

ON TWO THEOREMS OF A. ABIAN

Dedicated to Professor Kiiti Morita on his 60th birthday

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A (non-zero) ring without non-zero nilpotent elements is called a reduced ring. Recently, in his papers [1] and [2], A. Abian proved the following :

(I) A commutative reduced ring is a direct product of fields if and only if it is orthogonally complete and hyperatomic.

(II) A commutative reduced ring is a direct product of integral domains if it is orthogonally complete and superatomic.

In this paper, we shall prove that both these are still true for non-commutative reduced rings, more precisely,

Theorem 1. *The following conditions are equivalent :*

(1) *R is a reduced ring which is orthogonally complete and hyperatomic.*

(2) *R is a direct product of division rings.*

Theorem 2. *The following conditions are equivalent :*

(1) *R is a reduced ring which is orthogonally complete and superatomic.*

(2) *R is a direct product of integral domains and the annihilators of those integral domains exhaust the proper prime ideals of R .*

Although Theorem 1 has been obtained in [4] and our proofs of Theorems 1 and 2 are very similar to those of (I) and (II) in [1] and [2] respectively, we are much more skilful in performing those.

1. Definitions and lemmas. In a reduced ring, as is well-known, the intersection of prime ideals equals 0, namely, every non-zero element is excluded by some prime ideal (see for instance [5, p. 56]), and every idempotent is central. In what follows, R will represent always a reduced ring.

Lemma 1. *Let r and s be elements of a reduced ring R .*

(a) *If $rs = 0$, then $sr = 0$, and for every prime ideal P of R either r or s is contained in P .*

(b) *If $r^2s = r$, then there exists one and only one element r' such that $rr' = r'r$, $r^2r' = r$ and $rr'^2 = r'$. (r' will be called the semi-inverse*

of r .)

Proof. (a) $sr = 0$ is clear by $(sr)^2 = 0$. Moreover, $srR = 0$ yields $rRs = 0$. Hence, $r \in P$ or $s \in P$.

(b) This is only a consequence of [3, Lemma 1 and Theorem 1]. However, for the sake of completeness, we give here the proof. Evidently, $(rsr - r)^2 = 0$, and so $rsr = r$. By making use of this, we have $(sr^2 - r)^2 = 0$, which implies $sr^2 = r$ and $sr = sr^2s = rs$. Then, $r' = rs^2$ satisfies the relations requested. Now, let $rr'' = r''r$, $r^2r'' = r$ and $rr''^2 = r''$. Then, $r^2r' = r = r''r^2$ implies $r''r = r''r^2r' = rr'$. Hence, $r'' = r''^2r = r''rr' = rr'^2 = r'$.

Now, for $x, y \in R$ we define $x \leq y$ if and only if $xy = x^2$ (and $yx = x^2$ by Lemma 1 (a)). Then, the relation $\leq$ is a partial order in R . In fact, the reflexivity and the antisymmetry are easy, and the transitivity can be seen as follows: If $xy = x^2$ and $yz = y^2$ then $x^2z = xyz = xy^2 = x^2y = x^3$, i. e., $x(xz - x^2) = 0$, which implies $(xz - x^2)x = 0$ (Lemma 1 (a)). Hence, $(xz - x^2)^2 = 0$, and eventually $xz = x^2$.

Following [1], R is defined to be *orthogonally complete* if for every orthogonal subset T of R (i. e., a subset T such that $st = 0$ for every different $s, t \in T$) there exists $\sup T$ with respect to $\leq$ mentioned above. A non-zero element $a \in R$ is called an *atom* of R if $x \leq a$ implies $x = 0$ or $x = a$. An atom a is called a *hyperatom* if $ax \neq 0$ ($x \in R$) implies always $axs = a$ for some $s \in R$, and H will denote the set of all hyperatoms of R . R is defined to be *hyperatomic* if for every non-zero element $r \in R$ there exists $a \in H$ such that $a \leq r$. Finally, an element $a \in R$ is called a *superatom* if a is contained in every proper prime ideal except exactly one $P(a)$, and S will denote the set of all superatoms of R . (In [2], a superatom in our sense is called an atom.) Obviously, if a is in S then a is non-zero and $-a$ is a superatom with $P(-a) = P(a)$. R is defined to be *superatomic* if for every proper prime ideal P and every element $r \in R \setminus P$ there exists $a \in S$ such that $a \in R \setminus P$ and $a \leq r$.

Lemma 2. *If a is a hyperatom of a reduced ring R then aa' is an idempotent hyperatom, where a' is the semi-inverse of a .*

Proof. Since $a \in H$ and $aa' \neq 0$, by Lemma 1 (b) a has the semi-inverse a' and $e = aa'$ is a central idempotent. If $0 \neq er = aa'r$ ($r \in R$) then $a(a'r)t = a$ with some $t \in R$, and so $(er)(ta') = e$. It remains therefore to show that e is an atom. Assume that $x \leq e$, i. e., $ex = x^2$. Then, $ex^2 = e(ex) = x^2$, which yields $(ex - x)^2 = 0$. Hence,

$x = ex = x^2$. Recalling that x is then central, we have $xa \leq a$. Accordingly, $(xa)a' = xe = x$ is either 0 or $aa' = e$. This proves that e is an atom.

Now, let $E = \{e_r | r \in R\}$ be the set of all idempotent hyperatoms of R . We claim that $A_r = e_r R$ is a division ring. In fact, e_r is the identity of A_r , and for every non-zero element $e_r r (r \in R)$ there exists an element $s \in R$ such that $e_r = e_r r s = (e_r r) (e_r s)$ (Lemma 2).

Lemma 3. *If R is a hyperatomic reduced ring, then for every non-zero element $r \in R$ there holds $rE \neq 0$.*

Proof. By hypothesis, $ra = a^2$ with some $a \in H$. Then, by Lemma 2, $aa' \in E$ and $raa' = a^2 a' = a \neq 0$, where a' is the semi-inverse of a .

Lemma 4. *Let R be a reduced ring.*

(a) *Let $a \in S$, and $r \in R$. If $ar \neq 0$ then $ar, ra \in S$ and $P(ar) = P(ra) = P(a)$.*

(b) *Let $a, b \in S$. Then, $ab \neq 0$ if and only if $P(a) = P(b)$.*

(c) *Let $a, b \in S$. If $ab \neq 0$ and $a - b \neq 0$ then $a - b \in S$ and $P(a - b) = P(a)$.*

Proof. (a) Immediately, ar is contained in every proper prime ideal different from $P(a)$. On the other hand, there exists a proper prime ideal excluding ar . Hence, $ar \in S$ and $P(ar) = P(a)$. Similarly, by Lemma 1 (a) we see that $ra \in S$ and $P(ra) = P(a)$.

(b), (c) If $ab \neq 0$ then $P(a) = P(ab) = P(b)$ by (a), and $a - b$ is contained in every prime ideal different from $P(a)$. Hence, in case $a - b \neq 0$, $a - b \in S$ and $P(a - b) = P(a)$. Conversely, assume that $P(a) = P(b)$. If $ab = 0$ then by Lemma 1 (a) $a \in P(a)$ or $b \in P(b)$, a contradiction.

Corollary 1. *In a reduced ring R , every superatom a is an atom.*

Proof. Assume that $x \leq a$ and $x \neq 0$. By Lemma 4 (a), $xa = x^2 \in S$ and $P(xa) = P(a)$, whence it follows $x \notin P(a)$. Hence, $x(a - x) = 0$ implies $a - x \in P(a)$ (Lemma 1 (a)), and so $a(a - x) \in P(a)$. On the other hand, $a(a - x)$ is contained in every prime ideal different from $P(a)$. We obtain therefore $a(a - x) = 0$. Combining this with $x(a - x) = 0$, we readily obtain $(x - a)^2 = 0$, and hence $x = a$.

In virtue of Lemma 4 (b), we can define an equivalence relation $\sim$ in S , where $a \sim b$ if and only if $ab \neq 0$, or equivalently, $P(a) = P(b)$. Let

$S = \bigcup_{\lambda \in A} S_\lambda$ be the partition of S into the equivalence classes with respect to $\sim$. Obviously, $S_\lambda \mapsto P_\lambda = P(a)$ ($a \in S_\lambda$) is well-defined, and $S_\lambda \cap P_\lambda = \emptyset$. As a direct consequence of Lemma 4, we see that $B_\lambda = S_\lambda \cup \{0\}$ is an ideal of R which is an integral domain and $B_\lambda P_\lambda = 0$.

In the rest of this section, we assume further that R is superatomic. Then, $S_\lambda \mapsto P_\lambda$ gives a 1-1 correspondence between $\{S_\lambda | \lambda \in A\}$ and the set of all proper prime ideals of R . If r is in $R \setminus P_\lambda$ then by hypothesis there exists some $r_\lambda \in S_\lambda$ with $r_\lambda \leq r$. We claim here that such r_λ is unique. In fact, if $\bar{r}_\lambda \leq r$ and $\bar{r}_\lambda \in S_\lambda$ then $(r - r_\lambda)r_\lambda = 0 = (r - \bar{r}_\lambda)\bar{r}_\lambda$ implies $r - r_\lambda, r - \bar{r}_\lambda \in P_\lambda$ (Lemma 1 (a)). Hence, $\bar{r}_\lambda - r_\lambda \in P_\lambda \cap B_\lambda = 0$, namely, $\bar{r}_\lambda = r_\lambda$. On the other hand, if r is in P_λ then there is no $r_\lambda \in S_\lambda$ with $r_\lambda \leq r$. We define here the map $g_\lambda: R \rightarrow B_\lambda$ by

$$g_\lambda(r) = \begin{cases} r_\lambda & \text{if } r \notin P_\lambda, \\ 0 & \text{if } r \in P_\lambda. \end{cases}$$

Lemma 5. *Let R be a superatomic reduced ring. Then, g_λ is a ring homomorphism leaving every element of the integral domain B_λ invariant and $\text{Ker } g_\lambda = P_\lambda$. Accordingly, $R = P_\lambda \oplus B_\lambda$ and P_λ coincides with the annihilator of B_λ .*

Proof. Since R is a reduced ring, the (right and left) annihilator of B_λ has the intersection 0 with B_λ . It remains therefore to prove (i) $g_\lambda(r+s) = g_\lambda(r) + g_\lambda(s)$ and (ii) $g_\lambda(rs) = g_\lambda(r)g_\lambda(s)$ ($r, s \in R$). First, we consider the case $r \in P_\lambda$ and $s \notin P_\lambda$. Since $s_\lambda r = 0$, we obtain $s_\lambda(r+s) = s_\lambda s = s_\lambda^2$, which means $(r+s)_\lambda = s_\lambda$. Hence, we have (i), and readily (ii). Next, we consider the case $r \notin P_\lambda$ and $s \notin P_\lambda$. We claim that $rs_\lambda = r_\lambda s_\lambda$ and $sr_\lambda = s_\lambda r_\lambda$. In fact, by $r_\lambda \leq r$ and $s_\lambda \leq s$ it follows $r_\lambda rs_\lambda = r_\lambda^2 s_\lambda$. Since B_λ is an integral domain, we have $rs_\lambda = r_\lambda s_\lambda$, and similarly $sr_\lambda = s_\lambda r_\lambda$. Hence, we have $(rs)(r_\lambda s_\lambda) = rs_\lambda r_\lambda s_\lambda = (r_\lambda s_\lambda)^2$, namely, $(rs)_\lambda = r_\lambda s_\lambda$, proving (ii). In order to see (i), we shall distinguish between two cases. (1) $r + s \notin P_\lambda$: If $r_\lambda + s_\lambda = 0$ then $r_\lambda r = r_\lambda^2 = s_\lambda^2 = s_\lambda s = -r_\lambda s$, i. e., $r_\lambda(r+s) = 0$, whence it follows $r_\lambda \in P_\lambda$ or $r+s \in P_\lambda$ (Lemma 1 (a)). This contradiction means that $r_\lambda + s_\lambda \in S_\lambda$ (Lemma 4). Since $(r+s)(r_\lambda + s_\lambda) = rr_\lambda + sr_\lambda + rs_\lambda + ss_\lambda = r_\lambda^2 + s_\lambda r_\lambda + r_\lambda s_\lambda + s_\lambda^2 = (r_\lambda + s_\lambda)^2$, we have $(r+s)_\lambda = r_\lambda + s_\lambda$, proving (i). (2) $r + s \in P_\lambda$: Since $0 = (r+s)r_\lambda = r_\lambda^2 + sr_\lambda = r_\lambda^2 + s_\lambda r_\lambda$ and $0 = (r+s)s_\lambda = rs_\lambda + s_\lambda^2 = r_\lambda s_\lambda + s_\lambda^2$, we obtain $(r_\lambda + s_\lambda)^2 = 0$, and so $r_\lambda + s_\lambda = 0$, proving (i). Finally, in case $r \in P_\lambda$ and $s \in P_\lambda$, there is nothing to prove.

2. Proofs of theorems. The notations employed in the preceding section will be used here.

Proof of Theorem 1. (1) $\Rightarrow$ (2): Let $f: R \longrightarrow \prod_{r \in \Gamma} A_r$ be the map defined by $f(r) = (re_r)$. Then, f is a ring homomorphism, and by Lemma 3 $\text{Ker } f = \{r \in R \mid rE = 0\} = 0$. If f is shown to be an isomorphism, $A_r (r \in \Gamma)$ are adapted for the division rings in (2). Now, let (r') be an arbitrary element of $\prod_{r \in \Gamma} A_r$. By $e_r e_\delta \leq e_r$ and e_δ , we can easily see that $e_r e_\delta = 0$ for every $r' \neq \delta$. Hence, $T = \{r' \mid r' \in \Gamma\}$ is an orthogonal subset of R and there exists $r = \sup T$. We shall prove now $re_\delta = r^\delta$ for every $\delta \in \Gamma$. By $r^\delta \leq r$, we obtain $r^\delta re_\delta = (r^\delta)^2$, i. e., $r^\delta \leq re_\delta$. On the other hand, to be easily seen, $r'(r^\delta - re_\delta + r) = (r')^2$, i. e., $r' \leq r^\delta - re_\delta + r$ for every $r' \in \Gamma$. Hence, $r \leq r^\delta - re_\delta + r$, and so $(r^\delta - re_\delta + r)r = r^2$, whence it follows $r^\delta(re_\delta) = r^\delta r = r^2 e_\delta = (re_\delta)^2$. Combining this with $r^\delta \leq re_\delta$, we obtain $re_\delta = r^\delta$.

(2) $\Rightarrow$ (1): Let R be the direct product of division rings $R_\kappa (\kappa \in K)$. Then, it is clear that R is orthogonally complete. If $x = (x^\kappa)$ is an arbitrary non-zero element of R , then there exists $\alpha \in K$ with $x^\alpha \neq 0$. Then, we can easily see that x^α is a hyperatom and $x^\alpha \leq x$.

Proof of Theorem 2. (1) $\Rightarrow$ (2): Let $g: R \longrightarrow \prod_{\lambda \in \Lambda} B_\lambda$ be the map defined by $g(r) = (g_\lambda(r))$. Then, by Lemma 5, g is a ring homomorphism with $\text{Ker } g = \bigcap_{\lambda \in \Lambda} \text{Ker } g_\lambda = \bigcap_{\lambda \in \Lambda} P_\lambda = 0$, and P_λ coincides with the annihilator of the integral domain B_λ . If g is shown to be an isomorphism, $B_\lambda (\lambda \in \Lambda)$ are adapted for the integral domains in (2). Now, we shall show that g is a surjection. Let (r^λ) be an arbitrary non-zero element of $\prod_{\lambda \in \Lambda} B_\lambda$, $N = \{\lambda \in \Lambda \mid r^\lambda = 0\}$, and $M = \Lambda \setminus N$. Since the set $T = \{r^\lambda \mid \lambda \in M\}$ is an orthogonal subset of R , by hypothesis there exists $r = \sup T$. To our end, it suffices to show that $r \in P_\lambda$ if and only if $\lambda \in N$. Assume first that $r \in P_\lambda$. If $\lambda \in M$, then $r^\lambda = r_\lambda \in S_\lambda$, but then $r_\lambda^2 = rr_\lambda = 0$, a contradiction. Conversely, assume that $\lambda \in N$. If $r \notin P_\lambda$, then $r^\mu r_\lambda = 0 = r_\lambda r^\mu$ for every $\mu \in M$. Hence, $r^\mu(r_\lambda + r) = r^\mu r = (r^\mu)^2$, namely, $r^\mu \leq r_\lambda + r$ for every $\mu \in M$. This implies $r \leq r_\lambda + r$, and so $r(r_\lambda + r) = r^2$. However, the last contradicts $r(r_\lambda + r) = r_\lambda^2 + r^2$.

(2) $\Rightarrow$ (1): Assume that R is the direct product of the integral domains $R_\kappa (\kappa \in K)$ and $\prod_{\kappa \neq \alpha} R_\kappa (\alpha \in K)$ exhaust the proper prime ideals of R . Then, R is orthogonally complete evidently. Moreover, if $x = (x^\kappa)$ is an arbitrary element of $R \setminus \prod_{\kappa \neq \alpha} R_\kappa$ then we can easily see that x^α is a superatom of R not contained in $\prod_{\kappa \neq \alpha} R_\kappa$ and $x^\alpha \leq x$.

Corollary 2. *If R is a reduced ring with 1 which is orthogonally*

complete and superatomic, then R is a finite direct sum of integral domains.

Proof. In any rate, by the proof of Theorem 2, $R = \prod_{\lambda \in A} B_\lambda$ and $\prod_{\lambda \neq \alpha} B_\lambda$ ($\alpha \in A$) exhaust the proper prime ideals of R . If A is infinite, then the proper ideal $\bigoplus_{\lambda \in A} B_\lambda$ is contained in some maximal ideal, which is a contradiction.

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Added in proof. A recent result of O. Goldman [J. Algebra **34** (1975), 64—73] enables us to see that the following condition is equivalent to those in Theorem 1:

(3) *R is a reduced ring with 1 which is complete in its intrinsic topology.*