REMARKS ON MANIFOLDS OF NEGATIVE CURVATURE

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§ 1. Statement of results

Let M be an $n(\geqq 2)$-dimensional, connected and complete Riemannian manifold. A non-empty subset C of M is called *totally convex* if it contains every geodesic segment of M whose endpoints are in C.

Let $\widetilde{M}$ be a Riemannian covering manifold of M. If M contains a proper totally convex subset C, then $\pi^{-1}(C)$ is a proper totally convex subset of $\widetilde{M}$, where π is the covering map. Generally, the converse of this property does not hold. In the present note, we consider this problem when M is of negative sectional curvature. We prove, in § 2, the following

Proposition 1. *Let M be a connected and complete Riemannian manifold of negative sectional curvature. Let $\widetilde{M}$ be the regular Riemannian covering manifold of M corresponding to a proper normal subgroup of the fundamental group $\pi_1(M)$ of M. If $\widetilde{M}$ contains a proper closed totally convex subset and at least one closed geodesic, then there exists a proper closed totally convex subset in M.*

Let M be a connected and complete Riemannian manifold of non-positive sectional curvature. We say that a continuous function on M is *convex* if its restriction to any geodesic of M is convex. M admits a non-trivial convex function if and only if M contains a proper closed totally convex subset [1]. Furthermore, we suppose that M is compact. Then every convex function on M is constant. Hence M does not contain any proper closed totally convex subset. For any $\alpha\in\pi_1(M)$, $\alpha\neq 1$, a closed path belonging to α is freely homotopic to a closed geodesic. Thus every non simply connected regular Riemannian covering manifold of M contains closed geodesics. By these facts and Proposition 1, we have

Proposition 2. *Let M be a connected and compact Riemannian manifold of negative sectional curvature. Then every non simply connected regular Riemannian covering manifold of M does not admit any non-trivial convex function.*

Let M be a connected and complete Riemannian manifold of non-positive sectional curvature. Then the square of the length of a Killing vector field on M is a convex function on M. We see that a Killing vector field on M with bounded length is a parallel vector field and each sectional curvature for plane containing it is zero. Therefore, if M is of negative sectional curvature, every Killing vector field on M with bounded length vanishes identically. By the above facts and Proposition 2, we have

Corollary. *Under the condition of Proposition* 2, *every non simply connected regular Riemannian covering manifold of* M *does not admit any Killing vector field.*

Remark. There is a connected and compact Riemannian manifold of negative sectional curvature which has a non simply connected regular Riemannian covering manifold whose volume is infinite.

Finally, in § 3, we prove the following

Proposition 3. *Let* M *be a connected and complete Riemannian manifold of negative sectional curvature. Let* φ *be an isometry of* M. *If the displacement function* $d(p, \varphi(p))$, $p \in M$, *is constant on an open subset of* M, *then* φ *is the identity transformation.*

This result is a generalization of the following known fact: Let M be a simply connected and complete Riemannian manifold of negative sectional curvature, and φ an isometry of M. If φ is a Clifford translation, then φ is the identity transformation. Here an isometry of a metric space is called a Clifford translation if the distance between a point and its image is constant for every point.

Remark. After the author had proven the above results, he knew P. Eberlein and B. O'Neill's paper [2]. Proposition 2 is also obtained from their results; (9. 12) and (11. 12).

§ 2. Proof of Proposition 1

Let G be a proper normal subgroup of $\pi_1(M)$. Let $\pi: \widetilde{M} \longrightarrow M$ be the regular Riemannian covering corresponding to G. We first note that the isometry group Γ which is isomorphic to $\pi_1(M)/G$ acts transitively on each fibre $\pi^{-1}(p)$, $p \in M$. By the hypothesis $\widetilde{M}$ contains a proper closed totally convex subset. Let C be a proper closed totally convex subset of $\widetilde{M}$. Since $\widetilde{M}$ is of negative sectional curvature, C

contains every closed geodesic of $\widetilde{M}$ [1]. Let S be the intersection of all proper closed totally convex subsets of $\widetilde{M}$. Since, by the hypothesis, $\widetilde{M}$ contains at least one closed geodesic, we see that S is non-empty and a proper closed totally convex subset of $\widetilde{M}$. Let f be an element of Γ. Since f is isometric, $f(S)$ is also totally convex. We see that $f(S)=S$. Hence S is Γ-invariant. $\pi(S)\subset M$, because Γ acts transitively on each fibre $\pi^{-1}(p)$, $p\in M$, and S is Γ-invariant. Now we shall show that $\pi(S)$ is totally convex. Let p and q be any two points of $\pi(S)$ and σ a geodesic segment of M from p to q. Let $\tilde{\sigma}$ be a unique lift of σ which starts from $\tilde{p}\in\pi^{-1}(p)\cap S$ and $\tilde{q}$ the endpoint of $\tilde{\sigma}$. Then for $\tilde{q}'\in\pi^{-1}(q)\cap S$ there exists $f\in\Gamma$ such that $f(\tilde{q}')=\tilde{q}$. Since S is Γ-invariant, $\tilde{q}\in S$. Hence $\tilde{\sigma}\subset S$, because S is totally convex. Thus $\sigma=\pi(\tilde{\sigma})\subset\pi(S)$. This implies that $\pi(S)$ is totally convex. We complete the proof.

§ 3. Proof of Proposition 3

Let M be a connected and complete Riemannian manifold of negative sectional curvature, and φ an isometry of M. For a geodesic segment σ of M we denote by σ^* the geodesic extention of σ in both directions. We say that φ translates a geodesic $\tau: R\longrightarrow M$ if there exists a positive constant number c such that $\varphi\circ\tau(t)=\tau(t+c)$ for every $t\in R$.

We need the following

Lemma ([1], [3], and [4]). *Let $\widetilde{M}$ be a simply connected and complete Riemannian manifold of negative sectional curvature. Let τ_1 and τ_2 be distinct geodesics of $\widetilde{M}$. Then the function $t\longrightarrow d(\tau_1(t), \tau_2(R))$, $t\in R$, is unbounded.*

Now we shall prove Proposition 3. Let A be an open subset on which $d(p, \varphi(p))$ is constant, $p\in A$. Suppose for contradiction that φ is not the identity. Then we may assume that $d(p, \varphi(p))=a=\text{const}>0$ for every $p\in A$. Let q be a point of A and τ: $[0, a]\longrightarrow M$ a minimizing geodesic from q to $\varphi(q)$. Note that τ and $\varphi\circ\tau$ do not overlap each other. The angle between τ and $\varphi\circ\tau$ is π. Suppose that it is less than π. Since A is open, we can take an interior point q' of φ ([0, a]) which is contained in A. Then, by the triangle inequality, we have $d(q', \varphi(q'))<d(q, \varphi(q))=a$ which implies a contradiction. Since φ is an isometry, we see that τ^* is translated by φ. If M is simply

connected, we see that the above fact contradicts Lemma. From now on, we assume that M is not simply connected. Fix $p \in A$. Let $\{p_j\}$, $j=1, 2, \cdots$, be a sequence of points of A which converges to p. For each p_j we take a minimizing geodesic $\tau_j : [0, a] \longrightarrow M$ from p_j to $\varphi(p_j)$. Then $\{\tau_j\}$, $j=1, 2, \cdots$, uniformly converges to a minimizing geodesic $\sigma : [0, a] \longrightarrow M$ from p to $\varphi(p)$. Hence, if we choose a sufficiently large number k, for each $t \in [0, a]$ $\tau_k(t)$ and $\sigma(t)$ can be connected by a unique minimizing geodesic segment of M. Fix such a k and set $\tau_k = \tau$. Then τ is freely homotopic to σ. Let $\tilde{\sigma}$ be a lift of σ^* to the universal Riemannian covering manifold $\widetilde{M}$ of M. Noticing that σ^* and τ^* are translated by φ, we can lift τ^* to a geodesic $\tilde{\tau}$ of $\widetilde{M}$ such that the function $t \longrightarrow d(\tilde{\tau}(t), \tilde{\sigma}(R))$, $t \in R$, is bounded. But this contradicts Lemma. Hence we complete the proof.

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