# ON RIEMANNIAN MANIFOLDS OF NON-POSITIVE SECTIONAL CURVATURE ADMITTING A KILLING VECTOR FIELD

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Let $M$ be an $n$-dimensional connected and complete Riemannian manifold of non-positive sectional curvature. Let $X$ be a Killing vector field on $M$ and $\varphi_t$, $t \in R$, the 1-parameter subgroup of isometries generated by $X$. In this note, supposing that the length of $X$ is bounded, we shall study some properties of $M$.

**1. Killing vector field with bounded length.** We consider the function $f = \frac{1}{2}\langle X, X\rangle$ on $M$. Let $y \in M_p$ be a tangent vector at $p \in M$ and $\gamma$ the geodesic with initial velocity $y$. Since $X$ is a Jacobi field along $\gamma$ and $M$ is of non-positive sectional curvature, we have

$$(1) \qquad \frac{d^2}{dt^2} f \circ \gamma(0) = \|\nabla_y X\|^2 - \langle R(y, X)X, y\rangle \geqq 0.$$

**Proposition 1.** *Let $M$ be an $n$-dimensional connected and complete Riemannian manifold of non-positive sectional curvature. Let $X$ be a Killing vector field on $M$. If the length of $X$ is bounded, then $X$ is a parallel vector field on $M$. In particular, the orbit $\{\varphi_t(p);\ t \in R\}$ is a geodesic for each point $p \in M$.*

*Proof.* Since the length of $X$ is bounded, by (1) we see that $f$ must be constant. Thus, by (1) we have $\nabla_y X = 0$ for any point $p \in M$ and any tangent vector $y \in M_p$. Hence, $X$ is parallel.

**Remark 1.** In Proposition 1 we have $\langle R(Y, X)X, Y\rangle = 0$ for any tangent vector to $M$. If $\dim M = 2$, then $M$ is flat.

In the following sections, we assume that $M$ is always an $n$-dimensional connected and complete Riemannian manifold of non-positive sectional curvature admitting a Killing vector field $X$ with bounded length.

By Proposition 1 we may assume that $X$ is unit.

**2. An isometric immersion of the euclidean two-plane.**

**Proposition 2.** *For any plane section $\sigma$ at $p \in M$ such that $X_p \in \sigma$,* *the euclidean plane $E^2$ can be isometrically immersed in $M$ so that its image is tangent to $\sigma$ at $p$.*

*Proof.* Let $p$ be a given point of $M$ and $\tau: R \longrightarrow M$ a geodesic such that $\tau(0)=p$, $\langle \dot{\tau}(0), X\rangle=0$, and $\|\dot{\tau}(s)\|=1$. We define a mapping $g: E^2 \longrightarrow M$ by $g(s, t)=\varphi_t(\tau(s))$. We set $Y=\partial g/\partial s$. Clearly $X=\partial g/\partial t$. Then, we have $\nabla_Y Y=0$, because $\varphi_t$ is isometric. Clearly $\nabla_X X=0$. Since $\nabla_Y X=0$ and $T_Y X=T_X Y$, where $T_Y$ is the shape operator of $W=g(E^2)$ in $M$, we see that $W$ is totally geodesic. By Remark 1 and the equation of Gauss, $W$ is flat.

**Corollary 1.** *If $M$ is compact and the orbit $\{\varphi_t(p); t\in R\}$ is not periodic for any point $p$ of $M$, then the euclidean cylinder $S^1\times R$ can be isometrically immersed in $M$.*

*Proof.* Since $M$ is compact, there exists a closed geodesic $\rho: S^1 \longrightarrow M$. Then, we see that the mapping $h: S^1\times R \longrightarrow M$ defined by $h(s, t)=\varphi_t(\rho(s))$ is an isometric immersion.

**3. Decomposition of $M$.** We set $D_p=\{Y\in M_p;\ \langle Y, X_p\rangle=0\}$. Since $X$ is parallel, we can easily prove the following

**Lemma 1.** *The $(n-1)$-dimensional distribution $D: p \longrightarrow D_p$ is completely integrable.*

We denote by $S(p)$ the $(n-1)$-dimensional maximal connected integral submanifold of $D$ containing a point $p$ of $M$. Since $X$ is parallel, we have the following

**Lemma 2.** *$S(p)$ is a totally geodesic hypersurface of $M$ for each point $p$ of $M$.*

By Lemma 2, $M$ is locally isometric to $S(p)\times R$ in a neighbourhood of $p\in M$.

**Theorem 1.** *If $M$ is non-compact and $S(p_0)$ is compact for some point $p_0$ of $M$, then $M$ is isometric to $S(p_0)\times R$.*

To prove Theorem 1, we shall prepare some lemmas. Since $M$ is non-compact and $S(p_0)$ is compact, we can take a minimal divergent geodesic ray $\gamma: [0, \infty) \longrightarrow M$ such that $\gamma(0)=p'\in S(p_0)$, $\dot{\gamma}(0)=\varepsilon X_{p'}$, $\varepsilon=\pm 1$, and $\|\dot{\gamma}(t)\|=1$. Then $\gamma(t)=\varphi_{\varepsilon t}(p')$, $t\in[0, \infty)$, because the orbit $\{\varphi_t(p'); t\in R\}$ is geodesic. We may assume therefore that $\gamma(t)=\varphi_t(p')$, $t\in[0, \infty)$.

**Lemma 3.** *The orbit $\{\varphi_t(p');\ t\in R\}$ is a minimal geodesic line.*

*Proof.* For $t$ , $t_2 \in R$ such that $t<0$, $t_1<t_2$, let $\rho$ be a shortest geodesic segment from $\varphi_{t_1}(p')$ to $\varphi_{t_2}(p')$. Since $\varphi_{|t_1|}$ is isometric, $\varphi_{|t_1|} \circ \rho$ is a shortest geodesic segment from $p'$ to $\varphi_{|t_1|+t_2}(p')$. Then $\varphi_{|t_1|} \circ \rho \subset \{\varphi_t(p');\ t\in[0, \infty)\}$, because $\{\varphi_t(p');\ t\in[0, \infty)\}$ is a minimal geodesic ray. Thus $\rho \subset \{\varphi_t(p');\ t\in R\}$.

**Lemma 4.** *$\varphi_t(q) \notin S(p_0)$ holds for each point $q \in S(p_0)$ and $t \in (0, \infty)$.*

*Proof.* Suppose $\varphi_{t'}(q) \in S(p_0)$ for some point $q \in S(p_0)$ and some $t'>0$. Since $S(p_0)$ is maximal, $\varphi_{t'}(S(p_0)) \subset S(p_0)$, and so is $\varphi_{t'}(p') \in S(p_0)$. This is contradictory to Lemma 3.

From Lemma 4 the following is immediate.

**Lemma 5.** *$\varphi_s(q) \notin S(p_0)$ holds for each point $q \in S(p_0)$ and $s \in (-\infty, 0)$.*

*Proof of Theorem* 1. Since $S(p_0)$ is compact, there exists a perpendicular from each point of $M$ to $S(p_0)$. It is contained in some orbit $\{\varphi_t(p);t\in R\}$, $p\in S(p_0)$. Thus, by Lemmas 4 and 5, we see that the orbit $\{\varphi_t(p);t\in R\}$ is a minimal geodesic line for each point $p$ of $S(p_0)$. Therefore, we can define an isometry $\Phi : S(p_0)\times R \longrightarrow M$ by $\Phi(p, t) = \varphi_t(p)$. This completes the proof of Theorem 1.

In the above theorem, since $S(p_0)$ is compact and totally geodesic, there exists a closed geodesic $\tau : S^1 \longrightarrow S(p_0)$ which is also a closed geodesic of $M$. Hence, we have the following

**Corollary 2.** *The euclidean cylinder $S^1 \times R$ can be isometrically imbedded in $M$.*

**Theorem 2.** *If the orbit $\Gamma = \{\varphi_t(p_0); t\in R\}$ is periodic and $\Gamma \cap S(p_0) = \{p_0\}$ for some point $p_0$ of $M$, then $M$ is the bundle space of a fibre bundle whose base space is $S^1$ and fibre $S(p_0)$.*

*Proof.* Let $t_0$ be the minimal period of $\Gamma$. We set $\varphi_t(S(p_0)) = S_t$, $0 \leqq t < t_0$. Let $q$ be a point of $M\backslash\Gamma$ and $\tau$ a perpendicular from $q$ to $\Gamma$. Then $\tau \subset S_t$ for some $t \in [0, t_0)$, and clearly $q \in S_t$. Since each $S_t$ is maximal and $\Gamma \cap S_0 = \{p_0\}$, we see that $S_{t_1} \cap S_{t_2} = \emptyset$, $t_1 \neq t_2 \in [0, t_0)$. Thus, we have

$$M = \bigcup_{0 \leqq t < t_0} S_t, \quad S_{t_1} \cap S_{t_2} = \emptyset \quad \text{for } t_1 \neq t_2 \in [0, t_0). \tag{2}$$

Suppose $q \in S_t$. Then we can express a coordinate neighbourhood of $q$ in $M$ as $U_S \times (t-\varepsilon, t+\varepsilon)$, where $U_S$ is a coordinate neighbourhood of $q$ in $S_t$ and $0<\varepsilon<t_0/2$. Then the projection $\pi : M \longrightarrow S^1 = R/t_0Z$ can be

defined by $\pi(q')=t'$, $q'\in S_{t'}$, $t'\in(t-\varepsilon, t+\varepsilon)$. $\pi$ is a Riemannian submersion and by (2) $\pi^{-1}(t')=S_{t'}$. The mapping $\phi_k$: $(t-\varepsilon, t+\varepsilon)\times S(p_0)\longrightarrow \pi^{-1}(t-\varepsilon, t+\varepsilon)$ defined by $\phi_k(t', p)=\varphi_{kt_0+t'}(p)$, $k\in Z$, is an isometry. If we fix $t'$, then $\phi_k^{-1}\circ\phi_j$ is an isometry of $S(p_0)$. Now we set $\Phi=\varphi_{t_0}$. Then, $\Phi$ is an isometry of $S(p_0)$ and $\Phi(p_0)=p_0$. Let $G=\{\Phi^k\}$, $k\in Z$, $\Phi^0=\text{id.}$, be the discrete group generated by $\Phi$. Then $G$ is an effective topological group acting on $S(p_0)$. Therefore, we see that $M$ is the bundle space of a fibre bundle whose base space is $S^1$, fibre $S(p_0)$, and structure group is $G$.

**Corollary 3.** $\pi_1(M)/\pi_1(S(p_0))\cong Z$.

**Corollary 4.** *If $\varphi_{t_0}(p)=p$ for any point $p$ of $S(p_0)$, then $M$ is isometric to $S(p_0)\times S'$.*

**Remark 2.** If $M$ is compact, $S(p)$ is compact for any point $p$ of $M$.

**Remark 3.** If $\dim M=2$, then $\Phi(q)=q$ or $\Phi^2(q)=q$ for any point $q$ of $S(p_0)$.

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