideal I is *primary* if R/I is a primary ring. If I is a primary ideal of an AC-ring then $\boldsymbol{P}(I)$ is a prime ideal.

Theorem 6. *In an AC-ring R, the following statements are equivalent:*

1) *R has a unique prime ideal P.*
2) *R is local and $\boldsymbol{P}=\boldsymbol{J}$.*
3) *Every nonunit is nilpotent.*
4) *R is primary and all nonuits are zero divisors.*

Proof. Clearly 1) implies 2). If 2) then $\boldsymbol{P}=\boldsymbol{J}$ is the unique maximal ideal in R. Suppose a is not nilpotent. Then $a\notin \boldsymbol{J}$, and $aR+\boldsymbol{J}=R$. Hence a is a unit (cf. Theorem 2). Assume 3). Then every zero divisor is nilpotent and hence 4). If 4) holds then $\boldsymbol{P}$ is the set of all nonunits in R and hence the unique prime ideal of R.

As in the commutative case, if $Q_2, \cdots, Q_n$ are P-primary ideals then

$\cap Q_i$ is also P-primary.

Definition 2. A ring R is said to be a *strongly AC-ring* (*SAC-ring*) if for any ideal I of R and $a\notin I$, there exists x such that $ax\in C-I$.

An AC-ring in which every ideal is semi-prime, namely, a reduced regular ring (Theorem 4), is an SAC-ring. Any finite direct sum and homomorphic images of SAC-rings are SAC-rings. In an SAC-ring R, an ideal A is prime (semi-prime) if and only if $A\cap C$ is prime (semi-prime). Consequently, an SAC-ring is regular if and only if its center is regular (Theorem 4). Also $(A\cap C)R=A$. Hence for ideals A and B, $A=B$ if and only if $A\cap C=B\cap C$.

Hereafter, let R be an SAC-ring and R' be an *integral extension* of R with center C', i.e., R' is an overring of R such that $C\subset C'$ and for each $x\in R'$ there exist $a_0, \cdots, a_{n-1}$ in R such that $x^n+a_{n-1}x^{n-1}+\cdots+a_0=0$. Many properties in commutative integral extensions hold similarly in this situation. If P' is a prime (semi-prime) ideal in R' then it is easy to verify $P'\cap R$ is a prime (semi-prime) ideal in R. Using similar techniques as in the commutative case [2, 3], we obtain the following theorem.

Theorem 7. *If P is a prime ideal of R and P' is an ideal in R' which is maximal among the ideals having null intersection with $M=R-P$, then P' is a prime ideal in R' and $P'\cap R=P$.*

Proof. Since M is a multiplicatively closed system in R', P' does exist and is a prime ideal with $P'\cap R\subset P$. If $P'\cap R\neq P$ then there exists a central element $c\in P-(P'\cap R)$. By $(cR'+P')\cap M\neq\emptyset$, $cx+p'=m$, $x\in R'$, $p'\in P'$, and $m\in M$. Suppose $x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0$, $a_i\in R$. Then $x^nc^n+a_{n-1}x^{n-1}c^n+\cdots+a_1xc^n+a_0c^n=(m-p')^n+a_{n-1}(m-p')^{n-1}c+\cdots+a_1(m-p')c^{n-1}+a_0c^n=0$. There exist therefore $r\in R$ and $q'\in P'$ such that $m^n+rc+q'=0$. This shows $q'\in P'\cap R$, and hence $m^n\in P$, which is a contradiction.

Corollary 2. *Let I' be a proper ideal in R' and $I=I'\cap R$. If P is a prime ideal in R such that $I\subset P$ then there exists a prime ideal P' in R' such that $I'\subset P'$ and $P'\cap R=P$.*

Proof. Let $M=R-P$ and P' be an ideal of R' which is maximal with respect to $P'\supset I'$ and $P'\cap M=\emptyset$. By the above theorem $P'\cap R=P$.

Corollary 3. *Let I be a proper ideal in R. Then IR' is also a proper ideal in R'.*

Proof. Let P be a prime ideal in R containing I. By Theorem 7 there exists a prime ideal P' in R' such that $P'\cap R=P$. $IR'=(I\cap C)RR'=(I\cap C)R'\subset P'\neq R'$.

Theorem 8. *Let N' be a left ideal in R' and P' be a completely prime ideal in R' contained in N'. If $N'\cap R=P'\cap R$ then $N'=P'$.*

Proof. Suppose there exists $x\in N'-P'$. Let n be the smallest integer such that $x^n+a_{n-1}x^{n-1}+\cdots+a_0=p'\in P'$ $(a_i\in R)$. This implies $a_0\in N'\cap R=P'\cap R$. Since P' is completely prime, $(x^{n-1}+\cdots+a_1)x\in P'$ yields a contradiction $x^{n-1}+\cdots+a_1\in P'$.

Corollary 4. *Let P' be a completely prime ideal in R'. Then P' is a maximal left ideal in R' if and only if $P'\cap R$ is a maximal ideal in R.*

Proof. Suppose $P'\cap R$ is a maximal (left) ideal in R (cf. Theorem 2) and N' is a maximal left ideal in R' containing P'. Then $N'=P'$ by the above theorem. Conversely suppose P' is a maximal left ideal in R'. If P_0 is an arbitrary prime ideal in R containing $P'\cap R$, then by Corollary 2 there exists a prime ideal P_0' in R' such that $P'\subset P_0'$ and $P_0'\cap R=P_0$. But $P_0'=P'$ and hence $P_0=P'\cap R$.

In [4], this author proved that if R is a reduced regular ring and R' is an integral extension of R, then R' is also regular if R' is reduced. The following theorem includes this result.

Theorem 9. *If R' is reduced, then R' is regular if and only if R is regular.*

Proof. If R is regular then every prime ideal in R is maximal (Theorem 4). Let P' be a completely prime ideal in R'. Since $P'\cap R$ is maximal in R, P' is a maximal left ideal in R'. By Theorem 3 [4], R' is a regular ring. Conversely, if R' is regular then R' is an SAC-ring. For a prime ideal P in R, let P' be a prime ideal in R' such that $P=P'\cap R$ (Theorem 7). Since P' is a maximal left ideal, P is a maximal ideal in R (Corollary 4). R is regular again by Theorem 4.

For the remaining part of this paper, let R be a reduced regular ring (hence an SAC-ring). For any $a\in R$ there exists a unit u such that $au=ua=e$ where e is a (central) idempotent. If e is a nonzero idempotent of R then eR' is also an integral extension of the reduced regular ring eR.

Theorem 10. *For each $x\in R'$ there exist $y\in R'$ which can be expressed as a polynomial in x over R and a natural number n such that $yx^{n+1}=x^n$ (hence R' is a left π-regular ring).*

Proof. Let $A(x)=\{p(x)\mid p(x)$ is a monic polynomial in x over R and $p(x)x^m=0$ for some $m\}$. The theorem holds trivially for nilpotent elements. Let x be non-nilpotent in R'. In $A(x)$, we choose $p(x)=x^k+a_{k-1}x^{k-1}+\cdots+a_1x+a_0$ of least degree. $p(x)x^{n-1}=0$ for some $n>1$. Since x is non-nilpotent, $a_0\neq 0$. Let u be a unit in R such that $a_0u=ua_0=1-e$, $e^2=e$. If $k=1$ then $(x+a_0)x^{n-1}=0$ implies $(ux+(1-e))x^{n-1}=0$, whence it follows $uex^n=eux^n=0$. Hence, $ex^n=0$ and $-ux^{n+1}=x^n$. If $k>1$ then $uep(x)^{n-1}=eup(x)x^{n-1}=0$ implies $ep(x)^{n-1}=p(ex)(ex)^{n-1}=0$. Since $eup(x)=u\{(ex)^{k-1}+\cdots+(ea_1)\}(ex)$, $A(ex)$, considered as in eR', contains a polynomial of degree $k-1$. By inductive method, there exists a polynomial $T(ex)$ over eR such that $T(ex)\,(ex)^{m+1}=(ex)^m$ for some m. From $up(x)x^{n-1}=0$, we obtain $(-u)(x^{k-1}+\cdots+a_1)x^n+(ex)^{n-1}=x^{n-1}$. Multiplying on the right by x^{m+1}, we obtain $(-u)(x^{k-1}+\cdots+a_1)x^{n+m+1}+(ex)^{n+m}=x^{n+m}$, whence it follows $\{(-u)(x^{k-1}+\cdots+a_1)+eT(x)\}\,x^{n+m+1}=x^{n+m}$.

Actually to show R' is merely left π-regular is rather simple. J. W. Fisher and R. Snider [1] prove that a ring is left π-regular if and only if each of its prime factor ring is left π-regular. Let P' be a prime ideal in R' and $P=P'\cap R$. Since R is a reduced regular ring, R/P is a division ring. R'/P' is left π-regular, for it is algebraic over the division ring R/P.

Theorem 11. *If R is commutative then for each $x\in R'$ there exists a unit u which can be expressed as a polynomial in x over R such that $x^{2n}u=x^n$ for some n.*

Proof. By Theorem 10, there exists $y=p(x)$, a polynomial in x over R such that $yx^{n+1}=x^n$. Let $z=y^n$ and $e=1-x^nz$. Then it is easy to verify that $zx^{2n}=x^n$, $e^2=e$, and $ex^n=x^ne=0$. Finally, let $u=z(1-e)+e$. Then u is a polynomial in x over R, and $ux^{2n}=x^n$. If $vu=0$ then $vz(1-e)+ve=0$ implies $vz(1-e)=0$ and $ve=0$. Hence, $vx^nz=vzx^n=vz(1-e)x^n=0$ and $v=v(1-x^nz)=ve=0$. Since R' is left π-regular, u is a unit.

Corollary 5. *If R is commutative and R' is semi-prime then R' is isomorphic to a subdirect sum of prime algebraic algebras over fields.*

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