

ON THE SUBGROUPS H OF A GROUP G SUCH THAT $J(KH)KG \supset J(KG)$

Dedicated to Professor KIITI MORITA on his 60th birthday

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Let K be a field of characteristic $p > 0$, and G a finite group whose order is divisible by p . $J(KG)$ will denote the Jacobson radical of the group algebra KG , and all modules under consideration will be right modules. For a subset S of G , $\hat{S}$ will denote the sum of all elements of S . If T is a subset of KG then $l(T)$ (resp. $r(T)$) will denote the left (resp. right) annihilator of T in KG .

We consider the following classes of subgroups of G : $\mathfrak{P}(G) = \{H \subset G \mid H \text{ contains a Sylow } p\text{-subgroup of } G\}$, $\mathfrak{R}(G) = \{H \subset G \mid J(KG) \subset J(KH)KG\}$, and $\mathfrak{C}(G) = \{H \subset G \mid \text{the induced } KG\text{-module } \mathfrak{N}^G = \mathfrak{N} \otimes_{KH} KG \text{ is completely reducible for every irreducible } KH\text{-module } \mathfrak{N}\}$. By [3], it is known that the class $\mathfrak{P}(G)$ coincides with that of subgroups H of G such that every KG -module is (G, H) -projective.

Recently, in his paper [2], D. C. Khatri gave some sufficient conditions under which the above three classes are identical. In the present paper, we shall prove that $\mathfrak{R}(G) = \mathfrak{C}(G)$ without any restriction (Theorem 1) and that G is p -radical, namely, $\mathfrak{P}(G) = \mathfrak{R}(G)$ if and only if $\mathfrak{R}(G)$ contains a Sylow p -subgroup of G (Theorem 3). In §2, one will see that the results in [2] can be easily derived from Theorem 3. Furthermore, in case G is a Frobenius group and the order of its complement is divisible by p , we shall show that G is p -radical if and only if so is H (Theorem 8). Finally, as an application of Theorem 3, we shall present the condition for a p -solvable, p -radical group to have no blocks of p -defect 0 (Theorem 10).

1. At first, we shall prove the following which contains a result in [2]:

Theorem 1. $\mathfrak{R}(G) = \mathfrak{C}(G)$.

Proof. Let $H \in \mathfrak{R}(G)$, and $\mathfrak{N}$ an irreducible KH -module. Then, we have $\mathfrak{N}^G J(KG) = (\mathfrak{N} \otimes KG) J(KG) = \mathfrak{N} \otimes J(KG) \subset \mathfrak{N} \otimes J(KH)KG = \mathfrak{N} J(KH) \otimes KG = 0$. Hence, $\mathfrak{N}^G$ is completely reducible, and $H \in \mathfrak{C}(G)$. Conversely, assume that $H \in \mathfrak{C}(G)$. Let $G = \cup_{i=1}^n Hx_i$, $x_1 = 1$, be a

coset decomposition of G over H . Let $\alpha = \sum_{i=1}^n a_i x_i$ be an arbitrary element of $J(KG)$, where $a_i \in KH$. If $\mathfrak{N}$ is an arbitrary irreducible KH -module then $\mathfrak{N}^a \alpha = 0$, since $\mathfrak{N}^a$ is completely reducible. Therefore, $0 = (\mathfrak{N} \otimes 1) \alpha = \sum_{i=1}^n \mathfrak{N} a_i \otimes x_i$. This implies that $\mathfrak{N} a_i = 0$ for all i , and so $a_i \in J(KH)$. Hence $H \in \mathfrak{R}(G)$.

Remark 1. Since there exists a splitting field for G which is finite separable over K , $KG/J(KG)$ is a separable algebra. Therefore, for an arbitrary extension field F of K there holds $J(FG) = FJ(KG)$ and $J(KG) = J(FG) \cap KG$. Now, let L be an arbitrary field of characteristic p , and Z_p the prime field of characteristic p . If $H \in \mathfrak{R}(G)$, that is, $J(KG) \subset J(KH)KG$, then $J(Z_p G) = J(KG) \cap Z_p G \subset J(KH)KG \cap Z_p G = KJ(Z_p H)Z_p G \cap Z_p G = J(Z_p H)Z_p G$. Therefore, $J(LG) = LJ(Z_p G) \subset LJ(Z_p H)Z_p G = J(LH)LG$. This means that $\mathfrak{R}(G)$ is determined by G and p .

Khatri [2] proved that $\mathfrak{R}(G) \subset \mathfrak{P}(G)$ for any p -nilpotent group G . However, as was stated in [6], we obtain the following:

Theorem 2. $\mathfrak{R}(G) \subset \mathfrak{P}(G)$.

Proof. Let $H \in \mathfrak{R}(G)$. Then $\mathfrak{A} = K\hat{H} \otimes_{KH} KG$ is completely reducible by Theorem 1. Let $e = \sum_{i=1}^n \hat{H} \otimes x_i \in \mathfrak{A}$, and $\varphi: \mathfrak{A} \rightarrow Ke$ a KG -epimorphism defined by $\varphi(\sum_{i=1}^n k_i \hat{H} \otimes x_i) = \sum_{i=1}^n k_i e$ ($k_i \in K$), where $G = \cup_{i=1}^n Hx_i$ is a coset decomposition of G over H . Then $\text{Ker } \varphi = \{\sum_{i=1}^n k_i \hat{H} \otimes x_i \mid \sum_{i=1}^n k_i = 0\}$. We can easily see that Ke is the unique trivial KG -submodule of the completely reducible KG -module $\mathfrak{A}$, and so we have $\mathfrak{A} = \text{Ker } \varphi \oplus Ke$. If p divides $[G:H]$, then $e \in \text{Ker } \varphi$. This contradiction shows that $[G:H]$ is prime to p , and hence $H \in \mathfrak{P}(G)$.

If $\mathfrak{P}(G)$ coincides with $\mathfrak{R}(G)$, G will be called a p -radical group. (Any p' -group is p -radical by definition.)

Theorem 3. The following conditions are equivalent:

- (1) G is p -radical.
- (2) $\mathfrak{R}(G)$ contains a Sylow p -subgroup of G .
- (3) $J(KG) = \cap_{x \in G} J(KP^x)KG$, where P is a Sylow p -subgroup of G .
- (4) $\mathcal{K}(J(KG)) = \sum_{x \in G} KG\hat{P}^x$, where P is a Sylow p -subgroup of G .

Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3): Let σ_x be the inner automorphism of G by $x \in G$, and σ_x^* the algebra automorphism of KG

induced by σ_x . If $\mathfrak{R}(G)$ contains a Sylow p -subgroup P of G then $J(KG) \subset J(KP)KG$. Hence, $J(KG) = \sigma_x^*(J(KG)) \subset \sigma_x^*(J(KP)KG) = J(KP^x)KG$, whence it follows $J(KG) \subset \bigcap_{x \in G} J(KP^x)KG$. On the other hand, $\bigcap_{x \in G} J(KP^x)KG$ is a nilpotent ideal of KG by [7, Prop. 2], and so $J(KG) = \bigcap_{x \in G} J(KP^x)KG$. (3) $\Rightarrow$ (4): Since KG is a Frobenius algebra, we have $rl(R) = R$ for any right ideal R of KG . Hence, we have $l(J(KG)) = l(\bigcap_{x \in G} J(KP^x)KG) = \sum_{x \in G} l(J(KP^x)KG) = \sum_{x \in G} KG\hat{P}^x$. (4) $\Rightarrow$ (2): Since $l(J(KG)) \supset KG\hat{P}$, we have $J(KG) = rl(J(KG)) \subset r(KG\hat{P}) = J(KP)KG$. Hence, $P \in \mathfrak{R}(G)$. (3) $\Rightarrow$ (1): If $H \in \mathfrak{P}(G)$ then H contains a Sylow p -subgroup P of G . Let $G = \bigcup_{i=1}^n Hx_i$ be a coset decomposition of G over H . Then, we have $J(KG) = \bigcap_{x \in G} J(KP^x)KG \subset \bigcap_{x \in H} J(KP^x)KG = \bigcap_{x \in H} (\sum_{i=1}^n J(KP^x)KHx_i) = \sum_{i=1}^n (\bigcap_{x \in H} J(KP^x)KH)x_i = (\bigcap_{x \in H} J(KP^x)KH)KG \subset J(KH)KG$. Hence $H \in \mathfrak{R}(G)$. By Theorem 2, we have eventually $\mathfrak{P}(G) = \mathfrak{R}(G)$.

Remark 2. In [1], W.E. Deskins asserted that if G is a p -nilpotent group then $\mathfrak{R}(G)$ contains a Sylow p -subgroup of G . But this is false and a counter example is given by A. I. Saksonov [6]: Let $p=3$, and $G=SL(2, 3)$. Then, $G = \langle x, y, z | x^4=1, x^2=y^2, yxy^{-1}=x^{-1}, z^3=1, zxxz^{-1}=y, zyz^{-1}=xy \rangle$. Let c_a be the class sum of the conjugate class containing $a \in G$. Then, $c_x = x + x^3 + y + xy + x^2y + x^3y$, $c_z = z + zx^2y + zx^3 + zx^3y$ and $c_z^2 = 1 - c_x$. Since c_x is an idempotent, we have $(c_x + c_z - 1)^3 = c_x^3 + c_z^3 - 1 = 0$. Hence, $c_x + c_z - 1$ is a central nilpotent element of KG , and so an element of $J(KG)$. However, it is not contained in $J(K\langle z \rangle)KG$. The above example shows also that an extension of a p -radical group by a p -radical group is not always p -radical (cf. [2, p. 61]).

2. In virtue of Theorem 3, we can prove the results of Khatri [2] without effort.

Theorem 4. ([2, Th. 2]). *Let H be a normal subgroup of G . Then, H is in $\mathfrak{P}(G)$ if and only if H is in $\mathfrak{R}(G)$.*

Proof. If $H \in \mathfrak{P}(G)$ then G/H is a p' -group. Therefore, $J(KG) = J(KH)KG$, and so $H \in \mathfrak{R}(G)$. The converse is contained in Theorem 2.

Theorem 5. ([2, Th. 5]). *Let N be a normal subgroup of G such that G/N is a p' -group. Then, G is p -radical if and only if so is N .*

Proof. If G is p -radical then $J(KG) \subset J(KP)KG$, where P is a Sylow p -subgroup of G . Since G/N is a p' -group, N contains P . There-

fore, $J(KN) = J(KG) \cap KN \subset J(KP)KG \cap KN = J(KP)KN$. Hence, $P \in \mathfrak{R}(N)$, and so N is p -radical by Theorem 3. Conversely, if N is p -radical then $J(KN) \subset J(KP)KN$, where P is a Sylow p -subgroup of N . Since G/N is a p' -group, P is a Sylow p -subgroup of G and $J(KG) = J(KN)KG \subset J(KP)KG$. Hence, $P \in \mathfrak{R}(G)$, and so G is p -radical again by Theorem 3.

Theorem 6. *If G is a p -radical group, and H a normal subgroup of G , then G/H is p -radical.*

Proof. Let $\nu: G \rightarrow G/H$ be the natural homomorphism, and $\nu^*: KG \rightarrow K(G/H)$ the algebra homomorphism induced by ν . Since G is p -radical, $J(KG) \subset J(KP)KG$, where P is a Sylow p -subgroup of G . Hence, $J(K(G/H)) = \nu^*(J(KG)) \subset \nu^*(J(KP)KG) = J(K(PH/H)) \cdot K(G/H)$, and so G/H is p -radical by Theorem 3.

Theorem 7. ([2, Th. 7]). *Let M be a normal p -subgroup of G . Then, G is p -radical if and only if so is G/M .*

Proof. If G/M is p -radical then $J(K(G/M)) \subset J(K(P/M))K(G/M)$, where P is a Sylow p -subgroup of G . This together with $J(K(G/M)) \cong J(KG)/J(KM)KG$ and $J(K(P/M))K(G/M) \cong J(KP)KG/J(KM)KG$ implies that $J(KG) \subset J(KP)KG$. Hence, G is p -radical by Theorem 3. The converse is contained in Theorem 6.

The next contains [2, Th. 4].

Theorem 8. *Let G be a Frobenius group with kernel N and complement H .*

(1) *If p divides the order of N , then G is p -radical.*

(2) *In case p divides the order of H , G is p -radical if and only if so is H .*

Proof. (1) If p divides the order of N , then any Sylow p -subgroup of G is normal by Thompson's theorem. Hence, G is p -radical by Theorem 7.

(2) By Theorem 6, it remains only to prove the if part. If H is p -radical then $J(KH) \subset J(KP)KH$, where P is a Sylow p -subgroup of H . Since $J(KG) = J(KH)\hat{N}$ by [4, Th. 4], we have $J(KG) \subset J(KP)KH\hat{N} \subset J(KP)KG$, and so G is p -radical by Theorem 3.

Theorem 9. ([2, p. 61]). *Let $G = H_1 \times H_2$. Then, G is p -radical if and only if so are H_1 and H_2 .*

Proof. If H_1 and H_2 are p -radical then $J(KH_i) \subset J(KP_i)KH_i$, where P_i is a Sylow p -subgroup of H_i . Since $J(KG) = J(KH_1)KH_2 + J(KH_2)KH_1$ by [5, Th.], we have $J(KG) \subset J(KP_1)KG + J(KP_2)KG$. If we put $P = P_1 \times P_2$ then $J(KP_i) \subset J(KP)$. Hence, we have $J(KG) \subset J(KP)KG$, and so G is p -radical by Theorem 3. The converse is contained in Theorem 6.

3. For a moment let G be a permutation group on $\Omega = \{1, 2, \dots, n\}$, and $G_{i_1, i_2, \dots, i_l}$ be the stabilizer of the points $\{i_1, i_2, \dots, i_l\} \subset \Omega$. Then we have

Lemma. *If p divides the order of $G_{\alpha, \beta}$ for every $\{\alpha, \beta\} \subset \Omega$, then $(\sum_{\alpha \in \Omega} KG \hat{G}_\alpha)^2 = 0$, and conversely.*

Proof. Let $G_\beta = \bigcup_{i=1}^s G_{\alpha, \beta} x_i$ be a coset decomposition of G_β over $G_{\alpha, \beta}$. If p divides the order of every $G_{\alpha, \beta}$, then $\hat{G}_\alpha \hat{G}_\beta = \hat{G}_\alpha \hat{G}_{\alpha, \beta} (x_1 + x_2 + \dots + x_s) = |G_{\alpha, \beta}| \hat{G}_\alpha (x_1 + x_2 + \dots + x_s) = 0$. For any $x \in G$ we have $x G_\beta x^{-1} = G_{x(\beta)}$, and so $\hat{G}_\alpha x \hat{G}_\beta = 0$. Thus, we obtain $(\sum_{\alpha \in \Omega} KG \hat{G}_\alpha)^2 = 0$. Conversely, if $(\sum_{\alpha \in \Omega} KG \hat{G}_\alpha)^2 = 0$ then $0 = \hat{G}_\alpha \hat{G}_\beta = |\hat{G}_{\alpha, \beta}| \hat{G}_\alpha (x + x_2 + \dots + x_s)$ for every $\{\alpha, \beta\} \subset \Omega$. Since $G_\alpha x_i \cap G_\alpha x_j = \emptyset$ for $i \neq j$, p divides the order of every $G_{\alpha, \beta}$.

Theorem 10. *Let G be a p -solvable and p -radical group. Then, G has no blocks of p -defect 0 if and only if each pair of Sylow p -subgroups of G has a non-trivial intersection.*

Proof. We choose K a splitting field for G . Let c be the sum of all p -elements of G . Then, by [8, Th. 1] and [7, Cor. 4], c^2 is equal to the sum of the block idempotents of p -defect 0 and $l(J(KG)) = KGc$. Since G is p -radical, we have $l(J(KG)) = \sum_{x \in G} KG \hat{P}^x$ by Theorem 3, where P is a Sylow p -subgroup of G . Therefore, G has no blocks of p -defect 0 if and only if $(\sum_{x \in G} KG \hat{P}^x)^2 = 0$. Let $G = \bigcup_{i=1}^n Px_i$, $x_i = 1$, be a coset decomposition of G over P . Regarding G as a permutation group on $\Omega = \{P, Px_2, \dots, Px_n\}$, we see that the stabilizer of Px_i is $x_i^{-1}Px_i$. Hence, by Lemma, $(\sum_{x \in G} KG \hat{P}^x)^2 = 0$ if and only if $x_i^{-1}Px_i \cap x_j^{-1}Px_j \neq 1$ for $1 \leq i, j \leq n$. This completes the proof.

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