ON IMAGES OF TOPOLOGICAL ORDERED SPACES UNDER SOME QUOTIENT MAPPINGS

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A topological ordered space is a topological space equipped with a partial order. Since a topological space may be regarded as a topological ordered space equipped with the discrete order, i. e., $a\rho b$ if and only if $a=b$, the study of topological ordered spaces not only includes that of topological spaces but also reveals many generalizations of well-known results concerning topological spaces. From this point of view, the study of topological ordered spaces was first taken up by L. Nachbin [12]. In this paper, at first we survey some results obtained hitherto concerning the images of T_i-ordered spaces ($i=2, 3, 4$) under some quotient mappings[1], and then we establish the principal theorem which asserts that the image of a T_i-ordered space under a proper mapping is a T_i-ordered space ($i=2, 3$) and the image of a normally ordered space under a closed mapping is a normally ordered space. Needless to say, the theorem reduces to the known facts of topological spaces when the partial order concerned is the discrete order (cf. [5, §10, Corollaire 2 à Proposition 5 and Exercice 5] and [6, §4, Exercice 15]). Finally, we shall say a few words concerning the images of T_1-ordered spaces under quotient mappings.

The author wishes to express his gratitude to Professor O. Takenouchi for many useful comments and the encouragement during the preparation of this paper.

1. Throughout this paper, $\mathscr{U}$ and $\mathscr{V}$ denote topologies, and ρ and τ partial orders. The notation $(X, \mathscr{U}, \rho)$ is used to denote a set X endowed with a topology $\mathscr{U}$ and a partial order ρ. The notations $(X, \mathscr{U})$ and (X, ρ) are to be understood similarly. All mappings are assumed to be continuous.

Notation. In (X, ρ), for x, $y \in X$, $x \parallel y$ means that neither $x\rho y$ nor

[1] Let Y be an arbitrary set, X a topological space, and $f: X \to Y$ a surjection. The ***identification topology*** in Y determined by f is $\mathfrak{I}(f)=\{U \subset Y : f^{-1}(U)$ is open in $X\}$. For two topological spaces X, and Y, a continuous surjection $f: X \to Y$ is called an ***identification*** (or ***quotient***) ***mapping*** whenever the topology in Y is exactly $\mathfrak{I}(f)$ (cf. [7, pp. 120—121]). A mapping is said to be ***compact*** if the inverse image of a point is compact, and to be ***proper*** if it is closed and compact. It is elementary that a continuous open (closed or proper) mapping is a quotient mapping.

$y\rho x$. $x\bar{\rho}y$ if and only if $x\neq y$, $x\rho y$ or $x \parallel y$, and $x\rho' y$ stands for $x\neq y$, $y\rho x$ or $x \parallel y$. For $a\in X$ and $U, V\subset X$, $a\bar{\rho}U$ means that $a\bar{\rho}x$ for all $x\in U$, similarly $U\bar{\rho}V$ means that $x\bar{\rho}y$ for all $x\in U$ and all $y\in V$. (Note that these notations are different from those in [2].)

Definition 1. In (X, ρ), $[x, \longrightarrow]$ and $[\longleftarrow, x]$ denote the sets $\{y\in X : x\rho y\}$ and $\{y\in X : y\rho x\}$ respectively. In case $A\subset Y\subset X$, we put $i_Y(A) = \{\cup \{[a, \longrightarrow] : a\in A\}\} \cap Y$, $d_Y(A) = \{\cup \{[\longleftarrow, a] : a\in A\}\} \cap Y$, and A is said to be *increasing* (resp. *decreasing*) in Y if and only if $A=i_Y(A)$ (resp. $A=d_Y(A)$).

Definition 2. A space $(X, \mathscr{U}, \rho)$ is called a *T_1-ordered* (resp. *T_2-ordered*) *space* if for each pair $a, b\in X$ such that $a\rho' b$, there exist an increasing neighborhood U of a and a decreasing neighborhood V of b such that $b\notin U$ and $a\notin V$ (resp. $U\cap V=\emptyset$) (see [9]).

In these connections, the term O_i-space ($i=1, 2$) is used in [2] and [3] instead.

Definition 3. A space $(X, \mathscr{U}, \rho)$ is called a *regularly ordered space* if for each decreasing (resp. increasing) closed set $F\subset X$ and each element $a \notin F$, there exist disjoint neighborhoods U of a and V of F such that U is increasing (resp. decreasing) and V is decreasing (resp. increasing) in X. A space $(X, \mathscr{U}, \rho)$ is a *T_3-ordered space* if and only if $(X, \mathscr{U}, \rho)$ is both T_1-ordered and regularly ordered (cf. [9]).

Definition 4. A space $(X, \mathscr{U}, \rho)$ is called a *normally ordered space* if for each pair of disjoint closed sets $F_1, F_2\subset X$ where F_1 is increasing and F_2 is decrasing in X, there exist disjoint neighborhoods U_1 of F_1 and U_2 of F_2 such that U_1 is increasing and U_2 is decreasing in X. A space $(X, \mathscr{U}, \rho)$ is a *T_4-ordered space* if and only if $(X, \mathscr{U}, \rho)$ is both T_1-ordered and normally ordered (cf. [9]).

In D. Adnadjević [1], $(X, \mathscr{U}, \rho)$ is said to be T_3-ordered if $(X, \mathscr{U})$ is a T_3 space and for each closed set F and each point a such that $a\bar{\rho}F$ (resp. $F\bar{\rho}a$) there exist neighborhoods U of a and V of F such that $U\bar{\rho}V$ (resp. $V\bar{\rho}U$), and is said to be T_4-ordered if $(X, \mathscr{U})$ is a T_4 space and for each pair of closed sets F_1, F_2 such that $F_1\bar{\rho}F_2$ there exist neighborhoods U of F_1 and V of F_2 such that $U\bar{\rho}V$. (Note that $a\bar{\rho}F$ (resp. $U\bar{\rho}V$) implies $a\notin F$ (resp. $U\cap V=\emptyset$).) In these connections, the notion of "T_i-ordered in Adnadjević' sense" is properly stronger than ours ($i=3, 4$) (cf. [9, Example 3] and [10, Example 2]).

Definition 5. Let f be a mapping of (X, ρ) onto (Y, τ). Then τ is called a *quotient order of ρ induced by f* if $x\tau y$ for $x, y\in Y$ if and only if there exist $u\in f^{-1}(x)$, $v\in f^{-1}(y)$ such that $u\rho v$.

Definition 6. A mapping f of (X, ρ) onto (Y, τ) is said to be *isotonic* (resp. *dually isotonic*) if $x\rho y$ (resp. $f(x)\tau f(y)$) implies $f(x)\tau f(y)$ (resp. $x\rho y$) (cf. [1], [2]). In [12, p. 21], an isotonic mapping is cited as an increasing mapping.

Remark 1. In Definition 5, let R be the equivalence relation on X agreeing that x and y are equivalent if and only if $f(x)=f(y)$, identify Y with X/R, and regard f as the projection of X onto X/R. Then the order τ on Y is viewed as the order induced on X/R as follows: for $A, B\in X/R$, $A\tau B$ if and only if there exist $a\in A$, $b\in B$ such that $a\rho b$ (see [13, § 4]). Different orders on X/R were also considered. For instance, in [4, § 1, Exercice 2] $A\tau_1 B$ for $A, B\in X/R$ if and only if there exists $b\in B$ such that $a\rho b$ for all $a\in A$, and in [8] $A\tau_2 B$ for $A, B\in X/R$ if and only if $a\rho b$ for each $a\in A$ and each $b\in B$. In the latter case, the equivalence relation considered in [8][2)] is a very special one and τ_2 is then the quotient order of Definition 5. While, if f is dually isotonic then τ coincides with τ_2.

2. Suppose f is an open mapping of a T_2-ordered space $(X, \mathscr{U}, \rho)$ onto $(Y, \mathscr{V}, \tau)$ where $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ are T_2 spaces. As was shown in [2, Proposition 5], if f is isotonic and dually isotonic then $(Y, \mathscr{V}, \tau)$ is a T_2-ordered space. However, as the next example shows, if τ is the quotient order of ρ induced by f then the above does not hold generally, namely, the hypothesis that f is dually isotonic is indispensable.

Example 1. Let X be the set $\{(a, x, y): a=0 \text{ or } 1, x\in[0, \infty) \text{ and } y\in(-\infty, \infty)\}$. We define an equivalence relation R on X as follows: $(a, x, y)\ R\ (b, u, v)$ if and only if $a=b$, $x=u$. The topology $\mathscr{U}$ on X is the usual one. We define a partial order ρ in X as follows: $(a, x, y)\ \rho\ (b, u, v)$ if and only if $a=0$, $b=1$, $x=u\neq 0$, $y=v=\frac{1}{x}$; or $a=b$, $x=u$, $y=v$. Let $Y=X/R$, and f the projection of X onto Y. If we take the identification topology determined by f as the topology $\mathscr{V}$ of Y and the quotient order of ρ induced by f as the partial order τ in Y, then the mapping f is isotonic but not dually isotonic. And $(Y, \mathscr{V}, \tau)$ is not T_2-ordered. This is

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2) In (X, ρ), $(x, \rightarrow]$ and $[\leftarrow, x)$ denote the sets $\{y\in X: x\rho y \text{ and } x\neq y\}$ and $\{y\in X: y\rho x \text{ and } x\neq y\}$ respectively. Then the equivalence relation D on X used in [8] is defined by agreeing that for $x, y\in X$, $(x, y)\in D$ if and only if $(x, \rightarrow]=(y, \rightarrow]$ and $[\leftarrow, x)=[\leftarrow, y)$.

because $(0,0)^* \| (1,0)^*$ in Y where $(a,x)^* = f((a,x,y))$, but there do not exist an increasing neighborhood U of $(0,0)^*$ and a decreasing neighborhood V of $(1,0)^*$ such that $U \cap V = \emptyset$.

In $(X, \mathscr{U}, \rho)$, let R be an equivalence relation on X, and f the projection of X onto X/R. In X/R, suppose that $\mathscr{V}$ is the identification topology determined by f and τ is the quotient order of ρ induced by f. As a generalization of a well-known result in topological space (cf. [5, §8, Proposition 8]), Theorem 4 of [11] asserts that if $(X/R, \mathscr{V}, \tau)$ is T_2-ordered then the graph $G(R)$ is saturated order closed (s. o. closed) in X^2, namely, for each $(x,y) \notin G(R)$ with $f(x)\, \tau'\, f(y)$ there exist a saturated increasing neighborhood U of x and a saturated decreasing neighborhood V of y such that $(U \times V) \cap G(R) = \emptyset$, and conversely if f is open and $G(R)$ is s. o. closed in X^2 then $(X/R, \mathscr{V}, \tau)$ is T_2-ordered. By Example 1, we see that, in the latter half of the above assertion, the hypothesis that $G(R)$ is s. o. closed in X^2 is indispensable.

Now, we shall prove our principal theorem which includes Theorems 2.4 and 2.5 of [1].

Theorem. *Suppose f is a mapping of $(X, \mathscr{U}, \rho)$ onto $(Y, \mathscr{V}, \tau)$ where τ is the quotient order of ρ induced by f.*

(1) *If f is a proper mapping and $(X, \mathscr{U}, \rho)$ is a T_2-ordered space, then $(Y, \mathscr{V}, \tau)$ is a T_2-ordered space.*

(2) *If f is a proper mapping and $(X, \mathscr{U}, \rho)$ is a T_3-ordered space, then $(Y, \mathscr{V}, \tau)$ is a T_3-ordered space.*

(3) *If f is a closed mapping and $(X, \mathscr{U}, \rho)$ is a normally ordered space, then $(Y, \mathscr{V}, \tau)$ is a normally ordered space.*

Proof. (1) If $G(\rho)$ and $G(\tau)$ are the graphs of ρ and τ respectively, then $G(\rho)$ is closed in X^2 since $(X, \mathscr{U}, \rho)$ is T_2-ordered ([12, p. 26, Proposition 1]). Let g be a mapping of X^2 onto Y^2 defined by $g((x,y)) = (f(x), f(y))$. Then g is proper by [5, §10, Proposition 4]. Further $g(G(\rho)) = G(\tau)$. Therefore $G(\tau)$ is closed in Y^2. Thus $(Y, \mathscr{V}, \tau)$ is T_2-ordered by [12, p. 26, Proposition 1].

(2) Let F be an increasing closed set of Y, and a any element of Y not contained in F. (In case F is a decreasing set of Y and $a \notin F$, the proof will go as well.) Then $f^{-1}(F)$ is an increasing closed set of X, $f^{-1}(a)$ a compact set, and $f^{-1}(a) \cap f^{-1}(F) = \emptyset$. Hence for each $x \in f^{-1}(a)$, there exist a decreasing neighborhood $U(x)$ of x and an increasing neighborhood $V(x)$ of $f^{-1}(F)$ such that $U(x) \cap V(x) = \emptyset$. Since $f^{-1}(a)$ is compact, there exists a finite set $\{x_1, \cdots, x_n\} \subset f^{-1}(a)$ such that $f^{-1}(a) \subset \bigcup_{i=1}^n U(x_i) = U_1$ and U_1 is a neighborhood of $f^{-1}(a)$. Let $V_1 = \bigcap_{i=1}^n V(x_i)$, then V_1 is an

increasing neighborhood of $f^{-1}(F)$ and satisfies $U_1 \cap V_1 = \emptyset$. Let $U_2 = Y - f(X - U_1)$, $V_2 = Y - f(X - V_1)$, then U_2 and V_2 are disjoint neighborhood of a and F respectively such that $U_2 \bar{\tau} V_2$. Therefore $U = d_Y(U_2)$ is a decreasing neighborhood of a and $V = i_Y(V_2)$ is an increasing neighborhood of F such that $U \cap V = \emptyset$. Thus $(Y, \mathscr{V}, \tau)$ is T_3-ordered.

(3) Let F_1 and F_2 be disjoint closed sets of Y such that F_1 is decreasing and F_2 is increasing. Then $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are disjoint closed sets of X suchthat $f^{-1}(F_1)$ is decreasing and $f^{-1}(F_2)$ is increasing. Therefore there exist a decreasing neighborhood U_1 of $f^{-1}(F_1)$ and an increasing neighborhood V_1 of $f^{-1}(F_2)$ such that $U_1 \cap V_1 = \emptyset$. Let $U_2 = Y - f(X - U_1)$, $V_2 = Y - f(X - V_1)$. Then U_2 and V_2 are disjoint neighborhoods of F_1 and F_2 respectively such that $U_2 \bar{\tau} V_2$. Hence $U = d_Y(U_2)$ is a decreasing neighborhood of F_1 and $V = i_Y(V_2)$ is an increasing neighborhood of F_2 such that $U \cap V = \emptyset$. Thus $(Y, \mathscr{V}, \tau)$ is a normally ordered space. Q. E. D.

Remark 2. If $(X, \mathscr{U}, \rho)$ is a compact space $(X, \mathscr{U})$ equipped with a closed order ρ (i. e., the graph of ρ is closed in X^2), then $(X, \mathscr{U}, \rho)$ is called a compact ordered space. Therefore a compact ordered space is just the same as a compact T_2-ordered space (cf. [12, pp. 25, 44]). Now, suppose that f is a mapping of $(X, \mathscr{U}, \rho)$ onto $(Y, \mathscr{V}, \tau)$ where τ is the quotient order of ρ induced by f. Then the following results were obtained:

(1) In case $\mathscr{V}$ is the identification topology determined by f and $(X, \mathscr{U}, \rho)$ is a compact ordered space, $(Y, \mathscr{V}, \tau)$ is a compact ordered space if and only if $(Y, \mathscr{V})$ is a T_2 space ([13, Proposition 9]).

(2) If f is a proper mapping and $(X, \mathscr{U}, \rho)$ is a locally compact T_2-ordered space, then $(Y, \mathscr{V}, \tau)$ is a locally compact T_2-ordered space ([10, Theorem 1]).

Remark 3. The assertion (2) (resp. (3)) of the Theorem is still valid if "T_3-ordered" (resp. "normally ordered") is replaced by "T_3-ordered in Adnadjević' sense" (resp. "T_4-ordered in Adnadjević' sense").

In topological spaces, the closed image of a T_1 space is also T_1. However, in topological ordered spaces, even the proper image of T_1-ordered space is not necessarily T_1-ordered. We shall conclude our study with exhibiting an example for this.

Example 2. Let X be the set $\{(a, x) : a = 0 \text{ or } 1,\ x \text{ is a real number}\}$. The topology $\mathscr{U}$ on X is the usual one, and the partial order ρ on X is defined as follows: $(a, x)\ \rho\ (b, y)$ if and only if $a = 0$, $b = 1$, $x = y$ and x is a rational number; or $a = b$, $x = y$. Then $(X, \mathscr{U}, \rho)$ is T_1-ordered but not

T_2-ordered. Let $A=\{(0, x): x\in[0,1]\}$. We introduce an equivalence relation R on X as follows: $(r, s)\in R$ if and only if $r, s\in A$ or $r=s$. Let $Y=X/R$, and f the projection of X onto Y. If $\mathscr{V}$ is the identification topology determined by f and τ is the quotient order of ρ induced by f then $(Y, \mathscr{V}, \tau)$ is a topological ordered space and f is a proper mapping. But $(Y, \mathscr{V}, \tau)$ is not T_1-ordered. This is because, for $a^*=f(A)\in Y$, $b^*=\{(1, b)\}\in Y$ where b is an irrational number contained in $[0,1]$, $a^* \parallel b^*$, and every decreasing neighborhood of b^* should necessarily contain a^*.

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(*Received August 1, 1975*)