g-ADICAL ANALOGUES OF SOME ARITHMETICAL FUNCTIONS

IEKATA SHIOKAWA

1. Introduction

Let $g \geqq 2$ be a fixed integer. Any positive integer n can be expressed uniquely in the form

$$n = \sum_{i=1}^{k} a_i g^{k-i} = a_1 a_2 \cdots a_k$$

where each a_i is one of $0, 1, \cdots, g-1$, and a_1 is different from 0 or, equivalently,

$$k = k(n) = \left[\frac{\log n}{\log g}\right] + 1,$$

where $[z]$ denotes the integral part of the real number z. R. Bellman and H. N. Shapiro [1] pointed out the analogy existing between the dyadic decimal expression of a positive integer and the representation of the integer as the product of primes in the standard form. In this paper we attempt to develop their idea and exhibit some analogous facts in the g-adic representation system of positive integers.

A positive integer n is said to be g-adically divisible by a positive integer m if $k(m) \leqq k(n)$ and $a_i(m) \leqq a_i(n)$ for each i, $1 \leqq i \leqq k(n)$. We then say that m g-adically divides n or m is a g-adical divisor of n, and indicate this by $m \mid_g n$. We say also that n is a g-adical multiple of m. It will be natural to understand that the integer 0 is a g-adical divisor of any non-negative integer. We note that $d \mid_g n$ if and only if $n - d \mid_g n$. Let m, n be two non-negative integers. Their *greatest common g-adical divisor*, denoted by $(n, m)_g$, is defined to be the largest positive integer which g-adically divides both n and m. If $(n, m)_g = 0$, we say that n is *g-adically prime* to m. The *least common g-adic mutiple* $\{n, m\}$ of two non-negative integers n and m is the smallest positive integer which is g-adically divisible by both n and m. Then it is easy to see that $n + m = (n, m)_g + \{n, m\}_g$. According to our definitions, all powers of g will play in a sense a role of 'prime numbers' and this regulality of 'primes' brings us simpler and more interesting consequences than those in ordinary multiplicative number theory.

In this paper, by an arithmetical fnnction is meant a complex-valued function which is defined on the set of non-negative integers. An arithmetical function f is said to be *g-adically additive* if $f(m+n)=f(m)+f(n)$, whenever $(m, n)_g=0$. (This concept was first introduced by R. Bellman and H. N. Shapiro in [1] for the special case of $g=2$.) Furthermore if $f(m+n)=f(m)+f(n)$ for any non-negative integers m and n, f is called *completely* (*g-adically*) *additive*. Clearly $f(0)=0$ if f is g-adically additive. An arithmetical function f is *g-adically multiplicative* if (i) $f(0)=1$ and (ii) $f(m+n)=f(m)f(n)$, whenever $(m, n)_g=0$. If, further, the relatin $f(m+n)=f(m)f(n)$ holds for all non-negative integers m and n, f is called *completely* (*g-adically*) *multiplicative*. Now, the following four propositions can easily be verified:

Proposition 1. 1. *If f is g-adically multiplicative then*

$$\sum_{0\leq n<g^k} f(n) = \prod_{i=0}^{k-1} \sum_{j=0}^{g-1} f(jg^i), \quad r \geqq 1.$$

Propositin 1. 2. *If f is a g-adically multiplicative function whose sum $\sum_{n=0}^{\infty} f(n)$ is absolutely convergent then*

$$\sum_{n=0}^{\infty} f(n) = \prod_{k=0}^{\infty} \sum_{j=0}^{g-1} f(jg^k).$$

If, further, f is completely multiplicative then

$$\sum_{n=0}^{\infty} f(n) = \prod_{k=0}^{\infty} \frac{1-f(g^{k+1})}{1-f(g^k)}.$$

Proposition 1.3. *If f is g-adically multiplicative then $\sum_{d|_g n} f(d)$ is also g-adically multiplicative.*

Proposition 1.4. *If f is g-adically multiplicative then*

$$\sum_{d|_g n} f(d) = \prod_{i=1}^{k-1} \sum_{j=0}^{a_i} f(jg^{k-i})$$

where $k=k(n)$ and $a_i=a_i(n)$.

Some important arithmetical functions in multiplicative number theory have obvious analogues when translated into the language of the g-adic decimal expression. Thus, the function $\Omega_g(n)$, the g-adical analogue of the arithmetical function $\Omega(n)$ giving the total number of prime factors of n, is defined by

$$\Omega_g(n) = \sum_{\substack{j,i \\ jg^i \mid_g n}} 1 = \sum_{i=1}^{k(x)} a_i(n).$$

Similary we define the g-adical analogue of the function $\omega_g(n)$, the number of different prime factors of n, by setting

$$\omega_g(n) = \sum_{g^i \mid_g n} 1.$$

Obviously, both of the function $\Omega_g(n)$ and $\omega_g(n)$ are g-adically additive. The analogue of the Möbius function $\mu(n)$ can be written by

$$\mu_g(n) = \begin{cases} (-1)^{\omega_g(n)} & \text{if } \Omega_g(n) = \omega_g(n) \\ 0 & \text{otherwise.} \end{cases}$$

Moreover we define

$$\varphi_g(n) = \sum_{\substack{0 \leq m \leq n \\ (m,n)_g = 0}} 1$$

$$\tau_g(n) = \sum_{d \mid_g n} 1$$

and

$$\sigma_g(n) = \sum_{d \mid_g n} d$$

as the g-adical analogues of the Euler function $\varphi(n)$, the divisor function $\tau(n)$ and the sum $\sigma(n)$ of the divisors of n, respectively.

In this paper all O-constants may depend possively on g.

2. The function $\mu_g(n)$

Proposition 2.1. *The function $\mu_g(n)$ is g-adically multiplicative.*

Proof. *Let* $(m, n)_g = 0$. Suppose first that $\mu_g(m)\mu_g(n) = 0$. Then, from the g-adical additivity of Ω_g and ω_g, we have $\Omega_g(m+n) > \omega_g(m+n)$. This implies that $\mu_g(m+n) = 0$. If, however, $\mu_g(m)\mu_g(n) = 0$ then $\Omega_g(m+n) = \omega_g(m+n)$, and so

$$\mu_g(m+n) = (-1)^{\omega_g(m+n)} = (-1)^{\omega_g(m)}(-1)^{\omega_g(n)} = \mu_g(m) \cdot \mu_g(n).$$

Proposition 2.2.

$$\sum_{d|_g n} \mu_g(d) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geqq 1. \end{cases}$$

Proof. Put $f(n) = \sum_{d|_g n} \mu_g(d)$. Clearly $f(0) = 1$. For any integer j, $0 < j < g$, and $k \geqq 0$, we have

$$f(j\, g^k) = \mu_g(0) + \mu_g(g^k) = 1 - 1 = 0.$$

But, from Propositions 1.3 and 2.1, $f(n)$ is g-adically multiplicative. Hence, for any positive integer n we have

$$f(n) = f(\sum_{i=1}^{k} a_i\, g^{k-i}) = \prod_{i=1}^{k} f(a_i\, g^{k-i}) = 0.$$

Proposition 2.3. (*Inversion formula*) *Let f be an arithmetical function. If*

$$h(n) = \sum_{d|_g n} f(d), \quad n \geqq 0,$$

then

$$f(n) = \sum_{d|_g n} \mu_g(d)\, h(n-d) = \sum_{d|_g n} \mu_g(n-d) h(d), \quad n \geqq 0,$$

and conversely.

Proof. If $h(n) = \sum_{d|_g n} f(d)$, $n \geqq 0$, then

$$\sum_{d|_g n} \mu_g(d) h(n-d) = \sum_{d|_g n} \mu_g(d) \sum_{d'|_g n-d} f(d')$$
$$= \sum_{d+d'|_g n} \mu_g(d) f(d') = \sum_{d'|_g n} f(d') \sum_{d|_g n-d'} \mu_g(d) = f(n).$$

(using Proposition 2.2). Convesely, if $f(n) = \sum_{d|_g n} \mu_g(n-d) h(d)$, $n \geqq 0$, then

$$\sum_{d|_g n} f(d) = \sum_{d|_g n} f(n-d)$$
$$= \sum_{d|_g n} \sum_{d'|_g n-d} \mu_g(n-d-d')\, h(d')$$
$$= \sum_{d'|_g n} h(d') \sum_{d|_g n-d} \mu_g(n-d-d) = h(n).$$

Proposition 2.4. *Let $f(n)$ be g-adically multiplicative. Then*

$$\sum_{d|_g n} \mu_g(d)f(d) = \prod_{g^i|_g n} (1 - f(g^i))$$

Proof. From Propositions 1.4 and 2.1. we have

$$\sum_{d|_g n} \mu_g(d)f(d) = \prod_{j=0}^{k(n)-1} \sum_{j=0}^{a_i} \mu(j\, g^i)\, f(j\, g^i)$$
$$= \prod_{g^i|_g n} (1 + \mu_g g^i) f(g^i)) = \prod_{g^i|_g n} (1 - f(g^i)).$$

Propoition 2.5. *We have*

$$\sum_{0\leq n\leq x} \mu_g(n) = \begin{cases} \mu_g([x]) & \text{if } [x] \equiv 0 \pmod g \\ 0 & \text{otherwise}. \end{cases}$$

Proof. If $n \equiv 0 \pmod g$ then $a_k(n) = 0$, $a_k(n+1) = 1$ and $a_i(n) = a_i(n+1)$, $1 \leqq i \leqq k-1$, and so $\mu_g(n) + \mu_g(n+1) = 0$. But if $n \equiv j \pmod g$, $2 \leqq j \leqq g-1$, then $\Omega_g(n) > \omega_g(n)$; that is, $\mu_g(n) = 0$. Hence for any integer $m \geqq 0$ and l, $1 \leqq l \leqq g-1$,

$$\sum_{mg\leq n\leq mg+l} \mu_g(n) = 0.$$

As the result we obtain

$$\sum_{0\leq n\leq x} \mu_g(n) = \sum_{\left[\frac{x}{g}\right] g\leq n\leq [x]} \mu_g(n)$$
$$= \begin{cases} \mu_g([x]), & \text{if } [x] \equiv 0 \pmod g, \\ 0 & \text{othewise}. \end{cases}$$

We now write

$$\sum_{\substack{0\leq n<x \\ \mu_g(n)\neq 0}} 1 = M(x) \quad \text{and} \quad \sum_{\substack{0\leq n<x \\ \mu_g(n)=1}} 1 = M_1(x).$$

Proposition 2.6. *We have*

$$\limsup_{x\to\infty} x^{-\frac{\log 2}{\log g}} M(x) = (g-1)^{\frac{\log 2}{\log g}} \tag{2.1}$$

$$\liminf_{x\to\infty} x^{-\frac{\log 2}{\log g}} M(x) = 1, \tag{2.2}$$

and

$$\limsup_{x\to\infty} x^{-\frac{\log 2}{\log g}} M_1(x) = \frac{1}{2} (g-1)^{\frac{\log 2}{\log g}}, \tag{2.3}$$

(2.4) $$\liminf_{x\to\infty} x^{-\frac{\log 2}{\log g}} M_1(x) = \frac{1}{2}.$$

Proof. Let

$$n = \sum_{i=1}^{k} b_i 2^{k-i}$$

be the dyadic development of a positive integer n where $b_1 = 1$. We set

$$G(n) = \sum_{i=1}^{k} b_i g^{k-i}, \qquad G(0) = 0.$$

Clearly $G(n)$ is a g-adically additive and monotonically increasing arithmetical function. Note that $\mu_g(m) \neq 0$ if and only if $m = G(n)$ for some non-negative integer n. Thus we have

$$M(G(n)) = \sum_{\substack{m \\ 0 \leq G(m) < G(n)}} 1 = n, \quad n \geqq 0.$$

This implies that

(2.5) $$M(x) = \sum_{0 \leq m < G(n+1)} 1 = n + 1$$

for any x satisfying $G(n) < x \leqq G(n+1)$. Hence, by (2.5) we obtain

(2.6) $$\max_{G(n) < x \leq G(n+1)} x^{-\frac{\log 2}{\log g}} M(x) = G(n)^{-\frac{\log 2}{\log g}} n + O(\tfrac{1}{n}),$$

and

(2.7) $$\min_{G(n) < x \leq G(n+1)} x^{-\frac{\log 2}{\log g}} M(x) = G(n+1)^{-\frac{\log 2}{\log g}} (n+1).$$

Put

$$W(n) = G(n)^{-\frac{\log 2}{\log g}} n.$$

Then the following inequality holds for any integers j, $i \leqq j \leqq k$, and m, $0 \leqq m < 2^{k-j}$:

(2.8) $$W(2^k + 2^{k-1} + \cdots + 2^{k-j} + m) > W(2^k + 2^{k-1} + \cdots + 2^{k-j+1} + m).$$

Indeed (2.8) is true for $g = 2$. Suppose now that $g \geqq 3$. Then

$$\log W(2^k + 2^{k-1} + \cdots + 2^{k-j} + m) - \log W(2^k + 2^{k-1} + \cdots + 2^{k-j+1} + m)$$

$$= \log\frac{2^k+2^{k-1}+\cdots+2^{k-j}+m}{2^k+2^{k-1}+\cdots+2^{k-j+1}+m} - \frac{\log 2}{\log g}\log\frac{g^k+g^{k-1}+\cdots+g^{k-j}+G(m)}{g^k+g^{k-1}+\cdots+g^{k-j+1}+G(m)}$$

$$> \log\frac{2^k+2^{k-1}+\cdots+2^{k-j}+2^{k-j}}{2^k+2^{k-1}+\cdots+2^{k-j+1}+2^{k-j}} - \frac{\log 2}{\log g}\log\frac{g^k+g^{k-1}+\cdots+g^{k-j}}{g^k+g^{k-1}+\cdots+g^{k-j+1}}$$

$$= \log\frac{2^{j+1}}{2^{j+1}-1} - \frac{\log 2}{\log g}\log\frac{g^{j+1}-1}{g^{j+1}-g} \geqq \log\frac{2^{j+1}}{2^{j+1}-1} - \log\frac{3^{j+1}-1}{3^{j+1}-3} > 0,$$

since $g\geqq 3$ and $j\geqq 1$. Using (2. 8), we have, for any integer n with $2^k\leqq n<2^{k+1}-1$, $k\geqq 1$,

$$\begin{aligned} W(n) &= W(2^{k-1}+b_2 2^{k-2}+\cdots b_{k-1}2+b_k) \\ &\leqq W(2^{k-1}+2^{k-2}+b_3 2^{k-3}+\cdots+b_{k-1}2+b_k) \\ &\leqq \cdots \leqq W(2^{k-1}+2^{k-2}+\cdots+1) = W(2^{k+1}-1). \end{aligned}$$

Then

$$\text{(2. 9)} \qquad \max_{2^k\leqq n<2^{k+1}} W(n) = (g-1)^{\frac{\log 2}{\log g}}(g^k-1)^{-\frac{\log 2}{\log g}}(2_k-1), \quad k\geqq 1.$$

Therefore, from (2. 6) and (2. 9), we get

$$\begin{aligned} \limsup_{x\to\infty} x^{-\frac{\log 2}{\log g}} M(x) &= \limsup_{n\to\infty} G(n)^{-\frac{\log 2}{\log g}} n \\ &= \lim_{k\to\infty}\max_{2^k\leqq n<2^{k+1}} G(n)^{-\frac{\log 2}{\log g}} n = (g-1)^{\frac{\log 2}{\log g}}. \end{aligned}$$

Similary we obtain (2. 2) by using (2. 7) and the following equality ;

$$\max_{2^k\leqq n<2^{k+1}} W(n) = 1, \quad k\geqq 1,$$

which can be deduced from

$$n^{\frac{\log g}{\log 2}} = \Big(\sum_{b_i=0} 2^i\Big)^{\frac{\log g}{\log 2}} \geqq \sum_{b_i\neq 0} 2^{i\frac{\log g}{\log 2}} = G(n)$$

on noticing $W(2^k)=1$ where b_i is, of course, the i-th digits in the dyadic development of n. Finally (2. 3) (resp. (2. 4)) follows from (2. 1) (resp. (2. 2)) and Proposition 2. 5.

The proof of Proposition 2. 6 is now complete.

3. The function $\Omega_g(n)$

First we estimate the magnitude of the function Ω_g. By the definition we readily have

$$\text{(3. 1)} \qquad 1\leqq \Omega_g(n)\leqq (g-1)\,k\,(n), \quad n\geqq 1.$$

But at the same time

$$\Omega_g(g^k-1) = (g-1)(k-1) \text{ and } \Omega_g(g^k) = 1, \quad k \geqq 1.$$

Hence we have

$$\limsup_{n\to\infty} \frac{\Omega_g(n)}{\log n} = \frac{g-1}{\log g},$$

and

$$\liminf_{n\to\infty} \Omega_g(n) = 1.$$

Proposition 3.1. *We have for an arbitrary* $g \geqq 2$

$$\sum_{1\leqq n\leqq x} \Omega_g(n) = \frac{g-1}{2\log g} x \log x + O(x)$$

Moreover, for the particular case of $g=2$, *we have*

$$\limsup_{x\to\infty} \frac{1}{x}\left(\frac{x\log x}{2\log 2} - \sum_{0\leqq n\leqq x} \Omega_2(n)\right) = 1 - \frac{\log 3}{2\log 2}$$

and

$$\liminf_{x\to\infty} \frac{1}{x}\left(\frac{x\log x}{2\log 2} - \sum_{0\leqq n\leqq x} \Omega_2(n)\right) = 0.$$

Proof. Cf. [1], [3], [6] and [7].

Proposition 3.2. *The normal order of* $\Omega_g(n)$ *is* $\frac{(g-1)\log n}{2\log g}$.

Proof. Let $n = a_1 a_2 \cdots a_{k+1}$ be the g-adic decimal expression of n where a_1 is not 0 and let b be an integer with $0 \leqq b \leqq g-1$. We set

$$N(b, n) = \sum_{\substack{1\leqq i\leqq k+1\\ a_i=b}} 1 \text{ and } N^*(n, b) = \sum_{\substack{2\leqq i\leqq k+1\\ a_i=b}} 1.$$

Further we define

$$w(b, n) = \left|\frac{N(b,n)}{k+1} - \frac{1}{g}\right| \text{ and } w^*(b, n) = \left|\frac{N^*(b, n)}{k} - \frac{1}{g}\right|.$$

It follows from Lemma 8.8 in [4] that for any $\varepsilon > 0$ there exist a constant θ, $0<\theta<1$, depending only on g and ε such that the inequality

$$\sum_{\substack{g^k \leq n < g^{k+1} \\ w^*(b,n) > \frac{\varepsilon}{2g}}} 1 < k\, g^{\theta k}$$

holds for all sufficiently large k. But clearly if $w(b, n) > \frac{\varepsilon}{g}$ then $w^*(b, n) > \frac{\varepsilon}{2g}$ for all sufficiently large k. Hence we have

$$\sum_{\substack{g^k \leq n < g^{k+1} \\ w(b,n) > \frac{\varepsilon}{g}}} 1 \leqq \sum_{\substack{g^k \leq n < g^{k+1} \\ w^*(b,n) > \frac{\varepsilon}{2g}}} 1 < kg^{\theta k}$$

for all sufficiently large k. This implies that

$$\sum_{\substack{0 \leq n < x \\ w(b,\, n) > \frac{\varepsilon}{g}}} 1 = O(x^{\theta} \log x)\,.$$

In other wards the inequality $w(b, n) \leqq \frac{\varepsilon}{g}$ holds for almost all positive integers n. (Here 'almost all' means that such integers n form a set of density 1 in the set of all positive integers.) Recalling the definition of Ω_g we have

$$\Omega_g(n) = \sum_{b=0}^{g-1} b\, N(b,\ n)\,.$$

Hence the inequality

$$\frac{g-1}{2} \frac{\log n}{\log g}(1 - 2\varepsilon) < \Omega_g(n) < \frac{g-1}{2} \frac{\log n}{\log g}(1 + 2\varepsilon)$$

holds for almost all positive integer n, which is the assertion of our proposition.

Proposition 3.3 ([2]). *The nnmber of integers n, $n \leqq x$, satisfying the condition*

$$n \equiv (\text{mod } m)\,;\ \ \Omega_g(n) \equiv a \ (\text{mod } x)$$

where $m > 1$, $r > 1$, l, a are integers and $(r, g-1) = 1$, is given by the formula

$$T_0(x) = \frac{x}{mr} + O(x^{\lambda})\,, \quad \lambda < 1,$$

where λ does not depend on x, m, l, a.

4. The function $\omega_g(n)$

By a reasoning similar to that of the preceding section we have

$$\limsup_{n\to\infty} \frac{\omega_g(n)}{\log n} = \frac{1}{\log g}$$

and

$$\liminf_{n\to\infty} \omega_g(n) = 1.$$

Proposition 4.1.

$$\sum_{1\leq n\leq x} \omega_g(n) = \frac{g-1}{g\log g} x \log x + O(x)\,.$$

Proof. We first show, by induction on k, that

$$\sum_{1\leq n<g^k} \omega_g(n) = (g-1)\,k\,g^{k-1}, \quad k\geqq 1\,. \tag{4.1}$$

Indeed, this is true for $k=1$. Suppose now that the equality is true for some $k\geqq 1$. Then

$$\begin{aligned}\sum_{1\leq n<g^{k+1}} \omega_g(n) &= (g-1)g^k + g\sum_{0\leq n<g^k} \omega_g(n)\\ &= (g-1)\,(k+1)g^k\,.\end{aligned}$$

By considering the integral part of x we may suppose, without loss of generality, that x itself is an integer. Let $x = a_1\, a_2 \cdots a^k$, $a_1 \neq 0$, be the g-adic development of x and set

$$\delta_i = \begin{cases} 0 & \text{if } a_i = 0 \\ 1 & \text{otherwise.}\end{cases}$$

Then, using (4.1),

$$\begin{aligned}\sum_{1\leq n\leq x} \omega_g(n) &= \delta_1(x-g^{k-1}) + a_1\sum_{0\leq n<g^{k-1}} \omega_g(n) + \sum_{0\leq n\leq x-a_1g^{k-1}} \omega_g(n)\\ &= \delta_1(x-g^{k-1}) + a_1\sum_{0\leq n<g^{n-1}} \omega_g(n) + \delta_2(x-a_1g^{k-1}-g^{k-2})\\ &\quad + a_2\sum_{0\leq n<g^{k-2}} \omega_g(n) + \sum_{0\leq n\leq x-a_1g^{k-1}-a_2g^{k-2}} \omega_g(n)\\ &= \cdots = \sum_{i=1}^{k} \delta_i\Big(\sum_{l=i}^{k} a_l\, g^{k-l} - g^{k-i}\Big) + \sum_{i=1}^{k} a_i \sum_{0\leq n<g^{k-i}} \omega_g(n) + 1\end{aligned}$$

$$= \sum_{i=1}^{k} \delta_i \sum_{l=i}^{k} a_l\, g^{k-l} - \sum_{i=1}^{k} \delta_i\, g^{k-i} + \sum_{i=1}^{k} a_i(g-1)(k-i)\, g^{k-i-1} + 1$$

$$= \mathrm{I}_1 - \mathrm{I}_2 + \mathrm{I}_3 + 1, \quad \text{say.}$$

But

$$\mathrm{I}_1 \leqq (g-1) \sum_{i=1}^{k} \sum_{l=i}^{k} g^{k-l} = O(x)$$

$$\mathrm{I}_2 \leqq \sum_{i=1}^{k} g^{k-i} = O(x)$$

and

$$\mathrm{I}_3 = \frac{g-1}{g} k \sum_{i=1}^{k} a_i\, g^{k-i} - (g-1) \sum_{i=0}^{k} a_i\, i g^{k-i-1}$$

$$= \frac{g-1}{g} k\, x + O(x) = \frac{g-1}{g} \frac{\log x}{\log g} x + O(x)\,.$$

Summing up these results, we obtain

$$\sum_{1 \leqq n \leqq x} \omega_g(n) = \frac{g-1}{g \log g} x \log x + O(x)\,.$$

Proposition 4.2. *The normal order of* ω_g *is* $\frac{g-1}{g} \frac{\log n}{\log g}$.

Proof. Similar to that of Proposition 3.2.

Proposition 4.3. *The number of integers* n, $n \leqq x$ *satisfying the condition*

$$n \equiv l \pmod{m}\,; \quad \omega_g(n) \equiv a \pmod{r}$$

where $m > 1$, $r > 1$, l, a *are integers, is given by the formula*

$$U(x) = \frac{x}{mr} + O(x^{\nu}), \quad \nu < 1,$$

where ν *does not depend on* x, l, a.

Proof. Following A. O. Gelfond (see [2]), we have

$$U(x) = \frac{1}{mr} \sum_{t=0}^{m-1} \sum_{s=0}^{r-1} \sum_{n=0}^{x} e\left(\frac{n-l}{m} t + \frac{s}{r} (\omega_g(n) - a)\right)$$

$$= \frac{x}{mr} + \frac{1}{mr} \sum_{t=1}^{m-1} \sum_{n=1}^{x} e\left(\frac{n-l}{m} t\right)$$

$$+\frac{1}{mr}\sum_{t=0}^{m-1}\sum_{s=1}^{r-1}e\Big(-\frac{l}{m}t-\frac{s}{r}a\Big)\sum_{n=1}^{x}e\Big(\frac{n}{m}t+\frac{a}{r}\omega_g(n)\Big)$$

$$=\frac{x}{mr}+R_1+R_2,\ \text{say},$$

where $e(u)=e^{2\pi iu}$, $i^2=-1$. Apparently $|R_1|<1$. So we have to show $|R_2|=O(x^\nu)$. In order to do this, it is enough to prove that

$$\Big|\sum_{n=1}^{x}e\Big(\frac{n}{m}t+\frac{s}{r}\omega_g(n)\Big)\Big|=O(x^\nu),\quad \nu<1, \tag{4.2}$$

where $0\leqq t\leqq m-1$ and $1\leqq s\leqq r-1$. Let $x=a_1\,a_2\cdots a_k$, $a_1\neq 0$, be the g-adic development of x. Then, by Proposition 1.1 we have

$$\Big|\sum_{n=0}^{x}e\Big(\frac{t}{m}n+\frac{s}{r}\omega_g(n)\Big)\Big| \tag{4.3}$$

$$\leqq\sum_{i=0}^{k-1}\Big|\sum_{\sum_{h=0}^{i}a_k g^{k-h}\leqq n<\sum_{h=0}^{j+1}a_h g^{k-h}}e\Big(\frac{t}{m}n+\frac{s}{r}\omega_g(n)\Big)\Big|,\quad a_0=0,$$

$$\leqq g\sum_{j=0}^{k-1}\Big|\sum_{0\leqq n<g^j}e\Big(\frac{t}{m}+\frac{s}{r}\omega_g(n)\Big)\Big|$$

$$=g\sum_{j=0}^{k-1}\prod_{h=0}^{j-1}\Big|\sum_{b=0}^{g-1}e\Big(\frac{t}{m}bg^h+\frac{s}{r}\omega_g(bg^h)\Big)\Big|.$$

Put

$$S=\Big|\sum_{b=0}^{g-1}e\Big(\frac{t}{m}bg^h+\frac{s}{r}\omega_g(bg^h)\Big)\Big|$$

$$=\Big|1+\sum_{b=1}^{k-1}e\Big(\frac{t}{m}bg^h+\frac{s}{r}\Big)\Big|.$$

Then, if m divides (in the ordinary sense) tg^h we have

$$S=\Big|1+(g-1)e\Big(\frac{s}{r}\Big)\Big|$$

$$\leqq((g-1)^2+2(g-1)\cos\frac{2\pi}{r}+1)^{\frac{1}{2}}<g.$$

On the other hand, if m does not divide tg^h

$$S\leqq 1+\Big|\sum_{b=0}^{g-2}e\Big(\frac{t}{m}bg^h\Big)\Big|$$

$$=1+\left|\frac{\sin\pi\frac{t}{m}g^h(g-1)}{\sin\frac{t}{m}g^h}\right|\leqq 1+\frac{\sin\frac{\pi}{m}(g-1)}{\sin\frac{\pi}{m}}<g.$$

Thus we have a constant ν, $\frac{1}{2} < \nu < 1$, which depends at most on g, m and r, such that

(4. 4) $$S < g^{\nu}.$$

Therefore from (4. 3) and (4. 4) we obtain

$$\left| \sum_{n=0}^{x} e\left(\frac{tg^h}{m} + \frac{s}{r}\omega_g(n)\right) \right|$$
$$\leqq g \sum_{j=0}^{k-1} g^{\nu j} = \frac{g}{g^{\nu}-1}(g^{k\nu} - 1) < g^{1}x^{\nu}.$$

This implies (4. 2) and the proof of the proposition is now complete.

5. The function $\varphi_g(n)$

Proposition 5.1. $\varphi_g(n) = g^{k(n)-\omega_g(n)}$.

Proof. Cleary this formula is true for $n = 0$. Assume that n be a positive integer. Let $n = a_1 a_2 \cdots a_k$ be its g-adic development and let m be any integer such that $0 \leqq m \leqq n$. Then $(n, m)_g = 0$ if and only if $m = \sum_{\substack{1<i<k \\ a_i=0}} b^i g^{k-i}$ for some b_i, $0 \leqq b_i \leqq g-1$. Since for each i b_i may take g values $0, 1, \cdots, g-1$ independently and since the number of indices i satisfying $a_i = 0$ is just $k(n) - \omega_g(n)$, we have

$$\varphi_g(n) = g^{k(n)-\omega_g(n)}$$

as required.

By definition $1 \leqq \varphi_g(n) \leqq n$ for any $n \geqq 0$. Nevertheless, from proposition 5. 1 we have $\varphi_g(g^{k-1}) = g^{k-1}$ and $\varphi_g(g^k) = 1$. Then we get

$$\limsup_{n\to\infty} \frac{\varphi_g(n)}{n} = 1 \quad \text{and} \quad \liminf_{n\to\infty} \varphi_g(n) = 1.$$

Proposition 5.2.

$$\log \sum_{0\leqq n\leqq x} \varphi_g(n) = \frac{\log(2g-1)}{\log g} \log x + O(1).$$

Proof. Let k be any positive integer and let l be an integer satisfying $0 \leqq l \leqq k$. Denote by $B(k, l)$ the number of integers n, $0 \leqq n < g^k$ such that $\omega_g(n) = l$. Then we have

(5. 1) $$B(k, l) = \binom{k}{l}(g-1)^l, \quad k > 0, \quad 1 \leqq l < k,$$

which can be proved by induction on k and l. Indeed, the equality (5.1) is trivially true true for any $k \geqq 1$ if either $l=0$ or $l=k$. Suppose that (5.1) is true for some $k \geqq 1$ and all l, $0 \leqq l \leqq k$. Then

$$B(k+1,\ l) = B(k,\ l) + (g-1)B(k,\ l-1)$$
$$= \left(\binom{k}{l} + \binom{k}{l-1}\right)(g-1)^l = \binom{k+1}{l}(g-1)^l.$$

Now by (5.1) we have

$$\sum_{0 \leq n < g^k} g^{-\omega_g(n)} = \sum_{l=0}^{k} B(k,\ l) g^{-l}$$
$$= \sum_{l=0}^{k} \binom{k}{l} (g-1) g^{-l} = g^{-k}(2g-1)^k.$$

From this and Proposition 5.1 we get

$$\sum_{g^{k-1} \leq n < g^k} \varphi_g(n) = \sum_{g^{k-1} \leq n < g^k} g^{k-\omega_g(n)}$$
$$= (g-1) \sum_{0 \leq n < g^{k-1}} g^{k-1-\omega_g(n)} = (g-1)(2g-1)^{k-1}.$$

Hence

$$\sum_{0 \leq n < g^k} \varphi_g(n) = \varphi_g(0) + \sum_{i=0}^{k-1} \sum_{g^j \leq n < g^{j+1}} \varphi_g(n)$$
$$= 1 + (g-1) \sum_{j=0}^{k-1} (2g-1)^j = \frac{1}{2}(2g-1)^k + \frac{1}{2}.$$

Therefore

$$\frac{1}{2}(2g-1)^{k(x)-1} + \frac{1}{2} \leqq \sum_{0 \leq n < x} \varphi_g(n) < \frac{1}{2}(2g-1)^{k(x)} + \frac{1}{2}.$$

And the proposition follows at once from these inequalities.

Proposition 5.3.

$$\log \prod_{0 \leq n < x} \varphi_g(n) = \frac{1}{g} x \log x + O(x).$$

Proof. Since

$$\sum_{0 \leq n < x} k(n) = \sum_{i=1}^{k(x)-1} \sum_{g^{i-1} \leq n < g^i} k(n) + \sum_{g^{k(x)-1} \leq n < x} k(n)$$
$$= \sum_{i=1}^{k(x)-1} i(g-1) g^{i-1} + k(x)(x - g^{k(x)-1} + 1)$$

$$= \frac{1}{\log g} x \log x + O(x)$$

we have readily

$$\log \prod_{0 \leq n < x} \varphi_g(n)$$

$$= \log g(\sum_{0 \leq n < x} k(n) - \sum_{0 \leq n < x} \omega_g (n)) = \frac{1}{g} x \log x + O(x)$$

(usingProposition 4. 1).

6. The function $\tau_g(n)$

Proposition 6.1. $\tau_g (n) = \prod_{i=1}^{k} (a_i + 1)$,

where every a_i *is the i-th digit in the g-adic development of the non-negative integer* n. *Hence* $\tau_g(n)$ *is g-adically multiplicative.*

Proof. Every g-adical divisor d of n can be written, by definition, in the form

$$d = \sum_{i=1}^{k} b_i g^{k-i}, \quad 0 \leqq b_i \leqq a_i .$$

But in the summands of this representation b_i may take $a_i + 1$ values $0, 1, \cdots, a_i$ independently. This proves the proposition.

From Proposition 6. 1 we have easily

$$2 \leqq \tau_g(n) \leqq n + 1 ,$$

$$\tau_g(g^k - 1) = g^k \text{ and } \tau_g(g^k) = 2 .$$

Hence

$$\limsup_{n\to\infty} \frac{\tau_g(n)}{n} = 1 \quad \text{and} \liminf_{n\to\infty} \ \tau_g(n) = 2 . \tag{6. 1}$$

Proposition 6.2.

$$\log \sum_{0 \leq n \geqq x} \tau_g(n) = \frac{\log \frac{1}{2} g(g+1)}{\log g} \log x + O(1).$$

Proof. From Proposition 6. 1 we have

$$\sum_{0 \leq n \leq g^k} \tau_g(n) = \sum_{0 \leq n \leq g^k} \prod_{i=1}^{k} (a_i (n) + 1)$$

$$= \sum_{1\leq m_1\leq g} \sum_{1\leq m_2\leq g} \cdots \sum_{1\leq m_k\leq g} \sum_{i=1}^{k} m_i$$

$$= \sum_{r_1+r_2+\cdots+r_g=k} \frac{k!}{r_1!\, r_2! \cdots r_g!} 1^{r_1} 2^{r_2} \cdots g^{r_g} = \left(\frac{1}{2}g(g+1)\right)^k .$$

Hence

$$\left(\frac{1}{2} g(g+1)\right)^{k(x)-1} \leqq \sum_{0\leqq n\leqq x} \tau_g(x) < \left(\frac{1}{2} g(g+1)\right)^{k(x)},$$

and the proposition follows at once from this inequality.

Proposition 6.3. *For any fixed number* $\varepsilon > 0$ *the inequality*

$$(1-\varepsilon)\frac{(g-1)\log 2}{g\log g}\log n < \log \tau_g(n) < (1+\varepsilon)\frac{(g-1)\log 2}{2\log g}\log n$$

holds for almost all positive integers n.

Proof. This follows from Propositions 3. 2. and 4. 2. and the following obvious relation:

$$2^{\omega_g(n)} \leqq \tau_g(n) \leqq 2^{\Omega_g(n)}, \quad n \geqq 0 .$$

7. The function $\sigma_g(n)$

Proposition 7.1. $\sigma_g(n) = \frac{1}{2} n \tau_g(n), \quad n \geqq 1 .$

Proof. By induction on the value of $\omega_g(n)$. If $\omega_g(n) = 1$; that is, $n = ag^i$ for some a, $1 \leqq a \leqq g-1$, and $j \geqq 0$, then

$$\sigma_g(n) = \sum_{j=1}^{a} j\, g^i = \frac{1}{2} a(a+1)g^i = \frac{1}{2} n\tau_g(n)$$

(using Proposition 6. 1). Let $v \geqq 1$ be an integer. Suppose that the formula is true for all positive integer m satisfying $\omega_g(m) = v$. Let n be any integer such that $\omega_g(n) = v+1$. Then n can be written in the form

$$n = \sum_{i=1}^{v+1} b_i\, g^{r_i}$$

where $1 \leqq b_i \leqq g-1$ and $r_i \neq r_j$ provided $i \neq j$. Hence by the definition of $\sigma_g(n)$ we have

$$\sigma_g(n) = \sum_{l_1=0}^{b_1} \sum_{l_2=0}^{b_2} \cdots \sum_{l_{v+1}=0}^{b_{v+1}} \sum_{i=0}^{v+1} l_i g^{r_i}$$

$$= \sum_{l_1=0}^{b_1} (\tau_g(m)\, l_1 g^{r_1} + \sigma_g(m)),\quad m = \sum_{i=2}^{v+1} b_i g^{r_i}$$

$$= \frac{1}{2} b_1(b_1+1)\tau_g(m) g^{r_1} + (b_1+1)\frac{1}{2} m\, \tau_g(m)$$

$$= \frac{1}{2} b_1 g^{r_1}\tau_g(n) + \frac{1}{2} m\, \tau_g(n) = \frac{1}{2} n\, \tau_g(n).$$

By (6. 1) and the preceding proposition, we have

$$\limsup_{n\to\infty} \frac{\sigma_g(n)}{n^2} = \frac{1}{2} \text{ and } \liminf_{n\to\infty} \frac{\sigma_g(n)}{n} = 1.$$

Proposition 7.2.

$$\log \sum_{0<n<x} \sigma_g(n) = \Big(\frac{\log \frac{1}{2} g(g+1)}{\log g} + 1\Big) x + O(1).$$

Proof. By partial summation (using Propositions 6. 2 and 7. 1).

Proposition 7.3. *For any given nnmber* $\varepsilon > 0$ *we have*

$$(1-\varepsilon)\Big(\frac{(g-1)\log 2}{g\log g} + 1\Big)\log n$$
$$< \log \sigma_g(n) < (1+\varepsilon)\Big(\frac{(g-1)\log 2}{2\log g} + 1\Big)\log n$$

for almost all positive integers n.

Proof. This follows at once from Propositions 6. 3. and 7. 1.

8. *g*-adical analogue of the zeta function

As is widely recognized, the zeta function of Riemann plays substantial and important roles in analytic number theory, so it will be of some interest to make a *g*-adical analogue of this function. Along this line we shall treat power series as the generating functions of some arithmetical functions introduced in section 1. The *g*-adical zeta function is defined by the power series

$$\zeta_g(s) = \sum_{n=0}^{\infty} s^n = \frac{1}{1-s}, \quad |s| < 1.$$

Then from Proposition 1. 2. we have

$$\zeta_g(s) = \sum_{n=0}^{\infty} \frac{1 - s^{g^{k+1}}}{1-s^{g^k}}$$

which may be compared to the Euler factorization of the ordinary zeta function. (This fact was also pointed out in [1] for the special case of $g = 2$). Consider two convergent power series

$$f(s) = \sum_{n=0}^{\infty} a_n s^n \quad \text{and} \quad h(s) = \sum_{n=0}^{\infty} b_n s^n$$

where, to simplify the discussion we assume, $a_n \geqq 0$ and $b_n \geqq 0$. We define the product $f(s) * h(s)$ as follows;

$$f(s) * h(s) = \sum_{n=0}^{\infty} C_n^* s^n$$

$$\text{where} \quad C_n^* = \sum_{d|_g n} a_d b_{n-d}. \quad \text{Since}$$

$$a_0 b_n + a_n b_1 \leqq C_n^* \leqq \sum_{0 \leqq d \leqq n} a_d b_{n-d}$$

the convergence of the product series $f(s) * h(s)$ is assured. Taking account of this fact and (3. 1), Propositions 6. 1 and 7. 1, we can consider the corresponding generating functions and their product whenever $|s| < 1$. Now we shall exhibit several identities. The first is

$$\zeta_g(s) * \sum_{n=0}^{\infty} \mu_g(n) s^n = 1, \tag{8. 1}$$

which is obvious from Proposition 2. 1. Next, by the definition of $\tau_g(n)$ and $\sigma_g(n)$

$$\zeta_g(s) * \zeta_g(s) = \sum_{n=0}^{\infty} \tau_g(n) s^n, \tag{8. 2}$$

$$\zeta_g(s) * s\,\zeta_g'(s) = \sum_{n=0}^{\infty} \sigma_g(n) s^n. \tag{8. 3}$$

Put

$$\Lambda_g(n) = \begin{cases} g^i & \text{if } n = j\,g^i \text{ for some } j\ (0 < j < g) \text{ and } i \geqq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$\sum_{d|_g n} \Lambda_g(d) = n.$$

and so we have

$$(8.4)\qquad \zeta_g(s) * \sum_{n=0}^{\infty} \Lambda_g(n)\, s^n = s\zeta_g'(s).$$

By (8.2), (8.3) and (8.4) we find

$$(8.5)\qquad \sum_{n=0}^{\infty} \tau_g(n)s^n * \sum_{n=0}^{\infty} \Lambda_g(n)s^n = \sum_{n=0}^{\infty} \sigma_g(n)\, s^n.$$

The product $*$ being, by definition, associative, we have

$$(8.6)\qquad \zeta_g(s) * \sum_{n=0}^{\infty} \Omega_g\,(n)s^n = \frac{1}{2}\sum_{n=0}^{\infty} \tau_g(n)\,\Omega_g\,(n)\, s^n.$$

Indeed $\sum_{d|_g n} \Omega_g(n)$

$$= \sum_{0\leq m_1 \leq a_1(n)} \cdots \sum_{0\leq m_k \leq a_k(n)} \sum_{i=1}^{k(n)} m_i$$

$$= \sum_{i=1}^{k(n)} \prod_{j=i} (a_i\,(n)+1) \sum_{0\leq m_i \leq a_i(n)} m_i$$

$$= \frac{1}{2} \prod_{i-1}^{k(n)} (a_i\,(n)+1) \sum_{i=1}^{k(n)} a_i\,(n) = \tau_g(n)\,\sigma_g(n).$$

Also

$$(8.7)\qquad \zeta_g(s) * \sum_{n=0}^{\infty} |\,\mu_g(n)\,|\, s^n = \sum_{n=0}^{\infty} 2^{\omega_g(n)}\, s^n.$$

In fact $\sum_{d|_g n} |\,\mu_g(d)\,|$ is the number of such g-adical divisors d of n that $d = \sum_{i=1}^{k} \delta_i\, g^{k-1}$ where

$$\delta_i = \begin{cases} 0 \text{ or } 1 & \text{if } a_i(n) > 0 \\ 0 & \text{if } a_i(n) = 0, \end{cases}$$

and so this sum is equal to $2^{\omega_g(n)}$.

Finally we give two identities which can be readily verified:

$$(8.8)\qquad \prod_{k=0}^{\infty}(1 + t\, s^{g^k} + t^2\, s^{2g^k} + \cdots + t^{g-1}\, s^{(g-1)g^k}) = \sum_{n=0}^{\infty} t^{\Omega_g(n)}\, s^n,$$

$$(8.9)\qquad \prod_{k=0}^{\infty} (1+t\, s^{g^k} + t\, s^{2g^k} + \cdots + t\, s^{(g-1)g^k}) = \sum_{n=0}^{\infty} t^{\omega_g(n)}\, s^n$$

where t is any fixed positive number.

Remark. After we had completed this research the paper of G. C. Rota [4] came into our attention, in which he generalized the Möbius function μ to some partially ordered sets and obtained the corresponding inversion formula. Some of our settings and results (the definition of $\mu_0(n)$ and the product $*$ of power series and the inversion formula) are particular cases of Rota [4].

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DEPARTMENT OF THE FOUNDATIONS OF
MATHEMATICAL SCIENCES,
TOKYO UNIVERSITY OF EDUCATION, TOKYO

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