ON A GENERALIZED p-VECTOR CURVATURE AND ITS APPLICATIONS

TOSHIO NASU

Introduction

Let (M, g) be an n-dimensional Riemannian manifold with Riemannian metric g, and R be its curvature tensor ($n \geqq 3$). In this paper, we shall consider a special curvature structure of order p given by

$$\phi_r = \frac{1}{r!} g^r \wedge \omega \qquad (0 \leqq r \leqq n-2),$$

where ω is a curvature structure of order 2 and $p = r+2$. This was suggested to us by the work of S. Tachibana. Especially, if $\omega = R$, ϕ_r is nothing else but E. Cartan's notion of "p-vector curvature", which was formulated in the present form by R. S. Kulkarni.

In Theorem 1, we shall find a main property of this curvature structure. As simple application of this theorem, we shall give in Theorem 2 a sufficient condition for a Riemannian manifold with non-vanishing constant $2p^{\text{th}}$ sectional curvature to be of constant curvature in usual sense. In the last section, we shall study somewhat in detail the mean curvature ρ for p-plane, which was introduced by Tachibana [7] in connection with the work in K. Yano and S. Bochner [10]. As second application of Theorem 1, we shall prove in Theorem 3 that this curvature ρ generally determines the metric g itself of (M, g).

We shall assume, throughout this paper, that all manifolds are connected and all objects are of differentiability class C^∞. For the terminology and notation, we generally follow [4].

1. Preliminaries on curvature structures

In this section, let us recall some basic facts on the ring of double forms for later use (for the details, see [4]).

Let (M, g) be an n-dimensional smooth Riemannian manifold and let $\mathscr{F}(M)$ be the ring of smooth functions on M. Let $\Lambda^p(M)$ and $\Lambda^{*p}(M)$ denote the bundles of p-vectors and of p-forms on M, respectively. For simplicity, we denote the space of sections of a bundle by the same notation as the bundle space. We consider the spaces

$$\mathscr{D}^{p,q}=\Lambda^{*p}(M)\otimes\Lambda^{*q}(M),\ 0\leqq p,\ q\leqq n,\ \mathscr{D}=\sum_{p,q}\mathscr{D}^{p,q},$$

where the tensor product is taken over $\mathscr{F}(M)$. An element ω of $\mathscr{D}^{p,q}$ is an $\mathscr{F}(M)$-linear map $\omega: \Lambda^p(M)\times\Lambda^q(M)\longrightarrow\mathscr{F}(M)$ and the value of ω on decomposable elements $u=x_1\wedge x_2\wedge\cdots\wedge x_p$ and $v=y_1\wedge y_2\wedge\cdots\wedge y_q$ is denoted by

$$\omega(u\otimes v)=\omega(x_1\ x_2\cdots x_p\otimes y_1\ y_2\cdots y_q),$$

where $x_1,\cdots,x_p,y_1,\cdots,y_q$ are vector fields on M. $\mathscr{D}$ forms an associative ring with respect to the natural "exterior product" as follows: for $\omega\in\mathscr{D}^{p,q}$ and $\theta\in\mathscr{D}^{r,s}$, we define

$$\begin{aligned}(1.1)\quad &(\omega\wedge\theta)(x_1\cdots x_{p+r}\otimes y_1\cdots y_{q+s})\\ &=\sum_{\tau\in Sh(p,r)}\sum_{\mu\in Sh(q,s)}\varepsilon_\tau\,\varepsilon_\mu\,\omega(x_{\tau_1}\cdots x_{\tau_p}\otimes y_{\mu_1}\cdots y_{\mu_q})\theta(x_{\tau_{p+1}}\cdots x_{\tau_{p+r}}\otimes y_{\mu_{q+1}}\cdots y_{\mu_{q+s}})\end{aligned}$$

for any vector fields $x_1,\cdots,x_{p+r},y_1,\cdots,y_{q+s}$. Here, $Sh(p,r)$ denotes the set of all (p,r)-shuffles

$$Sh(p,r)=\{\tau\in S_{p+r};\tau_1<\cdots<\tau_p\quad\text{and}\quad\tau_{p+1}<\cdots<\tau_{p+r}\},$$

where S_{p+r} is the symmetric group of degree $p+r$. Then, we have

$$\omega\wedge\theta=(-1)^{pr+qs}\,\theta\wedge\omega$$

for any $\omega\in\mathscr{D}^{p,q}$ and $\theta\in\mathscr{D}^{r,s}$. A symmetric element of $\mathscr{D}^{p,p}$ is called the *curvature structure of order p* and the set of such elements is denoted by $\mathscr{C}^p$. $\mathscr{C}=\sum_p\mathscr{C}^p$ is a commutative subring of $\mathscr{D}$ called the ring of curvature structures on M.

The *first Bianchi sum* $\mathfrak{S}$ maps $\mathscr{D}^{p,q}$ into $\mathscr{D}^{p+1,q-1}$ and is defined as follows. Let $\omega\in\mathscr{D}^{p,q}$. If $q=0$, we set $\mathfrak{S}\omega=0$. If $q\geqq 1$, then we set

$$\mathfrak{S}\omega(x_1\cdots x_{p+1}\otimes y_1\cdots y_{q-1})=\sum_{j=1}^{p+1}(-1)^j\,\omega(x_1\cdots\hat{x}_j\cdots x_{p+1}\otimes x_j\,y_1\cdots y_{q-1})$$

for any vector fields $x_1,\cdots,x_{p+1},y_1,\cdots,y_{q-1}$, where as usual $\wedge$ denotes omission. Then, for any $\omega\in\mathscr{D}^{p,q}$ and $\theta\in\mathscr{D}^{r,s}$ we have

$$\mathfrak{S}(\omega\wedge\theta)=\mathfrak{S}\omega\wedge\theta+(-1)^{p+q}\omega\wedge\mathfrak{S}\theta.$$

We define $\mathscr{C}_1^p=\mathscr{C}^p\cap\text{kernel}\ \mathfrak{S}$ and set $\mathscr{C}_1=\sum_p\mathscr{C}_1^p$. Then, owing to the above formula, $\mathscr{C}_1$ is a subring of $\mathscr{C}$.

The *contraction c* maps $\mathscr{D}^{p,q}$ into $\mathscr{D}^{p-1,q-1}$ and is defined as follows. If $\omega\in\mathscr{D}^{p,q}$ and $p=0$ or $q=0$, we set $c\omega=0$. If both $p,\ q\geqq 1$, then for any vector fields $x_1,\cdots,x_{p-1},y_1,\cdots,y_{q-1}$, we set

$$c\omega(x_1\cdots x_{p-1}\otimes y_1\cdots y_{q-1})=\sum_{k=1}^{n}\omega(e_k\, x_1\cdots x_{p-1}\otimes e_k\, y_1\cdots y_{q-1}),$$

where $\{e_1, \cdots, e_n\}$ is a locally defined orthonormal frame field with respect to the metric g. Then, we have

$$\mathfrak{S}\cdot c=c\cdot\mathfrak{S}$$

on $\mathscr{D}$ and

$$c(g\wedge\omega)=g\wedge c\omega+(n-p-q)\omega \tag{1.2}$$

for any $\omega\in\mathscr{D}^{p,q}$.

Let ω^p denote the exterior product of $\omega\in\mathscr{C}$ with itself p times. Then, by the formula (1.1) we find

$$\omega^p(x_1\, x_2\cdots x_p\otimes y_1\, y_2\cdots y_p)=p!\ \det\|\,\omega(x_i\otimes y_j)\,\|$$

for any $\omega\in\mathscr{C}^1$. Particularly, the norm $\|\cdot\|$ of a p-vector induced by the metric g can be written as

$$\|x_1\wedge x_2\wedge\cdots\wedge x_p\|^2=\frac{1}{p!}\,g^p(x_1\, x_2\cdots x_p\otimes x_1\, x_2\cdots x_p) \tag{1.3}$$

for the decomposable p-vector $x_1\wedge x_2\wedge\cdots\wedge x_p$.

Let G_p denote the Grassmann bundle of p-planes on M, and $\pi: G_p\longrightarrow M$ be its projection. For $\omega\in\mathscr{C}^p$, we define the corresponding curvature function $K_\omega: G_p\longrightarrow \mathbf{R}$ as follows: for any $\sigma\in G_p$

$$K_\omega(\sigma)=\frac{\omega(x_1\cdots x_p\otimes x_1\cdots x_p)}{\|x_1\wedge\cdots\wedge x_p\|^2}, \tag{1.4}$$

where $\{x_1, \cdots, x_p\}$ is a base of σ. The value $K_\omega(\sigma)$ depends only on σ. We say a point $m\in M$ is *isotropic* with respect to ω if K_ω is identically constant on the fibre $\pi^{-1}(m)$; otherwise, we call m *non-isotropic.*

The curvature function K_ω generally determines ω, that is, if $K_\omega=K_\theta$ on $\pi^{-1}(m)$, then we have $\omega=\theta$ at $m\in M$, for any $\omega,\ \theta\in\mathscr{C}^p_1$. In particular, from (1.3) we have

Lemma. *$K_\omega\,|\pi^{-1}(m)\equiv$ const. κ if and only if $\omega=\dfrac{\kappa}{p!}\,g^p$ at m, for any $\omega\in\mathscr{C}^p_1$.*

Finally, let R_{xy} be the curvature operator defined by

$$R_{xy}=[\nabla_x, \nabla_y]-\nabla_{[x,y]}$$

for any vector fields x and y, where ∇ denotes the covariant differentiation with respect to the metric g. The curvature tensor R of type (0, 4) is defined by the formula

$$R(xy \otimes uv) = \langle R_{xy} u, v \rangle$$

for any vector fields x, y, u and v. It is well-known that $R \in \mathscr{C}_1^2$. Also, $-cR \in \mathscr{C}_1^1$ and $-c^2R \in \mathscr{F}(M)$ are the Ricci tensor Ric and the scalar curvature Sc of (M, g), respectively.

2. Generalized p-vector curvature structures

In this section, let us consider the generalized p-vector curvature structure

$$\phi_r = \frac{1}{r!} g^r \wedge \omega \qquad (0 \leq r \leq n-2),$$

where ω is an element of $\mathscr{C}_1^2$ and $p = r+2$. It is easy to see that for any p-plane σ we have

(2. 1) $$K_{\phi_r}(\sigma) = \sum_{1 \leq i < j \leq p} \omega(e_i\, e_j \otimes e_i\, e_j)$$

from (1. 1) and (1. 4), where $\{e_1, \cdots, e_p\}$ is an orthonormal base of σ. Thus, the value $K_{\phi_r}(\sigma)$ differs by constant factor from the average value of K_ω over all 2-planes spanned by e_i and e_j. Similarly, for any $(n-1)$-plane σ we have

(2. 2) $$\frac{1}{2} c^2 \omega = K_{\phi_{n-3}}(\sigma) + K_{c\omega}(v),$$

where v is the normal vector of σ in the tangent space $T_{\pi(\sigma)}(M)$.

One of the principal properties of the curvature structure ϕ_r is the following theorem, whose proof is essentially due to Kulkarni [4].

Theorem 1. *Suppose that $K_{\phi_r} \mid \pi^{-1}(m) \equiv$ const. a for some point $m \in M$ and for some fixed integer r such that $0 \leq r \leq n-4$. Then we have*

$$\omega = \frac{\kappa}{2n(n-1)} g^2 \qquad \text{at } m,$$

where $\kappa = 2an(n-1)/(r+1)(r+2)$. The converse is also true.

Proof. If $r=0$, Theorem 1 is trivial. Hence, we suppose $r \geq 1$. The assumption $K_{\phi_r} \mid \pi^{-1}(m) \equiv$ const. a implies

(2. 3) $$\phi_r = \frac{\kappa}{2n(n-1)(r!)} g^{r+2} \quad \text{at } m$$

by Lemma, from which we obtain easily

$$K_{\phi_s} | \pi^{-1}(m) \equiv \frac{\kappa(s+1)(s+2)}{2n(n-1)}$$

for any s satisfying $r \leqq s \leqq n-2$. Especially, we get

$$K_{\phi_{n-3}} | \pi^{-1}(m) \equiv \frac{\kappa(n-2)}{2n},$$

from which we find $K_{c\omega} \, | \pi^{-1}(m) \equiv \text{const.}$, by (2. 2). Hence, we have by Lemma

(2. 4) $$c\omega = \frac{\kappa}{n} g \quad \text{at } m.$$

On the other hand, from the identity (1. 2) we have inductively

$$c(g^r \wedge \omega) = g^r \wedge c\omega + r(n-r-3)\, g^{r-1} \wedge \omega \qquad (r \geqq 1).$$

Accordingly, we get by (2. 4)

(2. 5) $$c\phi_r = \frac{\kappa}{n(r!)} g^{r+1} + (n-r-3)\, \phi_{r-1} \quad \text{at } m.$$

Since $r \leqq n-4$, by substituting (2. 3) into (2. 5) and then using the identity

$$cg^t = t(n-t+1)\, g^{t-1} \qquad \text{for any } t \geqq 1,$$

we obtain

$$\phi_{r-1} = \frac{\kappa}{2n(n-1)\{(r-1)!\}} g^{r+1} \quad \text{at } m.$$

It is easy to check that, continuing this way, we have finally

$$\phi_0 = \frac{\kappa}{2n(n-1)} g^2 \quad \text{at } m.$$

It will be easily seen that the converse is true. q. e. d.

Suppose $a=0$ in Theorem 1. Then we have immediately a certain *cancellation law* in the ring $\mathscr{C}_1$ of curvature structures as follows (cf. Lemma 1 and Lemma 2 in Tachibana [7]):

Corollary. *Suppose that* $\omega \in \mathscr{C}_1^2$. *If* $g^r \wedge \omega = 0$ *at* $m \in M$ *for some* *r such that* $0 \leqq r \leqq n-4$, *then we have* $\omega=0$ *at* m.

3. Application to the Riemannian manifold with constant $2p^{\text{th}}$ sectional curvature

The $2p^{\text{th}}$ sectional curvature γ_{2p} of Thorpe [8] is given by the formula

$$(3.1) \qquad \gamma_{2p}(\sigma)=\frac{(-1)^p}{2^p\{(2p)!\}}\sum_{\tau,\mu\in S_{2p}} \varepsilon_\tau \varepsilon_\mu R(e_{\tau_1} e_{\tau_2} \otimes e_{\mu_1} e_{\mu_2}) \cdots R(e_{\tau_{2p-1}} e_{\tau_{2p}} \otimes e_{\mu_{2p-1}} e_{\mu_{2p}})$$

for any $2p$-plane $\sigma \in G_{2p}$, where $\{e_1, \cdots, e_{2p}\}$ is an orthonormal base of σ. In the case $p=1$, γ_2 is the usual sectional curvature of (M, g). Since we have from the formula (1.1)

$$\omega^p(x_1 \cdots x_{2p} \otimes y_1 \cdots y_{2p})$$
$$=\frac{1}{2^{2p}}\sum_{\tau,\mu\in S_{2p}} \varepsilon_\tau \varepsilon_\mu \omega(x_{\tau_1} x_{\tau_2} \otimes y_{\mu_1} y_{\mu_2}) \cdots \omega(x_{\tau_{2p-1}} x_{\tau_{2p}} \otimes y_{\mu_{2p-1}} y_{\mu_{2p}})$$

for any $\omega \in \mathscr{D}^{2,2}$ and any vector fields $x_1, \cdots, x_{2p}, y_1, \cdots, y_{2p}$, it follows that the formula (3.1) reduces to the expression

$$\gamma_{2p}(\sigma)=(-2)^p\{(2p)!\}^{-1} R^p(e_1 \cdots e_{2p} \otimes e_1 \cdots e_{2p}),$$

that is to say, γ_{2p} is the curvature function K_ω corresponding to the curvature structure

$$\omega=(-2)^p\{(2p)!\}^{-1} R^p.$$

Since $R^p \in \mathscr{C}_1^{2p}$, we have from Lemma

$$(3.2) \qquad \gamma_{2p} \equiv \text{const. } \kappa_{2p} \quad \text{iff} \quad R^p=(-2)^{-p} \kappa_{2p}\, g^{2p},$$

for any $p \geqq 1$.

Now, the condition $\gamma_{2p} \equiv$ const. ($p \geqq 2$) does not always imply $\gamma_2 \equiv$ const. (e. g. see A. Stehney [6, §2]). However, we have

Theorem 2. *Let (M, g) be an n-dimensional Riemannian manifold with non-vanishing constant $2p^{th}$ sectional curvature. If $0<2p \leqq n-4$ and its $2(p+1)^{th}$ sectional curvature is also identically constant, then (M, g) is of constant curvature in usual sense.*

Proof. The assumption $\gamma_{2p} \equiv$ const. κ_{2p} ($\neq 0$) in Theorem 2 implies

$$(3.3) \qquad R^p=(-2)^{-p} \kappa_{2p}\, g^{2p}$$

by (3.2). Furthermore, suppose $\gamma_{2(p+1)} \equiv$ const. $\kappa_{2(p+1)}$. Then we have similarly

$$R^{p+1}=(-2)^{-(p+1)}\,\kappa_{2(p+1)}\,g^{2(p+1)}. \tag{3.4}$$

Substituting (3. 3) into the left hand side of (3. 4) and applying Corollary to Theorem 1, we obtain $R=-\{\kappa_{2(p+1)}/2\kappa_{2p}\}\,g^2$. Hence, we find $\gamma_2\equiv\kappa_{2(p+1)}/\kappa_{2p}$, that is, (M,g) is of constant curvature. q. e. d.

4. Application to the mean curvature for p-plane

Let p be an integer such that $1<p<n$, and we put

$$\omega=2R-\frac{1}{p-1}\,g\wedge cR.$$

We consider, throughout this section, the generalized p-vector curvature structure ϕ_r defined by this $\omega\in\mathscr{C}_1^2$:

$$\phi_r=\frac{1}{r!}\,g^r\wedge\omega \qquad (r=p-2).$$

The *mean curvature* ρ *for* p*-plane* of Tachibana [7] is given by the formula

$$\rho(\sigma)=\frac{1}{p(n-p)}\sum_{i=1}^{p}\sum_{j=p+1}^{n}\gamma_2(e_i,e_j) \tag{4.1}$$

for any $\sigma\in G_p$, where $\{e_1,\cdots,e_n\}$ is an orthonormal base of the tangent space $T_{\pi(\sigma)}(M)$ such that $e_1,\cdots,e_p$ span σ, and $\gamma_2(e_i,e_j)$ denotes the sectional curvature of the 2-plane spanned by e_i and e_j. On the other hand, we get

$$\begin{aligned}K_{\phi_r}(\sigma)&=\sum_{i,j=1}^{p}R(e_ie_j\otimes e_ie_j)-\frac{1}{p-1}\sum_{1\le i<j\le p}(g\wedge cR)(e_ie_j\otimes e_ie_j)\\&=\sum_{i,j=1}^{p}R(e_ie_j\otimes e_ie_j)-\sum_{i=1}^{p}\sum_{k=1}^{n}R(e_ie_k\otimes e_ie_k)\\&=\sum_{i=1}^{p}\sum_{j=p+1}^{n}\gamma_2(e_i,e_j)\end{aligned}$$

by the formula (2. 1). Hence, ρ is a curvature function corresponding to the curvature structure $\{p(n-p)\}^{-1}\phi_r\in\mathscr{C}_1^p$, that is,

$$\rho=\frac{1}{p(n-p)}\,K_{\phi_r}:\ G_p\longrightarrow\mathbf{R}. \tag{4.2}$$

From (4. 1) and Theorem 1, we have the following proposition, which has been obtained by Tachibana (cf. Theorem in [7]).

Proposition. *Let $1<p<n$. Each point of M is isotropic with respect to the mean curvature ρ for p-plane if and only if*

(i) *(M, g) is Einsteinian for $p=n-1$,*
(ii) *(M, g) is of constant curvature, for $1<p<n-1$ and $2p\neq n$,*
(iii) *(M, g) is conformally flat, for $1<p<n-1$ and $2p=n$.*

Remark 1. It is interesting to compare this proposition in the case (iii) with the following (cf. Theorem 3.2 in Kulkarni [3]): (M, g) is conformally flat if and only if at every point of M we have

$$\gamma_2(e_1, e_2)+\gamma_2(e_3, e_4)=\gamma_2(e_1, e_4)+\gamma_2(e_2, e_3)$$

for every quadruple of orthogonal vectors $\{e_1, e_2, e_3, e_4\}$.

Now, let us assume $1<p<n-1$ and show the mean curvature ρ for p-plane generally determines the metric g. Let $(\overline{M}, \overline{g})$ be an another Riemannian manifold and $f:(M, g)\longrightarrow(\overline{M}, \overline{g})$ be a diffeomorphism. We indicate the corresponding quantities with respect to the metric $\overline{g}$ or the induced metric $g^*=f^*\overline{g}$ by bar overhead or asterisking, respectively. Suppose that f is K_{ϕ_r}-preserving, that is, for every $\sigma\in G_p$ we have

$$\overline{K}_{\bar{\phi}_r}(f_*\sigma)=K_{\phi_r}(\sigma). \tag{4.3}$$

Furthermore, if

($*$) *the set of non-isotropic points w.r.t. ϕ_r is dense in M,*

then f is conformal, that is, we have

$$g^*=e^{2\psi}g \qquad (\psi\in\mathscr{F}(M)) \tag{4.4}$$

by the well-known theorem of Kulkarni (see General Theorem 5.1 in [4]). Under these circumstances, we shall prove f is an isometry, that is, $\psi\equiv 0$.

First of all, we remark that the assumption ($*$) means

($*$)′ *the set of non-isotropic points w.r.t. ω is dense in M,*

by Theorem 1. Also, under the conformal change (4.4) of metric we have

$$R^*=e^{2\psi}\{R+g\wedge\kappa(\psi)\}, \tag{4.5}$$

$\kappa(\psi)$ being an element of $\mathscr{C}_1^1$, which depends on ψ. From (4.5) we obtain

$$c^*R^*=cR+(n-2)\kappa(\psi)+\text{Trace }\kappa(\psi)\, g. \tag{4.6}$$

Substituting (4. 4), (4. 5) and (4. 6) into ϕ_r^*, we have

$$\phi_r^* = \frac{1}{r!} e^{2(r+1)\psi} g^r \wedge \{\omega + \frac{2p-n}{p-1} g \wedge \kappa(\psi) - \frac{1}{p-1} \text{Trace } \kappa(\psi) g^2\}. \tag{4.7}$$

On the other hand, the condition (4. 3) can be written as

$$K_{\phi_r^*} = e^{2p\psi} K_{\phi_r},$$

which implies

$$\phi_r^* = e^{2p\psi} \phi_r, \tag{4.8}$$

because we have $\phi_r, \phi_r^* \in \mathscr{C}_1^p$. By (4. 4), (4. 8) and Corollary to Theorem 1 we obtain

$$\omega^* = e^{4\psi} \omega. \tag{4.9}$$

Eliminate ϕ_r^* from two equations (4. 7) and (4. 8). Then we have similarly

$$(p-1)(e^{2\psi}-1)\omega = (2p-n) g \wedge \kappa(\psi) - \text{Trace } \kappa(\psi) g^2. \tag{4.10}$$

Case (i) $n=2p$. Suppose that $M' = \{m \in M;\ \psi(m) \neq 0\}$ has non-empty interior. Then, each point of M' is isotropic w. r. t. ω by the equation (4. 10) and Lemma. But this contradicts the assumption $(*)'$. Hence, we have $\psi \equiv 0$.

Case (ii) $n \neq 2p$. In the case, it will be easily seen that the assumption $(*)'$ means

$(*)''$ *the set of non-isotropic points w. r. t. R is dense in M.*

By operating the contraction c to the equation (4. 10), we have

$$\begin{aligned}(e^{2\psi}-1)\{(2p-n)\, cR - c^2R\, g\} \\ = (2p-n)(n-2)\, \kappa(\psi) + (2p-3n+2) \text{ Trace } \kappa(\psi)\, g.\end{aligned} \tag{4.11}$$

Furthermore, operating c to (4. 11) we get

$$(e^{2\psi}-1)\, c^2R = 2(n-1) \text{ Trace } \kappa(\psi).$$

Substitute this into the left hand side of (4. 11). Then owing to $n \neq 2p$ we have

$$(e^{2\psi}-1)\, cR = (n-2)\, \kappa(\psi) + \text{Trace } \kappa(\psi)\, g,$$

which implies

$$c^*R^* = e^{2\psi}\, cR \tag{4.12}$$

by (4. 6). Substitute (4. 4) and (4. 12) into (4. 9). Then we find $R^* = e^{4\psi}R$.

Thus, f is K_R-preserving. Since we assume $(*)''$ and $n>3$, f is an isometry by Theorem 7.1 in [4]. Thus, we have proved the following theorem:

Theorem 3. *Let (M, g) and $(\overline{M}, \overline{g})$ be two Riemannian manifolds of dimension n. Let $f: (M, g) \longrightarrow (\overline{M}, \overline{g})$ be a diffeomorphism which preserves the mean curvature for p-plane, where $1<p<n-1$. Suppose that the set of non-isotropic points with respect to the mean curvature for p-plane is dense in M. Then f is an isometry.*

Remark 2. Theorem 3 is not true when $p=n-1$. In fact, if $p=n-1$, then the formula (4.1) reduces to the expression

$$\rho(\sigma)=\frac{1}{n-1} K_{Ric}(e_n).$$

The present author found a counterexample for corresponding local statement for the Ricci curvature K_{Ric} (cf. [5]).

REFERENCES

[1] E. CARTAN: Leçon sur les géométri des espaces de Riemann, 2nd ed., Gauthier-Villars, Paris, 1951.

[2] R. S. KULKARNI: Curvature and metric, Ann. of Math., **91** (1970), 311—331.

[3] R. S. KULKARNI: Curvature structures and conformal transformations, J. Differential Geometry, **4** (1970), 425—451.

[4] R. S. KULKARNI: On the Bianchi identities, Math. Ann., **199** (1972), 175—204.

[5] T. NASU: On Ricci curvatures and equivalence of Riemannian manifolds, Math. J. Okayama Univ., **17** (1974), 1—18.

[6] A. STEHNEY: Courbure d'ordre p et les classes de Pontrjagin, J. Differential Geometry, **8** (1973), 125—134.

[7] S. TACHIBANA: The mean curvature for p-plane, J. Differential Geometry, **8** (1973), 47—52.

[8] J. A. THORPE: Sectional curvatures and characteristic classes, Ann. of Math., **80** (1964), 429—443.

[9] J. A. THORPE: On the curvatures of Riemannian manifolds, Ill. J. Math., **10** (1966), 412—417.

[10] K. YANO & S. BOCHNER: Curvature and Betti numbers, Annal of Math. Studies, No. 32, Princeton University Press, Princeton, 1953.

FACULTY OF GENERAL EDUCATION
OKAYAMA UNIVERSITY

(Received May 7, 1974)