CENTRALIZERS OF A MODULE OVER A QUASI-FROBENIUS EXTENSION

Dedicated to Professor Kiiti Morita on his 60th birthday

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Introduction. This paper is a natural sequel to our previous paper [3]. The notation and terminology employed there will be used here.

In his paper [4], Y. Miyashita developed a commutor theory of a Frobenius extension. Successively we study a commutor theory of a Quasi-Frobenius (QF) extension in this paper.

Our main result of the present paper is the following:

Theorem A (cf. [4, Th. 2.10]). *Let A/B be a ring extension, M a right A-module such that $M\otimes_B A_A \mid M_A$, that is, $M\otimes_B A$ is A-isomorphic to an A-direct summand of a finite direct sum of copies of A, $A'=$* End (M_A), *the A-endomorphism ring of M, and $B'=$*End (M_B). *If A/B is a QF extension, then the following statements hold.*

1) *B'/A' is a QF extension.*

2) *${}_{B'}B'\otimes {}_{A'}M\mid {}_{B'}M$.*

3) *$\widetilde{A}/\widetilde{B}$ is a QF extension such that $\widetilde{A}\cong\widetilde{A}\otimes_B\widetilde{B}\cong\widetilde{B}\otimes_B A$ canonically. Here $\widetilde{A}$ and $\widetilde{B}$ denote respectively the double centralizers of M as A and B-modules.*

As a consequence of the above theorem, we can obtain the following in §2.

Theorem B (cf. [4, Th. 2.6]). *Let A be a ring, B a subring of A and T an intermediate subring of A/B. If T is a QF extension of B such that ${}_TT\otimes {}_BA_A\mid {}_TA_A$, then B', the centralizer of B in A, is a QF extension of T', the centralizer of T in A, such that ${}_AA\otimes {}_{T'}B'_{B'}\mid {}_AA_{B'}$, and moreover, T'', the double centralizer of T in A, is a QF extension of B'' such that $T''\cong T\otimes {}_BB''\cong B''\otimes {}_BT$ canonically.*

Also a direct proof of Morita [6, Ths. 1.1 and 1.3] is given in §1.

1. Throughout this note, we shall denote by End(X_R) the R-endomorphism ring of a right R-module X and consider End(X_R) as a left operator domain of X. Similarly, for a left R-module ${}_RY$, we shall consider $[\mathrm{End}({}_RY)]^\circ$, the opposite ring of End$({}_RY)$, as a right operator domain of Y. Further if X is a left R-right S-bimodule, we shall say X a

R-S-module simply.

Let $\varphi: B \longrightarrow A$ be a ring homomorphism.[1] Every A-module may be regarded as a B-module via φ. Especially Hom (A_B, B_B) has a B-A-module structure in a natural way:

$(b\cdot f)(x)=b\cdot f(x)$, $(f\cdot a)(x)=f(ax)$ $(a, x\in A, b\in B, f\in \text{Hom}(A_B, B_B))$.

Similarly Hom$({}_BA, {}_BB)$ may beregarded as an A-B-module.

Let X be an A'-A-module and Y an A-A''-module. If A is finitely generated (f. g., briefly) and projective as a left B-module, then a mapping

$$(1) \quad X\otimes_B A \longrightarrow \text{Hom}(\text{Hom}({}_BA, {}_BB)_B, X_B),\ x\otimes a \longmapsto (f\longmapsto xf(a))$$

is an A'-A-isomorphism whose inverse is given by $F \longmapsto \sum_i F(f_i)\otimes a_i$, where $(f_i, a_i)_{1\leq i\leq t}$ is a dual basis for ${}_BA$, that is, $f_i\in$ Hom $({}_BA, {}_BB)$, $a_i\in A(i=1,\cdots,t)$ such that $\sum_i f_i(a)\cdot a_i = a$ for all $a\in A$. Also, a mapping

$$(2) \quad \text{Hom}({}_BA, {}_BB)\otimes_B Y \longrightarrow \text{Hom}({}_BA, {}_BY),\ g\otimes y \longmapsto (a\longmapsto g(a)y)$$

is an A-A''-isomorphism whose inverse is given by $u\longmapsto \sum_i f_i\otimes u(a_i)$. Similarly if A is f. g. projective as a right B-module, then a mapping

$$(3) \quad A\otimes_B Y \longrightarrow \text{Hom}({}_B\text{Hom}(A_B, B_B), {}_BY),\ a\otimes y \longmapsto (f\longmapsto f(a)y)$$

is an A-A''-isomorphism whose inverse is given by $G\longmapsto \sum_j d_j\otimes G(g_j)$ and a mapping

$$(4) \quad X\otimes_B \text{Hom}(A_B, B_B) \longrightarrow \text{Hom}(A_B, X_B),\ x\otimes f \longrightarrow (a\longmapsto xf(a))$$

is an A'-A-isomorphism whose inverse is given by $v\longrightarrow \sum_j v(d_j)\otimes g_j$, where $(g_j, d_j)_{1\leq j\leq s}$ $(g_j\in \text{Hom}(A_B, B_B), d_j\in A)$ is a dual basis for A_B.

Now a ring extension $[A/B, \varphi]$, namely, a ring homomorphism $\varphi: B\longrightarrow A$, is called a Frobenius extension if A is f. g. projective as a right B-module and isomorphic to Hom(A_B, B_B) as a B-A-module (cf. Kasch [2]). If $[A/B, \varphi]$ is a Frobenius extension then there exist a B-B-homomorphism h of A to B and a finite number of elements r_i, l_i in A $(i=1,\cdots, n)$ such that

$$(5) \qquad a=\sum r_i h(l_i a)=\sum h(ar_i)\, l_i \qquad (a\in A),$$

and conversely (cf. Onodera [8]). When this is the case, we shall call such $(h; r_i, l_i)_{1\leq i\leq n}$ a Frobenius system for $[A/B, \varphi]$.

Next a ring extension $[A/B, \varphi]$ is called a right QF extension if

1) By a ring homomorphism, we shall mean one which sends an identity element to an identity element.

A is f. g. projective as a right B-module and isomorphic to a direct summand of a finite direct sum of copies of $\mathrm{Hom}(A_B, B_B)$ as a B-A-module (cf. Müller [7]). A left QF extension is defined symmetrically, and a QF extension is defined as an extension which is right QF as well as left QF.

If A_B is f. g. projective then from (3), we have an A-A-isomorphism

$$(6)\quad A\otimes_B A \longrightarrow \mathrm{Hom}({}_B\mathrm{Hom}(A_B, B_B), A_B),\ x\otimes y \longmapsto (f\longmapsto f(x)y),$$

and under this isomorphism, $\mathrm{Hom}({}_B\mathrm{Hom}(A_B, B_B)_A, {}_BA_A)$ is corresponding to the set $(A\otimes {}_BA)^A$ of casimir elements in $A\otimes_B A$. Here an element $\gamma\in A\otimes {}_BA$ is said to be casimir if $a\gamma=\gamma a$ for all $a\in A$. Thus $[A/B, \varphi]$ is right QF if and only if A_B is f. g. projective and there exist a finite number of B-B-homomorphisms α_i of A to B and casimir elements $\sum_j x_{ij}\otimes y_{ij}\in A\otimes {}_BA$ $(i=1, \cdots, p)$ such that

$$(7)\qquad \textstyle\sum_i \sum_j \alpha_j (a x_{ij})\, y_j = a \qquad (a\in A).$$

Such $(\alpha_i ; \sum_j x_{ij}\otimes y_{ij})_{1\leq i\leq p}$ is called a right QF system for $[A/B, \varphi]$. Similarly a left QF system for $[A/B, \varphi]$ is defined as a system $(\beta_k ; \sum_l w_{kl}\otimes z_{kl})_{1\leq k\leq q}$, where each β_k is a B-B-homomorphism of A to B and each $\sum_l w_{kl}\otimes z_{kl}$ is a casimir element in $A\otimes {}_BA$ such that

$$(8)\qquad \textstyle\sum_k \sum_l w_{kl}\, \beta_k (z_{kl}\, a) = a \qquad (a\in A).$$

Needless to say, a ring extension is QF if and only if it has a right QF system and a left QF system.

Let M be a right A-module, and set $A'=\mathrm{End}(M_A)$, $B'=\mathrm{End}(M_B)$, $\widetilde{A}=[\mathrm{End}({}_{A'}M)]^\circ$ and $\widetilde{B}=[\mathrm{End}({}_{B'}M)]^\circ$. Then M may be considered as a B'-$\widetilde{A}$-module, and further an inclusion mapping $\varphi' : A'\longrightarrow B'$ and ring homomorphisms $\rho_A : A\longrightarrow \widetilde{A}$ and $\rho_B : B\longrightarrow \widetilde{B}$ defined by $m\,\rho_A(a)=ma$ and $m\,\rho_B(b)=mb$ $(m\in M,\ a\in A,\ b\in B)$ respectevely are induced. Moreover $\varphi' : A'\longrightarrow B'$ induces a ring homomorphism $\widetilde{\varphi} : \widetilde{B}\longrightarrow \widetilde{A}$ in a natural way. The meaning of the above notations will be retained throughout this section.

Let us now assume that M satisfies the following condition:

$$(9)\qquad M\otimes {}_BA_A \mid M_A .$$

First suppose that $[A/B, \varphi]$ is a Frobenius extension with a Frobenius system $(h ; r_i, l_i)_i$. Then, by Miyashita [4, Prop. 2. 3], (9) is equivalent to the existence of $f_j, g_j\in B'$ $(j=1, \cdots, t)$ such that

$$(10)\qquad \sum_j g_j\,(f_j(m)\,a) = mh(a) \qquad (a\in A,\ m\in M).$$

So, let $f_j,\ g_j\in B'$ $(j=1,\cdots,t)$ be as above and define an A'-A'-homomorphism $Tr: B'\longrightarrow A'$ by

$$(11)\qquad [Tr\,(b')](m) = \sum_i (b'(mr_i))\,l_i \text{ for } b'\in B',\ m\in M.$$

Then we have

$$(12)\qquad \sum_j g_j\cdot Tr\,(f_jb') = b' \qquad (b'\in B').$$

Indeed, operating the left hand to $m\in M$, we have by (5) and (10)

$$\sum_j g_j\,(\sum_i f_j\,(b'(mr_i))\,l_i) = \sum_i b'\,(mr_i)h(l_i) = b'(m).$$

Similarly we have

$$(13)\qquad \sum_j Tr\,(b'\,g_j)\cdot f_j = b' \qquad (b'\in B').$$

Thus we have seen $(Tr;\ g_j,\ f_j)_{1\le j\le t}$ a Frobenius system for $[B'/A',\ \varphi']$. Furthermore, (11) implies ${}_{B'}B'\otimes{}_{A'}M|_{B'}M$. Therefore, applying the above argument to the left module M over the Frobenius extension $[B'/A',\ \varphi']$ with the Frobenius system $(Tr;\ g_j,\ f_j)_j$, we have by (11) that $(\tilde h;\ \rho_A(r_i),\ \rho_A(l_i))_i$ is a Frobenius system for $[\tilde A/\tilde B,\ \tilde\varphi]$, where $\tilde h\in \mathrm{Hom}({}_{\tilde B}\tilde A_{\tilde B},\ {}_{\tilde B}\tilde B_{\tilde B})$ is defined by

$$m\tilde h(\tilde a) = \sum g_j(f_j(m)\,\tilde a) \text{ for } \tilde a\in\tilde A, m\in M \qquad \text{(cf. (11))}.$$

Then, (10) shows $\tilde h(\rho_A(a)) = \rho_B(h(a))$ for $a\in A$.
It follows that a canonical mapping $\tilde B\otimes{}_B\tilde A\longrightarrow\tilde A,\ \tilde b\otimes a\longmapsto\tilde ba$, is an isomorphism whose inverse is given by $\tilde A\ni\tilde a\longmapsto\sum_i\tilde h(\tilde a\rho_A(r_i))\otimes l_i\in\tilde B\otimes{}_BA$. Similarly a canonical mapping $A\otimes{}_B\tilde B\longrightarrow\tilde A$ defined by $a\otimes\tilde b\longmapsto\widetilde{ab}$ is an isomorphism.

Next suppose that $[A/B,\ \varphi]$ is a QF extension having a right QF system $(\alpha_i;\ \sum_j x_{ij}\otimes y_{ij})_{1\le i\le p}$ and a left QF system $(\beta_k;\ \sum_l w_{kl}\otimes z_{kl})_{1\le k\le q}$. Then $(x_{ij}\cdot\alpha_i,\ y_{ij})_{i,j}$ and $(\beta_k\cdot z_{kl},\ w_{kl})_{k,l}$ are dual bases for ${}_BA$ and A_B by (7) and (8) respectively. By (9), there exist $u_n: M\otimes_B A_A\longrightarrow M_A$ and $v_n: M_A\longrightarrow M\otimes_BA_A$ $(n=1,\cdots,t)$ such that

$$(14)\qquad \sum v_n\cdot u_n = Id_{A\otimes_BM}.$$

Define $U_n\in B'$ and $V_n\in\mathrm{Hom}({}_{A'}B',\ {}_{A'}A')$ by

$$(15)\qquad U_n\,(m) = u_n(m\otimes 1) \text{ for } m\in M;\ U_n=u_n\cdot\iota$$

(16) $$V_n(b') = \pi \cdot (b' \otimes Id_A) \cdot v_n \quad \text{for} \quad b' \in B',$$

where ι and π denote respectively the mappings given by

$$\iota : M \longrightarrow M \otimes {}_B A, \quad \iota(m) = m \otimes 1,$$

$$\pi : M \otimes {}_B A \longrightarrow M, \quad \pi(m \otimes a) = ma.$$

It is then easy to see that $(V_n, U_n)_n$ is a dual basis for ${}_{A'}B'$. Similarly let us define $c_{k,n} \in B'$ and $c^*_{k,n} \in \text{Hom}(B'_{A'}, A'_{A'})$ by

(17) $$c_{k,n} = \gamma \cdot (Id_M \otimes \beta_k) \cdot v_n$$

(18) $$[c^*_{k,n}(b')]\,(m) = u_n(\textstyle\sum_l b'\,(mw_{kl}) \otimes z_{kl}) \qquad (b' \in B', \ m \in M),$$

where $\gamma : M \otimes {}_B B \longrightarrow M$, $\gamma(m \otimes b) = mb$.
Since $[\sum_{k,n} c_{k,n} \cdot c^*_{k,n}\ (b')]\,(m) = \sum_k \{\gamma \cdot (Id_M \otimes \beta_k)\} \{(\sum_n v_n \cdot u_n)\ (\sum_l b'\,(mw_{kl}) \otimes z_{kl})\} = b'\ (m \sum_{k,l} w_{kl}\ \beta_k(z_{kl})) = b'(m)$, $(c^*_{k,n}, c_{k,n})_{k,n}$ is a dual basis for $B'_{A'}$. Further let us define B'-A-homomorphisms

$$\Phi_i : M \otimes {}_B A \longrightarrow \text{Hom}(A_B, M_B), \quad \Psi_i : \text{Hom}(A_B, M_B) \longrightarrow M \otimes {}_B A$$

by

(19) $$[\Phi_i\,(m \otimes a)]\,(x) = m\alpha_i\,(ax) \quad \text{for} \quad m \in M, \ a, \ x \in A,$$

(20) $$\Psi_i(g) = \textstyle\sum_j g(x_{ij}) \otimes y_{ij} \quad \text{for} \quad g \in \text{Hom}(A_B, M_B)$$

and mappings

$$\Psi'_i : B' \longrightarrow \text{Hom}(B'_{A'}, A'_{A'}), \quad \Phi'_i : \text{Hom}\,(B'_{A'}, A'_{A'}) \longrightarrow B'$$

by

(21) $$\Phi'_i(f')(m) = \textstyle\sum_n [f'\,(\gamma \cdot (Id_M \otimes \alpha_i) \cdot v_n)]\ (u_n\,(m \otimes 1))$$
$$\text{for} \quad f' \in \text{Hom}(B'_{A'}, A'_{A'}), \ m \in M,$$

(22) $$[\Psi'_i(b')\,(b'_1)]\,(m) = \textstyle\sum_j ((b'\,b'_1)\,(mx_{ij}))\,y_{ij} \quad \text{for} \quad b', \ b'_1 \in B', \ m \in M.$$

Then it is easy to see

(23) $$\textstyle\sum_i \Phi'_i \cdot \Psi'_i = Id_{B'}.$$

Obviously, every Φ'_i is a left A'-homomorphism. Moreover we shall claim that for any $h \in \text{Hom}\,({}_B A_B, {}_B B_B)$, $\sum_n (\gamma \cdot (Id_M \otimes h) \cdot v_n) \otimes (u_n \cdot \iota)$ is casimir in $B' \otimes {}_{A'}B'$. In fact, noting $(V_n, U_n)_n$ to be a dual basis for ${}_{A'}B'$ and an isomorphism $B' \otimes {}_{A'}B' \ni x' \otimes y' \longmapsto (g' \longmapsto x'\,g'(y'))$ $\in \text{Hom}\,(\text{Hom}({}_{A'}B', {}_{A'}A')_{A'}, B'_{A'})$, we have the conclusion if we show the following equation for all $b' \in B'$ and all n':

$$\sum_n (\gamma \cdot (Id_M \otimes h) \cdot v_n) \cdot V_{n'}, (u_n \cdot \iota \cdot b') = \sum_n b' \cdot (\gamma \cdot (Id_M \otimes h) \cdot v_n) \cdot V_{n'} (u_n \cdot \iota).$$

However this equation can be verified directly. Especially $\sum_n (\gamma \cdot (Id_M \otimes \alpha_i) \cdot v_n) \otimes (u_n \cdot \ )$ is casimir in $B' \otimes {}_{A'}B'$, which implies Φ'_i a right B'-homomorphism, and so, it is an A'-B'-homomorphism. On the otherhand, Ψ'_i is obviously an A'-B'-homomorphism. Accordingly we have ${}_{A'}B'_{B'} \mid {}_{A'}\mathrm{Hom}(B'_{A'}, A'_{A'})_{B'}$ by (23). Therefore, recalling $(c^*_{k,n}, c_{k,n})_{k,n}$ to be a dual basis for $B'_{A'}$, if we set for each i, k, n

$$\alpha'_i = \Psi'_i (1_{B'})^{2)},$$
$$d^{(i)}_{k,n} = \Phi'_i (c^*_{k,n}),$$

then we can see that

(24) $$(\alpha'_i ; \sum_{k,n} c_{k,n} \otimes d^{(i)}_{k,n})_{1\leq i\leq p}$$

is a right QF system for $[B'/A', \varphi']$ (see (3)). We have further

(25) $$\alpha'_i (b') (m) = \sum_j (b' (mx_{ij})) y_{ij}$$

(26) $$d^{(i)}_{k,n}(m) = u_n (m \otimes \sum_l \alpha_i (w_{kl}) z_{kl}) \qquad (b' \in B', \ m \in M).$$

In fact, (25) is obvious by (22), and further the left hand of (26) is equal to $\sum_{n'} [c^*_{k,n} (\gamma \cdot (Id_M \otimes \alpha_i) \cdot v_{n'})] (u_{n'} (m \otimes 1))$

$$= u_n (\sum_l (\gamma \cdot (Id_M \otimes \alpha_i)) ((\sum_{n'} v_{n'} \cdot u_{n'}) (m \otimes w_{kl})) \otimes z_{kl}) \quad \text{(by (18))}$$
$$= u_n (m \otimes \sum_l \alpha_i (w_{kl}) z_{kl}) \quad \text{(by (14))}.$$

This shows (26).

Similarly we shall define B'-A-homomorphisms

$$\psi_k : \mathrm{Hom}(A_B, M_B) \longrightarrow M \otimes_B A, \ \varphi_k : M \otimes_B A \longrightarrow \mathrm{Hom}(A_B, M_B)$$

by

$(19)_1$ $$[\varphi_k (m \otimes a)] (x) = m\beta_k (ax) \text{ for } a, \ x \in A, \ m \in M,$$

$(20)_1$ $$\psi_k (g) = \sum_l g(w_{kl}) \otimes z_{kl} \text{ for } g \in \mathrm{Hom} (A_B, M_B).$$

Then $$\sum_k \varphi_k \cdot \psi_k = Id .$$

Further if mappings

$$\varphi'_k : \mathrm{Hom}(B'_{A'}, A'_{A'}) \longrightarrow B', \ \psi'_k : B' \longrightarrow \mathrm{Hom}(B'_{A'}, A'_{A'})$$

are defined by

2) Here $1_{B'}$ denotes the identity element of a ring B'.

$(21)_1$ $\varphi'_k(f') = \sum_n f'(\gamma \cdot (Id_M \otimes \beta_k) \cdot v_n) \cdot (u_n \cdot \iota)$ for $f' \in \mathrm{Hom}(B'_{A'}, A'_{A'})$

$(22)_1$ $[\psi'_k(b')(b'_1)](m) = \sum_l ((b' \cdot b'_1)(mw_{kl}))z_{kl}$ for $b', b'_1 \in B'$, $m \in M$,

then φ'_k and ψ'_k are A'-B'-homomorphisms such that

$(23)_1$ $$\sum_k \psi'_k \cdot \varphi'_k = Id.$$

Indeed, the last equation can be seen as follows:

$$
\begin{aligned}
&[\{(\textstyle\sum_k \psi'_k \cdot \varphi'_k)(c^*_{k',n'})\}(b')](m)\\
&= \textstyle\sum_{k,l} [\varphi'_k(c^*_{k',n'}) \cdot b'](mw_{kl}) \cdot z_{kl} \quad (\text{by } (22)_1)\\
&= \textstyle\sum_{k,l,n} [c^*_{k',n'}(\gamma \cdot (Id_M \otimes \beta_k) \cdot v_n)](u_n(b'(mw_{kl}) \otimes 1)) \cdot z_{kl} \qquad (\text{by } (21)_1)\\
&= \textstyle\sum_{k,l,n} u_{n'} [\sum_{l'} (\gamma \cdot Id_M \otimes \beta_k) \cdot v_n)(u_n(b'(mw_{kl}) \otimes 1) \cdot w_{k'l'}) \otimes z_{k'l'}] \cdot z_{kl}\\
&\hfill (\text{by } (18))\\
&= \textstyle\sum_{k,l,l'} u_{n'} [(b'(mw_{kl}) \cdot \beta_k(w_{k'l'})) \otimes z_{k'l'}] \cdot z_{kl} \quad (\text{by } (14))\\
&= \textstyle\sum_{l'} u_{n'} (b'(m \sum_{k,l} w_{kl} \beta_k(z_{kl}w_{k'l'})) \otimes z_{k'l'})\\
&= u_{n'}(\textstyle\sum_{l'} b'(mw_{k'l'}) \otimes z_{k'l'}) = [c^*_{k',n'}(b')](m),
\end{aligned}
$$

where the fifth equation is followed from the fact that $\sum_{l'} w_{k'l'} \otimes z_{k'l'} \in A \otimes_B A$ is casimir. Since $\{c^*_{k',n'}\}_{k',n'}$ is a generating set for the left A'-module $\mathrm{Hom}(B'_{A'}, A'_{A'})$, the above equation yields $(23)_1$.

Now let us consider the following sequence

$$B' \xrightarrow[\sim]{\sigma'} \mathrm{Hom}({}_{A'}\mathrm{Hom}(B'_{A'}, A'_{A'}), {}_{A'}A') \underset{\varphi^*_k}{\overset{\psi^*_k}{\rightleftarrows}} \mathrm{Hom}({}_{A'}B', {}_{A'}A'),$$

where $\varphi^*_k = \mathrm{Hom}(\varphi'_k, Id_{A'})$, $\psi^*_k = \mathrm{Hom}(\psi'_k, Id_{A'})$ and σ' denotes a canonical isomorphism. Since $(V_n, U_n)_n$ is a dual basis for ${}_{A'}B'$, if we set

$$\beta'_k = (\psi^*_k \cdot \sigma')(1_{B'})$$
$$T^{(k)}_n = (\sigma'^{-1} \cdot \varphi^*)(V_n),$$

$(24)_1$ $$(\beta'_k\,;\ \textstyle\sum_n T^{(k)}_n \otimes U_n)_{1 \leq k \leq q}$$

is then a left QF system for B'/A' (see (1)). Further we have

$(25)_1$ $$[\beta'_k(b')](m) = \textstyle\sum_l (b'(mw_{kl}))z_{kl} \qquad (b' \in B',\ m \in M),$$

$(26)_1$ $$T^{(k)}_n = \gamma \cdot (Id_M \otimes \beta_k) \cdot v_n\ (= c_{k,n}).$$

In fact, $(25)_1$ is obvious and $(26)_1$ can be seen as follows: It is enough to show that

$$\mathrm{Hom}\,(\varphi_k', Id_{A'})\,(V_n) = \sigma'\,(\gamma\cdot(Id_M\otimes\beta_k)\cdot v_n).$$

Since $[\{\mathrm{Hom}\,(\varphi_k', Id_{A'})\,(V_n)\}\,(c^*_{k',n'})]\,(m)$

$= [\pi\cdot(\varphi_k'\,(c^*_{k',n'})\otimes Id_A)\cdot v_n]\,(m)$ (by (16))

$= \sum_r [\varphi_k'\,(c^*_{k',n'})]\,(m_r)\cdot a_r \quad (v_n(m) = \sum_r m_r\otimes a_r \textit{ in } M\otimes{}_BA)$

$= u_{n'}\,(\sum_{r,l'} m_r\otimes\beta_k\,(w_{k'l'})\,z_{k'l'})\cdot a_r$ (by (18) and $(21)_1$)

$= u_{n'}\,(\sum_{r,l'} m_r\,\beta_k\,(a_r w_{k'l'})\otimes z_{k'l'})$

$= [c^*_{k',n'}\,(\gamma\cdot(Id_M\otimes\beta_k)\cdot v_n)]\,(m)$

(where the fourth equation is followed from the casimirness of $\sum_{l'} w_{k'l'}\otimes z_{k'l'}\in A\otimes{}_BA$) the fact that $\{c^*_{k',n'}\}_{k',n'}$ is a generating set of ${}_{A'}\mathrm{Hom}(B'_{A'}, A'_{A'})$ yields the desired equation.

Let us consider the following :

$$B'\otimes{}_{A'}M \xrightarrow{\lambda\otimes Id_M} \mathrm{Hom}(M_A, \mathrm{Hom}(A_B, M_B)_A)\otimes{}_{A'}M$$

$$\xrightarrow{\nu} \mathrm{Hom}(A_B, M_B)$$

$$\underset{\mathrm{Hom}(\beta_k\cdot z_{kl},\, Id_M)}{\overset{\mathrm{Hom}(\hat{w}_{kl},\, Id_M)}{\rightleftarrows}} \mathrm{Hom}\,(B_B, M_B) \xrightarrow{\nu_1} M,$$

where λ, ν and ν_1 are canonical isomorphisms, and for each $a\in A$, $\hat{a}$ denotes a mapping $B\longrightarrow A$ given by $\hat{a}(b) = ab$.
Setting

(28) $\quad f_{k,l} = \nu_1\cdot\mathrm{Hom}(\hat{w}_{kl}, Id_M)\cdot\nu\cdot(\lambda\otimes Id_M) : B'\otimes{}_{A'}M\longrightarrow M$

(29) $\quad g_{k,l} = (\lambda^{-1}\otimes Id_M)\cdot\nu^{-1}\cdot\mathrm{Hom}(\beta_k\cdot z_{kl}, Id_M)\cdot\nu_1^{-1} : M\longrightarrow B'\otimes{}_{A'}M,$

$f_{k,l}$ and $g_{k,l}$ are left B'-homomorphisms such that

(14′) $\quad \sum_{k,l} g_{k,l}\cdot f_{k,l} = Id\ ;\ {}_{B'}B'\otimes_{A'}M\,|\,{}_{B'}B',$

(30) $\quad f_{k,l}\,(b'\otimes m) = b'\,(mw_{kl}) \qquad (b'\in B',\ m\in M),$

(31) $\quad g_{k,l}\,(m) = \sum_{k',l',n}(\gamma\cdot(Id_M\otimes\beta_{k'})\cdot v_n)\otimes u_n(m\otimes\beta_k'\,(z_{kl}w_{k'l'})\,z_{k'l'})$

From now on, we shall denote ρ_A and ρ_B by ρ simply. Corresponding to U_n, $T_n^{(k)}\in B'$, let us define $U_{k,l}$, $T_{k,l}^{(i)}\in\mathrm{End}\,({}_{A'}M)$ by

(15′) $\quad U_{k,l}\,(m) = f_{k,l}\,(1\otimes m)$ for $m\in M,$

(26′) $$T^{(i)}_{k,l}=\gamma' \cdot (\alpha'_i \otimes Id_M) \cdot g_{k,l},$$

where γ': $A' \otimes_{A'} M \longrightarrow M$ is the canonical isomorphism. Then we have

(32) $$[U_{k,l}]^{\circ} = \rho(w_{kl})$$

(33) $$[T^{(i)}_{k,l}]^{\circ} = \rho\ (\textstyle\sum_j \beta_k\ (z_{kl}\ x_{ij})\ y_{ij}).$$

In fact, (32) is obvious by (30), and (33) can be seen as follows:

$$\begin{aligned}(T^{(i)}_{k,l})\ (m) &= \textstyle\sum_{k',l',n,j} [\gamma \cdot (Id_M \otimes \beta_{k'}) \cdot v_n)\ (u_n\ (m \otimes \beta_k\ (z_{kl}\ w_{k'l'})\ z_{k'l'})\ x_{ij})]\ y_{ij} \\ &= \textstyle\sum_{k',l',j} m\ \beta_{k'}\ (\beta_k(z_{kl}\ w_{k'l'})\ z_{k'l'}\ x_{ij})y_{ij} \\ &= \textstyle\sum_j m\beta_k\ (z_{kl} \sum_{k',l'} w_{k'l'}\ \beta_{k'}\ (z_{k'l'}\ x_{ij}))\ y_{ij} \\ &= m \textstyle\sum_j \beta_k\ (z_{kl}\ x_{ij})y_{ij} \quad \text{(by (8))},\end{aligned}$$

which shows (33). Further using (8), (32) and (33), we have

$$\textstyle\sum_{k,l}[U_{k,l}]^{\circ} \otimes [T^{(i)}_{k,l}]^{\circ} = \sum_j \rho(x_{ij}) \otimes \rho(y_{ij})\ (\in \widetilde{A} \otimes {}_{\widetilde{B}}\widetilde{A}).$$

Similarly, corresponding to $d^{(i)}_{k,n} \in B'$, let us define $d^{(k')}_{i,k,l} \in \mathrm{End}\ ({}_{A'}M)$ by

$$[d^{(k')}_{i,k,l}\ (m)] = f_{k,l}((\textstyle\sum_{k_1,n} c_{k_1,n}\beta'_{k'}\ (d^{(i)}_{k_1,n})) \otimes m) \text{ for } m \in M \qquad \text{(cf. (26))}.$$

Then a calculation using (17), (26), $(25)_1$ and (30) shows that the right hand of the above equation is equal to $m\sum_{l'} w_{kl}\, w_{k'l'}\, \alpha_i(z_{k'l'})$, and so,

(35) $$[d^{(k')}_{i,k,l}]^{\circ} = \rho(\textstyle\sum_{l'} w_{kl}\ w_{k'l'}\ \alpha_i\ (z_{k'l'})).$$

Accordingly, $\sum_{i,k,l} [d^{(k')}_{i,k,l}]^{\circ} \otimes [T^{(i)}_{k,l}]^{\circ} = \sum_{l'}\rho\ (w_{k'l'}) \otimes \rho\ (z_{k'l'})\ (\in \widetilde{A} \otimes {}_{\widetilde{B}}\widetilde{A})$.

Therefore, applying the above argument that leads a left QF and a right QF systems for A/B to those for B'/A' to the left modules M over the QF extension B'/A' with the left QF system $(24)_1$ and the right QF system (24), we know that

(24*) $$(\widetilde{\alpha}_i\ ;\ \textstyle\sum_j \rho(x_{ij}) \otimes \rho(y_{ij}))_i$$

is a right QF system and

$(24^*)_1$ $$(\widetilde{\beta}_k\ ;\ \textstyle\sum_l \rho\ (w_{kl}) \otimes \rho(z_{kl}))_k$$

is a left QF system for $\widetilde{A}/\widetilde{B}$, where $\widetilde{\alpha}_i$, $\widetilde{\beta}_k \in \mathrm{Hom}({}_{\widetilde{B}}\widetilde{A}_{\widetilde{B}},\ {}_{\widetilde{B}}\widetilde{B}_{\widetilde{B}})$ are defined

by $$m\widetilde{\alpha_i(\widetilde{a})} = \textstyle\sum_{k,n} c_{k,n}(d^{(i)}_{k,n}(m)\widetilde{a})$$

$$m\widetilde{\beta}_k(\widetilde{a}) = \sum_n T_n^{(k)} (U_n (m)\widetilde{a}) \qquad (\widetilde{a} \in \widetilde{A},\ m \in M).$$

To be easily seen, we have

$$(36) \qquad \widetilde{\alpha}_i (\rho(a)) = \rho (\alpha_i (a)),\ \widetilde{\beta}_k (\rho (a)) = \rho (\beta_k (a)) \qquad (a \in A).$$

It follows therefore that mappings $A \otimes {}_B\widetilde{B} \longrightarrow \widetilde{A}$, $a \otimes \widetilde{b} \longmapsto a\widetilde{b}$, and $\widetilde{A} \longrightarrow A \otimes {}_B\widetilde{B}$, $\widetilde{a} \longmapsto \sum_{k,l} w_{kl} \otimes \widetilde{\beta}_k (z_{kl} \widetilde{a})$, are mutually inverse isomorphisms. The same holds for mappings $\widetilde{B} \otimes {}_BA \longrightarrow \widetilde{A}$, $\widetilde{b} \otimes a \longmapsto \widetilde{b}a$, and $\widetilde{A} \longrightarrow \widetilde{B} \otimes {}_BA$, $\widetilde{a} \longmapsto \sum_{i,j} \widetilde{\alpha}_i (\widetilde{ax_{ij}}) \otimes \widetilde{y}_{ij}$.

Summarizing the above, we obtain the following:

Theorem 1.1 (cf. [4, Th. 2.10]). *Suppose that $[A/B, \varphi]$ is a QF (resp. Frobenius) extension and M is a right A-module with $M \otimes \otimes {}_BA_A \mid M_A$. Then there holds the following:*

1) *$[B'/A', \varphi']$ is a QF (resp. Frobenius) extension.*

2) ${}_{B'}B' \otimes {}_{A'}M \mid {}_{B'}M$.

3) *$[\widetilde{A}/\widetilde{B}, \widetilde{\varphi}]$ is a QF (resp. Frobenius) extension such that $\widetilde{A} \cong A \otimes_B \widetilde{B} \cong \widetilde{B} \otimes_B A$ canonically.*

Proposition 1.2. *Suppose that $[A/B, \varphi]$ is a QF extension. Then the following statements hold.*

1) *If A/B is H-separable (i. e., ${}_AA \otimes {}_BA_A \mid {}_AA_A$) then ${}_{A'}B'_{A'} \mid {}_{A'}A'_{A'}$. Furthermore assume $M \otimes {}_BA_A \mid M_A$.*

2) *If A/B is separable (i. e., ${}_AA_A \mid {}_AA \otimes {}_BA_A$), then ${}_{A'}A'_{A'} \mid {}_{A'}B'_{A'}$.*

3) *If ${}_BA_B \mid {}_BB_B$ then B'/A' is H-separable.*

4) *If ${}_BB_B \mid {}_BA_B$ then B'/A' is separable.*

Proof. Let $(\alpha_i : \sum_j x_{ij} \otimes y_{ij})_i$ and $(\beta_k : \sum_l w_{kl} \otimes z_{kl})_k$ be respctive right QF and left QF systems for A/B. 1): Since A/B is H-separable, there exist $f_n : {}_AA \otimes {}_BA_A \longrightarrow {}_AA_A$ and $g_n : {}_AA_A \longrightarrow {}_AA \otimes_B A_A$ $(n = 1, \cdots, n_0)$ such that $\sum g_n \cdot f_n = Id$. Let us consider a sequence of B'-A-modules

$$M \otimes {}_BA \xrightarrow[\sim]{\gamma_1} M \otimes {}_AA \otimes {}_BA \underset{\bar{g}_n = Id_M \otimes g_n}{\overset{\bar{f}_n = Id_M \otimes f_n}{\rightleftarrows}} M \otimes {}_AA \xrightarrow[\sim]{\gamma_2} M$$

where γ_1 and γ_2 are both the canonical isomorphisms.

Setting $\qquad u_n = \gamma_2 \cdot \bar{f}_n \cdot \gamma_1,\ v_n = \gamma_1^{-1} \cdot \bar{g}_n \cdot \gamma_2^{-1}$

then $$\sum v_n \cdot u_n = Id .$$

Therefore we can use Theorem 1. 1 and its proof in any case. So, we shall employ the notations as before throughout the proof. First a right QF system (24) yields

$$\sum_i \sum_{k,n} (\alpha_i'(b' \cdot c_{k,n})) \cdot d_{k,n}^{(i)} = b' \qquad (b' \in B').$$

However, to be easily verified, $c_{k,n}$ and $d_{k,n}^{(i)}$ are contained in the centralizer $V_{B'}(A')$ of A' in B' at the present case. It follows that ${}_{A'}B'_{A'} \mid {}_{A'}A'_{A'}$. Furthermore assume $M \otimes_B A_A \mid M_A$. Hence B'/A' is QF by Theorem 1. 1. 2): Suppose ${}_A A_A \mid A \otimes_B A_A$. Thus

$${}_{A'}M_A \cong {}_{A'}M \otimes_A A_A \mid {}_{A'}M \otimes_A A \otimes_B A_A \cong {}_{A'}M \otimes_B A_A \sim {}_{A'} \operatorname{Hom}(A_B, M_B)_A,$$

which yields ${}_{A'}\operatorname{Hom}(M_A, M_A)_{A'} \mid {}_{A'}\operatorname{Hom}(M_A, \operatorname{Hom}(A_B, M_B)_A)_{A'}$

$$\cong {}_{A'}\operatorname{Hom}(M_B, M_B)_{A'}.$$

This shows 2). 3): Let us assume ${}_B A_B \mid {}_B B_B$. Thus there exist $\delta_r \in \operatorname{Hom}({}_B A_B, {}_B B_B)$ and $a_r \in V_A(B)$ $(r = 1, \cdots, s)$ such that $\sum a_r \delta_r (a) = a$ for all $a \in A$. Replacing $f_{k,n}$ and $g_{k,n}$ in the proof of Theorem 1. 1 respectively by

$$f_r = \nu_1 \cdot \operatorname{Hom}(\hat{a}_r, Id_M) \cdot \nu \cdot (\lambda \otimes Id_M)$$

$$g_r = (\lambda^{-1} \otimes Id_M) \cdot \nu^1 . \operatorname{Hom}(\delta_r, Id_M) \cdot \nu_1^{-1},$$

we have ${}_B B' \otimes_{A'} M_B \mid {}_{B'} M_B$ (see (27)). This yields ${}_{B'}B' \otimes_{A'} B'_{B'} \sim {}_{B'}\operatorname{Hom}(B'_{A'}, B'_{A'}) \cong {}_{B'}\operatorname{Hom}(B' \otimes_{A'} M_B, M_B)_{B'} \mid {}_{B'}\operatorname{Hom}(M_B, M_B)_{B'} = {}_{B'}B'_{B'}$, which shows 3). 4): Finally suppose ${}_B B_B \mid {}_B A_B$. Then there exist $f_r \in \operatorname{Hom}({}_B A_B, {}_B B_B)$ and $a_r \in V_A(B)$ $(r = 1, \cdots, t)$ such that $\sum f_r (a_r) = 1$.

Setting $$c_i = \sum_{j,r} a_r x_{ij} f_r(y_{ij})$$

then $$c_i \in V_A(B), \quad \sum_i \alpha_i (c_i) = 1.$$

Thus a mapping $\rho_{c_i}: M \longrightarrow M$ defined by $\rho_{c_i}(m) = mc_i$ is in $V_{B'}(A')$, and so, $$\sum_i \sum_{k,n} c_{k,n}, \ (\rho_{c_i} \cdot d_{k,n}^{(i)}) = 1_{B'} \qquad \text{(see (36))}.$$

On the other hand, every $\sum_{k,n} c_{k,n} \otimes d_{k,n}^{(i)}$ is casimir in $B' \otimes_{A'} B'$ and hence so is $\sum_i \sum_{k,n} c_{k,n} \otimes \rho_{c_i} \cdot d_{k,n}^{(i)}$. It follows that B'/A' is separable. Thus our proof is complete.

Proposition 1.3. *Assume that* $[A/B, \varphi]$ *is right* QF *(resp. left* QF*, Frobenius) and* M_A *is a generator. Then* $[A(=\tilde{A})/\tilde{B}, \tilde{\varphi}]$ *is right* QF *(resp. left* QF*, Frobenius), and moreover right* QF *(resp. left* QF*, Frobenius) system for* $[A/B, \varphi]$ *gives a right* QF *(resp. left* QF*, Fro-*

benius) system for $[A/\widetilde{B}, \widetilde{\varphi}]$ *naturally.*

Proof. First we shall show that any $f \in \mathrm{Hom}(A_B, B_B)$ is contained in $\mathrm{Hom}(A_{\widetilde{B}}, B_{\widetilde{B}})$ (note that M_A is a generator). In fact, $mf(u(m')\widetilde{b}) = (\lambda_m \cdot f \cdot u)(m' \widetilde{b}) = m(f(u(m'))\widetilde{b}$ $(m, m' \in M, \widetilde{b} \in \widetilde{B}, u \in \mathrm{Hom}(M_A, A_A))$ means the conclusion, where λ_m denotes a mapping $A \longrightarrow M$ given by $\lambda_m(a)=ma$. Let $(\alpha_i; \sum_j x_{ij} \otimes y_{ij})_i$ be a right QF system for A/B. The mention just above shows that every α_i is in $\mathrm{Hom}(A_{\widetilde{B}}, \widetilde{B}_{\widetilde{B}})$ and $A_{\widetilde{B}} \mid \widetilde{B}_{\widetilde{B}}$. We claim further every α_i is contained in $\mathrm{Hom}({}_{\widetilde{B}}A, {}_{\widetilde{B}}\widetilde{B})$. To see this, let $x \in A$, $\widetilde{b} \in \widetilde{B}$ and $f \in \mathrm{Hom}(A_B, B_B)$ be arbitrary elements. Then $f(\alpha_i(\widetilde{b}x) - \widetilde{b}\alpha_i(x)) = f(1)\alpha_i(\widetilde{b}x) - f(\widetilde{b})(\alpha_i(x)) = \alpha_i(f(1)\widetilde{b}x - f(\widetilde{b})x) = 0$, which implies $\alpha_i(\widetilde{b}x) - \widetilde{b}\alpha_i(x) = 0$ as desired, since A_B torsionless. On the other hand, every casimir element in $A \otimes_B A$ is mapped canonically into a casimir element in $A \otimes_{\widetilde{B}} A$. Therefore we have shown the proposition for a right QF extension.

Next let $(\beta_k; \sum_l w_{kl} \otimes z_{kl})_k$ be a left QF system for A/B. By the mention at the begining of the proof, every β_k is in $\mathrm{Hom}(A_{\widetilde{B}}, \widetilde{B}_{\widetilde{B}})$. Further we can regard as $\mathrm{Hom}({}_BA, {}_BB) \subset \mathrm{Hom}({}_{\widetilde{B}}A, {}_{\widetilde{B}}\widetilde{B})$; $[(g(\widetilde{b}x) - \widetilde{b}g(x))\beta_k](y) = \beta_k(yg(\widetilde{b}x)) - \beta_k(y\widetilde{b}g(x)) = g(\beta_k(y)\widetilde{b}x) - g(\beta_k(y\widetilde{b})x) = 0$ $(g \in \mathrm{Hom}({}_BA, {}_BB), x, y \in A) \Longrightarrow (g(\widetilde{b}x) - \widetilde{b}(g(x)))\beta_k = 0 \Longrightarrow g(\widetilde{b}x) - \widetilde{b}g(x) = 0$ as desired. In particular, $\beta_k \in \mathrm{Hom}({}_{\widetilde{B}}A_{\widetilde{B}}, {}_{\widetilde{B}}B_{\widetilde{B}})$ and ${}_{\widetilde{B}}A \mid {}_{\widetilde{B}}\widetilde{B}$. It follows that the proposition for a left QF extension has be shown. The case of Frobenius extesion is now obvious. Thus our proof is complete.

Let B_0 be the subring of B generated as a ring by the identity element of B and by all the elements of the form $f(a)$ $(a \in A, f \in \mathrm{Hom}(A_B, B_B))$, $\overline{B}$ the double centraizer of A_B (i. e., $\overline{B} = [\mathrm{End}({}_{\mathrm{End}(A_B)}A)]^{\circ}$), and T an arbitrary intermediate subring of $\overline{B}/\varphi(B_0)$. Then the proof of the above proposition (taking A_A as M_A) shows the following assertion obtained in Morita [6, Ths 1.1 and 1.3].

Proposition 1.4. *Under the same notations as above, if A/B is right QF (resp. left QF, Frobenius) then so is A/T.*

2. In this section, B denotes a subring of a ring A, $\varphi: B \longrightarrow A$ the inclusion mapping, T an intermediate subring of A/B and Z the ring of intergers. Moreover, for a subset S of A, S' denotes the centra-

lizer of S in A.

Under the above notations, we shall prove the following as an application of Theorem 1. 1.

Theorem 2.1 (cf. [4, Th. 2. 6]). *If A/B is a QF extension with ${}_TT \otimes {}_BA_A \mid {}_TA_A$ then B'/T' is a QF extension with ${}_AA \otimes {}_{T'}B'_{B'} \mid {}_AA_{B'}$, and moreover T'' $(= (T')')/B''$ is a QF extension such that*

$$T \otimes {}_BB'' \longrightarrow T'', \ t \otimes b'' \longmapsto tb''$$

and

$$B'' \otimes {}_BT \longrightarrow T'', \ b'' \otimes t \longmapsto b''t$$

are isomorphisms.

Before going to prove the theorem, we note some facts. For a T-A-module X, we can give X a left $T \otimes_{\mathbf{Z}} A^{\circ}$-module structure in a natural way: $(t \otimes a^{\circ}) \cdot x = txa$ $(a \in A, \ t \in T, \ x \in X)$. Conversely if X is a left $T \otimes_{\mathbf{Z}} A^{\circ}$-module then X has a T-A-module structure. Hence the $T \otimes_{\mathbf{Z}} A^{\circ}$-endomorphism ring of left $T \otimes_{\mathbf{Z}} A^{\circ}$-module A can be identified with the centralizer T' of T in A naturally. The following lemma can be easily verified.

Lemma 2.2. 1) *$T \otimes {}_BA \cong (T \otimes A^{\circ}) \otimes {}_{B \otimes A} A^{\circ}$ as left $T \otimes A^{\circ}$-modules by the correspondence $t \otimes a \longmapsto t \otimes 1^{\circ} \otimes a$, where the unspecified tensor products of the right hand are taken over $\mathbf{Z}$.*

2) *For each element $\sum t_i \otimes a_i \in (T \otimes {}_BA)^T = \{\gamma \in T \otimes_B A \mid t\gamma = \gamma t$ for all $t \in T\}$, a mapping*

$$\eta\,(\textstyle\sum t_i \otimes a_i): \ B' \longrightarrow T', \ \ [\eta\,(\sum t_i \otimes a_i)](b') = \sum t_i b' a_i$$

is a left T'-homomorphism.

Lemma 2.3. 1) *${}_TT \otimes_B A_A \mid {}_TA_A$ is equivalent to the existence of a finite number of elements $b'_m \in B'$ and $\sum_n t_{mn} \otimes a_{mn} \in (T \otimes {}_BA)^T$ $(m = 1, \cdots, m_0)$ such that*

$$\textstyle\sum_m \sum_n t_{mn} \otimes a_{mn} b'_m = 1 \otimes 1 \ (\in T \otimes {}_BA). \tag{37}$$

2) *Assume ${}_TT \otimes {}_BA_A \mid {}_TA_A$ and let $(b'_m\,;\ \sum_n t_{mn} \otimes a_{mn})_m$ be as above 1). Then there holds the following:*

i) $\sum_{m,n} f(tt_{mn})\, a_{mn}\, t_1\, b'_m = f(tt_1)$ $(t, \ t_1 \in T, \ f \in \mathrm{Hom}(T_B, \ T_B))$.

ii) *${}_{T'}B'$ is f. g. projective with a dual basis $(\eta_m, \ b'_m)_m$, where $\eta_m = \eta\,(\sum_n t_{mn} \otimes a_{mn})$ $(m=1, \cdots, m_0)$.*

Proof. 1) and 2-i) can be verified easily. Let $b' \in B'$ be an arbitrary element. A mapping $\lambda_{b'}: A \longrightarrow A$ defined by $\lambda_{b'}(a) = b'a$ is a left B-homomorphism obviously. Operating $Id_T \otimes \lambda_{b'}: T \otimes {}_B A \longrightarrow T \otimes {}_B A$ to (37), we have

$$\sum_{m,n} t_{mn} \otimes b' a_{mn} b'_m = 1 \otimes b',$$

which yields $\quad b' = \sum_{m,n} t_{mn}\, b'_m = \sum_m \eta_m(b') b'_m .$

This implies ii) from Lemma 2. 2 2).

We are now ready for proving the theorem.

Proof of Theorem 2. 1. Assume that T/B is a QF extension with ${}_T T \otimes {}_B A_A \mid {}_T A_A$. Let $(\alpha_i;\ \sum_j x_{ij} \otimes y_{ij})_i$ and $(\beta_k;\ \sum_k w_{kl} \otimes z_{kl})_k$ be respective right QF and left QF systems for T/B and let $(b'_m;\ \sum_n t_{mn} \otimes a_{mn})_m$ be as Lemma 2. 3 1). Since ${}_{T'}B'$ is f. g. projective by Lemma 2. 3 2-ii), a mapping $\sigma: A \otimes {}_{T'}B' \longrightarrow \mathrm{Hom}\,(\mathrm{Hom}({}_{T'}B', {}_{T'}T')_{T'}, A_{T'})$ defined by $[\sigma(a \otimes b')](g) = a.\,g(b')$ is an A-B'-isomorphism (see (1)).

Setting $\quad \gamma_{k,l} = \sum_{m,n} \beta_k(z_{kl}\, t_{mn}) a_{mn} \otimes b'_m \in A \otimes_{T'} B',$

we have by Lemma 2. 3 2-i)

$$[\sigma(\gamma_{k,l} b')](\eta_{m'}) = \sum_{n'} \beta_k(z_{kl}\, t_{m'n'})\, b' a_{m'n'} = [\sigma(b' \gamma_{k,l})](\eta_{m'}).$$

However, $\{\eta_{m'}\}_{m'}$ is a generating set of $\mathrm{Hom}({}_{T'}B', {}_{T'}T')_{T'}$ by Lemma 2. 3 2-ii). Therefore the above equation shows $\sigma(\gamma_{k,l}\, b') = \sigma(b' \gamma_{k,l})$, that is, $\gamma_{k,l} \in (A \otimes {}_{T'}B')^B$ for each k, l.

Similarly we have $\quad [\sigma(\sum_{k,l} w_{kl} \gamma_{k,l})](\eta_{m'}) = [\sigma(1 \otimes 1)](\eta_{m'})$, and so,

$$\sum_{k,l} w_{kl} \gamma_{k,l} = 1 \otimes 1 \in A \otimes_{T'} B'.$$

It follows that ${}_A A \otimes {}_{T'}B'_{B'} \mid {}_A A_{B'}$ and $(\eta_{k,l},\ w_{kl})_{k,l}$ is a dual basis for $T''_{B''}$ by Lemma 2. 3 (left and right are replaced), where $\eta_{k,l}: T'' \longrightarrow B''$ is defined by $\eta_{k,l}(t'') = \sum_{m,n} \beta_k(z_{kl}\, t_{mn})\, a_{mn}\, t''\, b'_m \quad (t'' \in T'')$. Then $\eta_{k,l}(t) = \beta_k(z_{kl}\, t)$ for $t \in T$. Thus, if $\sum t_r \otimes a_r \in T \otimes {}_B B''$ is an element with $\sum t_r\, a_r = 0$, then

$$\begin{aligned} \sum t_r \otimes a_r &= \sum \sum_{k,l} w_{kl}\, \beta_k(z_{kl}\, t_r) \otimes a_r \\ &= \sum_{k,l} w_{kl} \otimes \eta_{k,l}(\sum t_r a_r) = 0, \end{aligned}$$

which implies that a canonical mapping $T \otimes {}_B B'' \ni t \otimes b'' \longmapsto t b'' \in T''$ is an injection. However, recalling the above dual basis for $T''_{B''}$, it is obviously surjective, and so, it is a bijection. Moreover our assumptions, T/B being QF and ${}_T T \otimes {}_B A_A \mid {}_T A_A$, yield

$$ {}_A A \otimes {}_B T_T \sim {}_A\mathrm{Hom}(T_B, A_B)_T \cong {}_A\mathrm{Hom}(T \otimes {}_B A_A, A_A)_T \\ \mid {}_A\mathrm{Hom}(A_A, A_A)_T \cong {}_A A_T. $$

Therefore a symmetric argument shows that a mapping $B'' \otimes_B T \ni b'' \otimes t \longmapsto b'' t \in T''$ is a bijection.

On the other hand, it is easy to see that $(\alpha_i \otimes Id_{A^\circ} : \sum_j (x_{ij} \otimes 1^\circ) \otimes (y_{ij} \otimes 1^\circ))_i$ and $(\beta_k \otimes Id\ A^\circ ; \sum_l (w_{kl} \otimes 1^\circ) \otimes (z_{kl} \otimes 1^\circ))_k$ are respective right QF and left QF systems for $T \otimes_Z A^\circ / B \otimes {}_Z A^\circ$. Since $(T \otimes_Z A^\circ) \otimes_{B \otimes A^\circ} A \cong T \otimes_B A \mid A$ as left $T \otimes_Z A^\circ$-modules by Lemma 2. 2 1) andour assumption, we can apply Theorem 1. 1 to the left module A over the QF extension $T \otimes_Z A^\circ / B \otimes_Z A^\circ$. Thus $\mathrm{End}({}_{B \otimes A^\circ} A) / \mathrm{End}({}_{T \otimes A^\circ} A)$, and hence B'/T' is a QF extension. Finally, a similar argument shows that T''/B'' is QF. Thus our proof is complete.

3. Let $\varphi : B \longrightarrow A$ be a ring homomorphism, U a right B-module and V a right A-module. Put $B^* = \mathrm{End}(U_B)$ and $A^* = \mathrm{End}(V_A)$. In a natural way, $\mathrm{Hom}(A_B, U_B)$ is a B^*-A-module. In the subsequent study, we assume always the following conditions.

(38) There exists an A-isomorphism $\alpha : V \longrightarrow \mathrm{Hom}(A_B, U_B)$.

(39) $V_B \mid U_B$; $\sum g_t \cdot f_t = Id_V$ for some $f_t : V_B \longrightarrow U_B$ and $g_t : U_B \longrightarrow V_B$ $(t=1, \cdots, t_0)$.

By (38), V may be regarded as a B^*-A-module. Further α induces a ring homomorphism $\varphi^* : B^* \longrightarrow A^*$ defined by $\varphi^*(b^*)(v^*) = \alpha^{-1}(b^*\ \alpha(v))$ or $\varphi^*(b^*) = \alpha^{-1} \cdot b^* \cdot \alpha$ $(b^* \in B^*,\ v \in V)$. On the other hand, (39) gives the canonical isomorphism

$$ \pi : V \longrightarrow \mathrm{Hom}({}_{B*}\mathrm{Hom}(V_B, U_B)),\ {}_{B*}U),\ [\pi(v)](f) = f(v). $$

Moreover, we have a chain of B^*-A^*-isomorphisms

$$ \begin{array}{ccl} & \mathrm{Hom}(Id_V, \alpha) & \\ A^* & \xrightarrow{\hspace{6em}} & \mathrm{Hom}(V_A, \mathrm{Hom}(A_B, U_B)_A) \\ & \searrow & \quad \downarrow\wr \\ & & \mathrm{Hom}(V \otimes {}_A A_B, U_B) \\ & \xi & \quad \downarrow\wr \\ & & \mathrm{Hom}(V_B, U_B). \end{array} $$

where the vertical mappings are canonical. Then the composite $\xi : A^* \longrightarrow \mathrm{Hom}(V_B, U_B)$ of these isomorphisms is a B^*-A^*-isomorphism such that

$$[\xi(a^*)](v) = [\alpha(a^* v)](1) \qquad (a^* \in A^*,\ v \in V).$$

Therefore, we have a left A^*-isomorphism

$$(38^*) \qquad \alpha^* = \mathrm{Hom}(\xi,\ Id_V) \cdot \pi : V \longrightarrow \mathrm{Hom}({}_{B*}A_*,\ {}_{B*}U)$$
$$[\alpha^*(v)](a^*) = [\alpha(a^* v)](1).$$

Accordingly, a ring homomorphism $\tilde{\varphi}: \tilde{B} = [\mathrm{End}({}_{B*}U)]^\circ \longrightarrow \tilde{A} = [\mathrm{End}({}_{A*}V)]^\circ$ can be induced by α^* in a similar way. For convenience, the canonical ring homomorphisms $A \longrightarrow \tilde{A}$ and $B \longrightarrow \tilde{B}$ will be denoted by ρ. Under the above notations, we can obtain the following which corresponds to Theorem 1. 1.

Theorem 3.1. *Assume that U_B and V_A satisfy* (38) *and* (39). *If $[A/B,\ \varphi]$ is QF (resp. Frobenius), then there hold the followings*:

1) $[A^*/B^*, \varphi^*]$ *is QF (resp. Frobenius).*
2) ${}_{A*}V \cong {}_{A*}\mathrm{Hom}({}_{B*}A^*,\ {}_{B*}U)$.
3) ${}_{B*}V \mid {}_{B*}U$.
4) $[\tilde{A}/\tilde{B},\ \tilde{\varphi}]$ *is QF (resp. Frobenius) such that*

$$\tilde{B} \otimes_B A \longrightarrow \tilde{A},\ \tilde{b} \otimes a \longmapsto \tilde{b}\rho(a)$$

and

$$A \otimes_B \tilde{B} \longrightarrow \tilde{A},\ a \otimes \tilde{b} \longmapsto \rho(a)\tilde{b}$$

are isomorphisms.

Proof. The assertion 2) has be shown already. Let us set $V' = \mathrm{Hom}(A_B,\ U_B)$ and $\Delta = \mathrm{End}(A_B)$. Then we can consider A as a subring of Δ and V' a B^*-Δ-module in a natural way. Put $\Lambda = \mathrm{End}(V'_A)$ and $\Gamma = \mathrm{End}(V'_\Delta)$. We have then a ring homomorphism $\psi : B^* \longrightarrow \Gamma$ defined by $\psi(b^*)(v') = b^* v'$ ($b^* \in B^*$, $v' \in V'$) and a ring isomorphism $\bar{\psi} : A^* \longrightarrow \Lambda$ induced by α. Then it is easy to see $\bar{\psi} \cdot \varphi^* = \iota \cdot \psi$, where ι denotes the inclusion mapping of Γ to Λ.

Let $(\alpha_i\,;\ \sum_j x_{ij} \otimes y_{ij})_i$ and $(\beta_k\,;\ \sum_l w_{kl} \otimes z_{kl})_k$ be respective right QF and left QF systems for A/B and let $(\delta_n,\ a_n)_{1 \le n \le n_0}$ be a dual basis for A_B. Define $u_n : A \otimes {}_BA_A \longrightarrow A_A$ and $v_n : A_A \longrightarrow A \otimes {}_BA_A$ by $u_n(x \otimes y) = \delta_n(x)y$ and $v_n(x) = a_n \otimes x$ respectively. Then $\sum_n v_n \cdot u_n = Id$, and so $A \otimes {}_BA_A \mid A_A$. Therefore we can apply Theorem 1.1 to the right module A over the QF extension A/B. Thus we know by its proof that Δ/A is a QF extension with a right QF system $(\alpha'_i\,;\ \sum_{k,n} c_{k,n}\otimes d^{(i)}_{k,n})_i$ and a left QF system $(\beta'_k\,;\ \sum_n c_{k,n}\otimes U_n)_k$,

where
$$\begin{aligned}
[\alpha'_i(\delta)](a) &= \textstyle\sum_j \delta(ax_{ij})y_{ij} && \text{(see (25))}\\
[\beta'_k(\delta)](a) &= \textstyle\sum_l \delta(aw_{kl})z_{kl} && \text{(see (25))}\\
c_{k,n}(a) &= a_n\beta_k(a) && \text{(see (17))}\\
d^{(i)}_{k,n}(a) &= \delta_n(a)\textstyle\sum_l \alpha_i(w_{kl})z_{kl} && \text{(see (26))}\\
U^n(a) &= \delta_n(a) \qquad (\delta\in\Delta,\ a\in A) && \text{(see (15))}.
\end{aligned}$$

Further define $\bar{f}_t : V'\otimes_A\Delta_\Delta \longrightarrow V'_\Delta$ and $\bar{g}_t : V'_\Delta \longrightarrow V'\otimes_A\Delta_\Delta$ by

$$[\bar{f}_t(w\otimes\delta)](a) = (f_t\cdot\alpha^{-1})(w\delta(a))$$
$$\bar{g}_t(w) = \textstyle\sum_n((\alpha\cdot g_t)(w(a_n)))\otimes\delta_n \qquad (w\in V',\ \delta\in\Delta,\ a\in A)$$

Then
$$\textstyle\sum \bar{g}_t\cdot\bar{f}_t = Id\,;\quad V'\otimes_A\Delta_\Delta\,|\,V'.$$

Hence we can apply Theorem 1.1 to the right module V' over the QF extension Δ/A with the above left QF and right QF systems. Thus we know that Λ/Γ is a QF extension with a right QF system

(40) $$(\alpha^+_i : \textstyle\sum_{k,t} c^+_{k,t}\otimes d^{+(i)}_{k,t})_i$$

and a left QF system

(41) $$(\beta^+_k\,;\ \textstyle\sum_t c^+_{k,t}\otimes U^+_t)_k$$

where
$$\begin{aligned}
[\alpha^+_i(\lambda)](w) &= \textstyle\sum_{k,n}(\lambda(w\cdot c_{k,n}))\cdot d^{(i)}_{k,n}\\
[\beta^+_k(\lambda)](w) &= \textstyle\sum_n(\lambda(w\cdot c_{k,n}))\cdot U_n\\
c^+_{k,t}(w) &= (\gamma\cdot(Id_{V'}\otimes\beta'_k)\cdot\bar{g}_t)(w)\\
d^{+(i)}_{k,t}(w) &= \bar{f}_t(w\otimes\textstyle\sum_n\alpha'_i(c_{k,n})\cdot U_n)\\
U^+_t(w) &= \bar{f}_t(w\otimes 1)
\end{aligned}$$

$(w\in V',\ \lambda\in\Lambda,\ \gamma : V'\otimes_A A\ni w\otimes a\longmapsto wa\in V')$.

Moreover we can see that

$$\begin{aligned}
[\alpha^+_i(\lambda)(w)](a) &= [\lambda(w\cdot a\cdot\alpha_i)](1)\\
[\beta^+_k(\lambda)(w)](a) &= [\lambda(w\cdot a\cdot\beta_k)](1)\\
c^+_{k,t}(w) &= \textstyle\sum_l((\alpha\cdot g_t)(w(w_{kl})))\cdot z_{kl}\\
[d^{+(i)}_{k,t}(w)](a) &= (f_t\cdot\alpha^{-1})(w\cdot a\textstyle\sum_j\beta_k(x_{ij})\,y_{ij})\\
[U^+_t(w)](a) &= (f_t\cdot\alpha^{-1})(w\cdot a) \qquad (w\in V',\ a\in A).
\end{aligned}$$

Let us define further α_i^*, $\beta_k^* \in \mathrm{Hom}({}_{B*}A^*{}_{B*}, {}_{B*}B^*{}_{B*})$ by

$$(42) \qquad [\alpha_i^*(a^*)](u) = [\alpha(a^* \alpha^{-1}(\lambda_u \cdot \alpha_i))](1)$$

$$(43) \qquad [\beta_k^*(a^*)](u) = [\alpha(a^* \alpha^{-1}(\lambda_u \cdot \beta_k))](1),$$

where for $u \in U$, λ_u denotes a mapping $B \ni b \longmapsto ub \in U$. We have then $\psi \cdot \alpha_i^* = \alpha_i^+ \cdot \bar{\psi}$ and $\psi \cdot \beta_k^* = \beta_k^+ \cdot \bar{\psi}$. It follows that

$$(40^*) \qquad (\alpha_i^* \; ; \; \textstyle\sum_{k,t} c_{k,t}^* \otimes d_{k,t}^{*(i)})_i$$

is a right QF system and

$$(41^*) \qquad (\beta_k^* \; ; \; \textstyle\sum_t c_{k,t}^* \otimes U_t^*)_k$$

is a left QF system for $[A^*/B^*, \varphi^*]$, where $c_{k,t}^* = \bar{\psi}^{-1}(c_{k,t})$, $d_{k,t}^{*(i)} = \bar{\psi}^{-1}(d_{k,t}^{+(i)})$ and $\bar{\psi}^{-1}(U_t^+)$. Moreover we can see that

$$(44) \qquad c_{k,t}^*(v) \; (= \alpha^{-1}(c_{k,t}^+(\alpha(v))))) = \textstyle\sum_l (g_t(\alpha(v)(w_{kl}))) z_{kl}$$

$$(45) \qquad (\alpha(d^{*(i)}_{k,t}(v)))(a) \; (= [d_{k,t}^{+(i)}(a(v))](a)) = f_t(va\textstyle\sum_j \beta_k(x_{ij}) y_{ij})$$

$$(46) \qquad [\alpha(U_t^*(v))](a) \; (= [U_t^+(\alpha(v))](a)) = f_t(va).$$

On the other hand, if we define $f_n^* : {}_{B*}V \longrightarrow {}_{B*}U$ and $g_n^* : {}_{B*}U \longrightarrow {}_{B*}V$ by $f_n^*(v) = [\alpha(v)](a_n)$ and $g_n^*(u) = \alpha^{-1}(\lambda_u \cdot \delta_n)$, then $\sum g_n^* \cdot f_n^* = Id_V$, which shows 3).

Now 1), 2) and 3) enable us to use the above argument to the QF extension A^*/B^* with the left QF system (41*) and the right QF system (40*). So, let $\tilde{c}_{i,n}$, $\tilde{d}_{i,n}^{(k)}$ and $\tilde{a}_n$ be in $\tilde{A}$ such that

$$(44') \qquad v\tilde{c}_{i,n} = \textstyle\sum_{k,t} c_{k,t}^* \{g_n^*((\alpha^*(v))(d_{k,t}^{*(i)}))\}$$

$$(45') \qquad [\alpha^*(v\tilde{d}_{i,n}^{(k)})](a^*) = f_n^*(\textstyle\sum_t (c_{k,t}^* \cdot \alpha_i^*(U_t^*))(a^*v))$$

$$(46') \qquad [\alpha^*(v\tilde{a}_n)](a^*) = f_n^*(a^*v).$$

Then
$$[\alpha(a^*v\tilde{a}_n)](1) = \alpha(a^*va_n)(1),$$

and so,
$$[\xi(a^*)](v\tilde{a}_n) = [\xi(a^*)](va_n).$$

Since $\xi(A^*) = \mathrm{Hom}(V_B, U_B)$ and $V_B \mid U_{B'}$ the last implies $\tilde{a}_n = \rho(a_n)$. Furthermore, the right hands of (44′) and (45′) are equal to $v \sum_j \delta_n(x_{ij}) y_{ij}$ and $v \sum_l w_{kl} \alpha_i(z_{kl} a_n)$, respectively, and so,

$$\tilde{c}_{i,n} = \rho(\textstyle\sum_j \delta_n(x_{ij}) y_{ij}),$$

$$\tilde{d}_{i,n}^{(k)} = \rho(\textstyle\sum_l w_{kl} \alpha_i(z_{kl} a_n)).$$

It follows that

$$\textstyle\sum_{i,n} \widetilde{d}^{(k)}_{i,n} \otimes \widetilde{c}_{i,n} = \sum_l \rho(w_{kl}) \otimes \rho(z_{kl}),$$

$$\textstyle\sum_n \widetilde{a}_n \otimes \widetilde{c}_{i,n} = \sum_j \rho(x_{ij}) \otimes \rho(y_{ij}) \ \text{ in } \widetilde{A} \otimes {}_{\widetilde{B}}\widetilde{A}.$$

Therefore, if we define $\widetilde{\alpha}_i$, $\widetilde{\beta}_k \in \mathrm{Hom}({}_{\widetilde{B}}\widetilde{A}_{\widetilde{B}},\ {}_{\widetilde{B}}\widetilde{B}_{\widetilde{B}})$ by

(42′) $$u\widetilde{\alpha}_i(\widetilde{a}) = \{\alpha^*((\alpha^{*-1}(\rho_u \cdot \alpha_i^*))\widetilde{a})\}(1_{A*})$$

(43′) $$u\widetilde{\beta}_k(\widetilde{a}) = \{\alpha^*((\alpha^{*-1}(\rho_u \cdot \beta_k^*)) \cdot \widetilde{a})\}(1_{A*})$$

($u \in U$, $\rho_u : B^* \ni b^* \longmapsto b^*u \in U$), we know that $(\widetilde{\beta}_k : \sum_l \rho(w_{kl}) \otimes \rho(z_{kl}))_k$ and $(\widetilde{\alpha}_i : \sum_j \rho(x_{ij}) \otimes \rho(y_{ij}))_i$ are respective left QF and right QF systems for $[\widetilde{A}/\widetilde{B}, \widetilde{\varphi}]$. Moreover it is easy to see $\widetilde{\alpha}_i(\rho(a)) = \rho(\alpha_i(a))$ and $\widetilde{B}_k(\rho(a)) = \beta_k(a)$ for $a \in A$. Therefore, as was mentioned in the proof of Theorem 1.1, we have the latter half of 4). Similarly we can show the assertion for a Frobenius extension. Thus our proof is complete.

Remark. Under the same assumption as the theorem, let us consider the following diagram of functors:

$$\begin{array}{ccc}
 & \overline{\mathrm{D}}_1 & \\
\mathrm{Mod}_A & \rightleftarrows & {}_{A^*}\mathrm{Mod} \\
\mathrm{S},\ \mathrm{T}\ \uparrow & \overline{\mathrm{D}}_2 & \uparrow\ \mathrm{S}^*,\ \mathrm{T}^* \\
 & \mathrm{D}_1 & \\
\mathrm{Mod}_B & \rightleftarrows & {}_{B^*}\mathrm{Mod} \\
 & \mathrm{D}_2 &
\end{array}$$

where Mod_B (resp. ${}_{B^*}\mathrm{Mod}$) denotes the category of right B (resp. left B^*)-modules and

$$\begin{array}{ll}
\mathrm{D}_1 = \mathrm{Hom}_B(-,\ U_B), & \mathrm{D}_2 = \mathrm{Hom}_{B^*}(-,\ {}_{B^*}U) \\
\overline{\mathrm{D}}_1 = \mathrm{Hom}_A(-,\ V_A), & \overline{\mathrm{D}}_2 = \mathrm{Hom}_{A^*}(-,\ {}_{A^*}V) \\
\mathrm{S} \ = - \otimes {}_BA_A \quad , & \mathrm{T} \ = \mathrm{Hom}_B(A_B,\ -) \\
\mathrm{S}^* = {}_{A^*}A^*{}_{B^*} \otimes - \ , & \mathrm{T}^* = \mathrm{Hom}_{B^*}({}_{B^*}A^*,\ -).
\end{array}$$

Then the functors $\overline{\mathrm{D}}_1 \circ \mathrm{S}$, $\mathrm{T}^* \circ \mathrm{D}_1 : \mathrm{Mod}_B \longrightarrow {}_{A^*}\mathrm{Mod}$ are eqivalent. The same holds for the functors $\overline{\mathrm{D}}_2 \circ \mathrm{S}^*$, $\mathrm{T} \circ \mathrm{D}_2 : {}_{B^*}\mathrm{Mod} \longrightarrow \mathrm{Mod}_A$. In fact, $(\overline{\mathrm{D}}_1 \circ \mathrm{S})(X) = \mathrm{Hom}(X \otimes {}_BA_A,\ V_A)$

$$\cong \mathrm{Hom}(X_B,\ \mathrm{Hom}(A_A,\ V_A)_B)$$
$$\cong \mathrm{Hom}(X_B,\ V_B)$$
$$\mathrm{Hom}(Id_X,\ \alpha^*)$$
$$\cong \mathrm{Hom}(X_B,\ \mathrm{Hom}({}_{B^*}A^*,\ {}_{B^*}U)_B)$$
$$\cong \mathrm{Hom}\ ({}_{B^*}A,\ {}_{B^*}\mathrm{Hom}(X_B,\ U_B))$$
$$= (\mathrm{T}^* \circ \mathrm{D}_1)\ (X) \qquad (X \in \mathrm{Mod}_B).$$

Thus, if we define $\eta_X : (\overline{\mathrm{D}}_1 \circ \mathrm{S})\ (X) \longrightarrow (\mathrm{T}^* \circ \mathrm{D}_1)(X)$ by the composite of these isomorphisms, then η_X is a left A^*-isomorphism which is natural in $X \in \mathrm{Mod}_B$ obviously. The second assertion can be shown similarly.

Corollary. *Suppose that U_B is an f. g. injective cogenerator and B is right artinian. Put $V = \mathrm{Hom}(A_B,\ U_B)$. If $[A/B,\ \varphi]$ is QF (resp. Frobenius) then $[A^*/B^*,\ \varphi^*]$ is QF (resp. Frobenius).*

Proof. By our assumption A is right artinian and V is an f. g. injective cogenerator as an A, hence as a B-module. Thus $V_B \mid U_B$, and so this corollary is a direct consequence of the theorem.

Remark 1) This corolllary also can be seen as follows: In what follows, for a ring R, we shall denote by $\mathfrak{D}_R$ (resp. ${}_R\mathfrak{D}$) the full subcategory of Mod_R (resp. ${}_R\mathrm{Mod}$) consisting of all f. g. right (resp. left) R-modules. By our assumption, A^* and B^* are both left artinian, ${}_{B^*}U$ and ${}_{A^*}V$ are both f. g. injective cogenerators, $A = [\mathrm{End}({}_{A^*}V)]^\circ$ and $B = [\mathrm{End}({}_{B^*}U)]^\circ$, and further $\mathrm{D} = (\mathrm{D}_{10},\ \mathrm{D}_{20})$ (resp. $\overline{\mathrm{D}} = (\overline{\mathrm{D}}_{10},\ \overline{\mathrm{D}}_{20})$) gives a duality between $\mathfrak{D}_B$ and $\mathfrak{D}_{B^*}$ (resp. between $\mathfrak{D}_A$ and ${}_{A^*}\mathfrak{D}$), where $(\ \ )_0$ denotes the restriction of $(\ \ )$ to $\mathfrak{D}$. [3] However, as was mentioned in the proof of the corollary, we have $V_B | U_B$ and ${}_{B^*}V \mid {}_{B^*}U$, and so, Remark to Theorem 3. 1 yields that

$$\overline{\mathrm{D}}_{20} \circ \mathrm{S}^*{}_0 \simeq \mathrm{T}_0 \circ \overline{\mathrm{D}}_{20}, \qquad \overline{\mathrm{D}}_{10} \circ \mathrm{S}_0 \simeq \mathrm{T}^*{}_0 \circ \overline{\mathrm{D}}_{10}.$$

Moreover the assumption that A/B is QF (resp. Frobenius) yields $\mathrm{S}_0 \sim$ (resp. $\simeq$) T_0. [4] Thus

$$\mathrm{S}^*{}_0 \simeq \overline{\mathrm{D}}_{10} \circ \overline{\mathrm{D}}_{20} \circ \mathrm{S}^*{}_0 \simeq \overline{\mathrm{D}}_{10} \circ \mathrm{T}_0 \circ \overline{\mathrm{D}}_{20} \sim (\text{resp.} \simeq)\ \overline{\mathrm{D}}_{10} \circ \mathrm{S}_0 \circ \overline{\mathrm{D}}_{20}$$
$$\simeq \mathrm{T}^*{}_0 \circ \overline{\mathrm{D}}_{10} \circ \overline{\mathrm{D}}_{20} \simeq \mathrm{T}^*{}_0,$$

and so A^*/B^* is QF (resp. Frobenius).

3) See [1, Th. 6].
4) See [6, Th. 5. 1].

2) With the same assumptions as in the corollary, if $[A/B, \varphi]$ is QF then φ and φ^* are both monic : Let $b \in B$ and $b^* \in B^*$ be elements with $\varphi(b)=0$ and $\varphi^*(b^*)=0$. Since B is right artinian, U_B is f.g. injective and V_B is a cogenerator, we have therefore $U_B | V_B$. As $Vb = V\varphi(b)=0$, $Ub=0$. Thus the faithfulness of U_B implies $b=0$. Finally $\varphi^*(b^*)=0 \Longrightarrow b^*\mathrm{Hom}(A_B, U_B)=0 \Longrightarrow b^*(\mathrm{Hom}(A_B, U_B)(A)) = b^*U = 0$ as desired.

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(Received October 12, 1973)
(Revised June 25, 1974)