ON ISOMORPHISMS OF WEAKLY GALOIS EXTENSIONS

MIGUEL A. FERRERO

In [5], O. E. Villamayor and D. Zelinsky developed a Galois theory for weakly Galois extensions over a commutative ring, which generalizes the Galois theory of S. U. Chase, D. K. Harrison and A. Rosenberg [1] (in this paper a Galois extension in the sense of [1] is called a strongly Galois extension). The purpose of this paper is to give a generalization of [1 ; Theorem 3. 4] and some related results. The author wishes to express his hearty thanks to Professor O. E. Villamayor for several suggetions.

Throughout the paper, we suppose that every ring has an identity element, every extension of a ring has the same identity element, and that every module is unital. Let $S \supset R$ be commutative rings ($S \supset R$ will mean always a ring extension), and $G^* = G(S/R)$ the group of all R-algebra automorphisms of S. As in [5], $B(R)$ will represent the Boolean ring of R which consists of all idempotents in R, and Spec $B(R)$ the spectrum space of $B(R)$ consisting of all prime ideals in $B(R)$. For any $x \in$ Spec $B(R)$, we consider $S_x = T^{-1}S = S/xS$ and $R_x = T^{-1}R = R/xR$, where $T = \{1 - e\,;\ e \in x\}$ (see [5 ; (2. 6)]).

By [5] S is said to be a *weakly Galois extension* of R, if the following conditions are satisfied :

(a) S is an f. g. (finitely generated) faithfully projective R-module.

(b) S is a separable R-algebra (abbr. R-separable).

(c) $\rho(S)G^* = \mathrm{Hom}_R(S, S)$, where $\rho : S \longrightarrow \mathrm{Hom}_R(S, S)$ is the regular representation of S defined by $\rho(s)(t) = st$.

The following remark is due to K. Oshiro :

Remark 1. In the definition of a weakly Galois extension, (a) and (c) imply (b).

Proof. As is well-known, S is R-separable if and only if $S/\mathfrak{m}S$ is $R/\mathfrak{m}$-separable for every maximal ideal $\mathfrak{m}$ of R. Let G' be the group of all $R/\mathfrak{m}$-algebra automorphisms of $S/\mathfrak{m}S$, and $\overline{G}$ the subgroup of G' consisting of the automorphisms induced by elements of G^*. Then, by (c) we have $\rho(S)\otimes_R R/\mathfrak{m} = \mathrm{Hom}_R(S, S)\otimes_R R/\mathfrak{m} = \mathrm{Hom}_{R/\mathfrak{m}}(S/\mathfrak{m}S, S/\mathfrak{m}S)$ for every maximal ideal $\mathfrak{m}$ of R, since S is an f. g. projective R-module. Therefore, we have $\overline{\rho}(S/\mathfrak{m}S)\,G' \subset \mathrm{Hom}_{R/\mathfrak{m}}(S/\mathfrak{m}S,\ S/\mathfrak{m}S) = \overline{\rho}(S/\mathfrak{m}S)\overline{G} \subset \overline{\rho}(S/\mathfrak{m}S)G'$, where $\overline{\rho}$ is

the regular representation of $S/\mathfrak{m}S$ over $R/\mathfrak{m}$. Henceforth, we may assume that R is a field. Then, S is semi-simple. In fact, if the Jacobson radical J of S were non-zero, $\rho(J)G^*$ would be a non-zero nilpotent ideal of the simple ring $\mathrm{Hom}_R(S, S)$. Let $S=Se_1\oplus\cdots\oplus Se_n$, where $e_1, \cdots, e_n$ are primitive orthogonal idempotents. From (a) and (c) we have $S^{G^*}=R$ and $(Se_i)^{H_i}=Re_i$ for $H_i=\{\sigma\in G^*;\ \sigma(e_i)=e_i\}$ $(i=1, \cdots, n)$. It follows then the field Se_i is a strongly Galois extension of Re_i, and so R-separable. Therefore, S is R-separable.

Lemma 1. *Let x be in* Spec $B(R)$. *If $f_1, \cdots, f_n$ are orthogonal idempotents (especially such that $\sum_i f_i=1$) in S_x, then there exist some orthogonal idempotents $e_1, \cdots, e_n$ in S such that $(e_i)_x=f_i (i=1, \cdots, n)$, where $(e_i)_x$ is the image of e_i by the canonical homomorphism $S\longrightarrow S_x$. If $\sum_i f_i=1$ then $\sum_i e_i=1$.*

Proof. We shall proceed by the induction on n. The case $n=1$ is known by [5; (2.12)]. Now, let $f_1, \cdots, f_n$ be orthogonal idempotents in S_x. Then, by the induction hypothesis there exist orthogonal idempotents $e_1, \cdots, e_{n-1}$ in S such that $(e_i)_x=f_i$ $(i=1, \cdots, n-1)$. Furthermore, by [5; (2.12)] there exists $e_n'\in S$ such that $(e_n')_x=f_n$. Since $((e_1+\cdots+e_{n-1})e_n')_x=(f_1+\cdots+f_{n-1})f_n=0$, by [5; (2.9)] we have $(e_1+\cdots+e_{n-1})e_n'(1-e)=0$ for some $e\in x$. Putting $e_n=e_n'(1-e)$, $e_1, \cdots, e_n$ are orthogonal idempotents in S and $(e_i)_x=f_i$ $(i=1, \cdots, n-1)$, $(e_n)_x=(e_n'(1-x))_x=(e_n')_x=f_n$.

Lemma 2. *Let $S\supset R$, and x in* Spec $B(R)$. *Let $f_1, \cdots, f_n$ be orthogonal idempotents in S_x, $\tau_1, \cdots, \tau_n$ elements of G^*, and σ an element of $G(S_x/R_x)$ such that $\sigma|(S_xf_i)=(\tau_i)_x|(S_xf_i)$ $(i=1, \cdots, n)$, where $(\tau_i)_x$ is the automorphism of S_x induced by τ_i. Then, $\tau_x=\sigma$ for some $\tau\in G^*$.*

Proof. By Lemma 1, there exist orthogonal idempotents $e_1', \cdots, e_n'\in S$ such that $(e_i')_x=f_i$ $(i=1, \cdots, n)$. By making use of [5; (2.9) and (2.12)], we can choose inductively orthogonal idempotents $e_1'', \cdots, e_p''\in S$ $(p\leqq n)$ such that $e_1'', \cdots, e_p'', e_{p+1}', \cdots, e_n'$ are orthogonal, $(e_i'')_x=f_i$ $(i=1, \cdots, p)$, and $\tau_1(e_1''), \cdots, \tau_p(e_p'')$ are orthogonal. Especially, we can find orthogonal idempotents $e_1'', \cdots, e_n''\in S$ such that $\tau_1(e_1''), \cdots, \tau_n(e_n'')$ are orthogonal and $(e_i'')_x=f_i$ $(i=1, \cdots, n)$. Since $(\sum_i e_i'')_x=\sum_i f_i=1=\sum_i \sigma(f_i)=(\sum_i \tau_i(e_i''))_x$, there exists some $e\in x$ such that $\sum_i e_i''(1-e)=\sum_i \tau_i(e_i'')(1-e)=\sum_i \tau_i(e_i''(1-e))$. If we put $e_i=e_i''(1-e)$ $(i=1, \cdots, n)$ and $e_{n+1}=1-\sum_{i=1}^n e_i$, then $e_1, \cdots, e_{n+1}$ are orthogonal idempotents in S such that $(e_i)_x=f_i$ $(i=1, \cdots, n)$, $(e_{n+1})_x=0$, and that $\tau_1(e_1), \cdots, \tau_n(e_n)$, e_{n+1} are orthogonal idempotents with $\tau_1(e_1)+\cdots+\tau_n(e_n)+e_{n+1}=1$. Then we can define an automorphsm $\tau\in G^*$ by $\tau(s)=\sum_{i=1}^n \tau_i(se_i)+se_{n+1}$, which satisfies $\tau_x=\sigma$.

Lemma 3. *Let S be a weakly Galois extension of R. Let G be a subgroup of G^* such that for every $x \in \mathrm{Spec}\, B(R)$ there exists a subgroup $H(x)$ of G satisfying $S_x^{H(x)_x} = R_x$, where $H(x)_x = \{\sigma_x ;\ \sigma \in H(x)\}$. Then, there exists a finite subgroup H of G which verifies $S^H = R$.*

Proof. From [4; Proposition 1.3], $G(S_x/R_x)$ is a finite group. By [5; (2.14)] we may assume that $H(x)$ is a finte group. As in [5; (3.6)], we can choose here $x_1, \cdots, x_n \in \mathrm{Spec}\, B(R)$ such that there exist respective neighborhoods $V(x_i)$ of x_i verifying $S_y^{H(x_i)_y} = R_y$ for all $y \in V(x_i)$ $(i = 1, \cdots, n)$ and $\bigcup_i V(x_i) = \mathrm{Spec}\, B(R)$. Let H be the subgroup of G^* generated by $\bigcup_i H(x_i)$. Then, by [5; Theorem (2.16)], H satisfies the conditions of our lemma.

Let S be a weakly Galois extension of R, and x in Spec $B(R)$. Then, S_x is a weakly Galois extension of R_x by [5; (3.2)]. Since R_x has no idempotents other than 0 and 1, by [5; (3.15)] S_x is a strongly Galois extension of R_x and its Galois group is a subgroup of $G(S_x/R_x)$. For a subgroup G of G^*, we say that G verifies (I) if it satisfies the following condition:

(I) Every S_x $(x \in \mathrm{Spec}\, B(R))$ is a strongly Galois extension of R_x with a Galois group $H(x) \subset G_x$.

The following corollaries are easily obtained:

Corollary 1. *If S is a weakly Galois extension of R and $G \subset G^*$ satisfies* (I), *then there exists a finite subgroup H of G which verifies* (I) and $S^H = R$.

Corollary 2. *If S is a weakly Galois extension of R, then there is a finite subgroup G of G^* which satisfies* (I) *and* $S^G = R$.

Proposition 1. *Let S be a weakly Galois extension of R, and $S' \supset R'$ commutative rings. Let $f: S \longrightarrow S'$ be a ring-homomorphism which induces a monomorphism $f|R: R \longrightarrow R'$, and G a subgroup of G^* verifying* (I). *If for each $\sigma \in G$ there exists some $\sigma' \in G(S'/R')$ such that $\sigma' f = f\sigma$ then f is a monomorphism. Furthermore, if there exists a group-homomorphism $\psi: G \longrightarrow G(S'/R')$ such that $\psi(\sigma) f = f\sigma$ for all $\sigma \in G$, then f and ψ are monomorphisms.*

Proof. If R has no idempotents other than 0 and 1, then S is a strongly Galois extension with a Galois group $H \subset G$ by the condition (I). There exist $s_1, \cdots, s_n,\ t_1, \cdots, t_n \in S$ such that $\sum_i s_i\, \sigma(t_i) = \begin{cases} 1 & \text{if } \sigma = I \\ 0 & \text{if } \sigma \neq I \end{cases}$ $(\sigma \in H)$. If $f(s) = 0$ for some $s \in S$, then $f(\mathrm{tr}_H(t_i s)) = \sum_{\sigma \in H} f(\sigma(t_i s)) = \sum_{\sigma'} \sigma'(f(t_i s))$

$=\sum_{\sigma'} \sigma'(f(t_i)\, f(s))=0$, and by $\mathrm{tr}_H(t_i\, s)\in S^H=R$ we have $\mathrm{tr}_H(t_i s)=0$, since $f|R$ is monic. Accordingly, $s=\sum_{\sigma\in H}\sum_i s_i\,\sigma(t_i)\,\sigma(s)=\sum_i s_i\,\mathrm{tr}_H(t_i s)=0$. In the general case, it suffices to prove the same for the localizations S_x, R_x, $S'_x=S'\otimes_R R_x$, $R'_x=R'\otimes_R R_x$ and $f_x: S_x\longrightarrow S'_x$ by arbitrary $x\in \mathrm{Spec}\, B(R)$. But, R_x has no idempotents other than 0 and 1, therefore the first part is obtained. Next, let $\psi: G\longrightarrow G(S'/R')$ be a group-homomorphism with $\psi(\sigma)\, f=f\sigma$ for all $\sigma\in G$. If $\psi(\sigma)=I$ for some $\sigma\in G$ then $f(\sigma(s))=\psi(\sigma)(f(s))=f(s)$, and so $\sigma(s)=s$ for every $s\in S$. Therefore, ψ is a monomorphism.

Proposition 2. *Let S be a strongly Galois extension of R with Galois group H, and $S'\supset R'$ commutative rings. Let $f: S\longrightarrow S'$ be a ring-homomorphism which induces an epimorphism $f|R: R\longrightarrow R'$, and G' a finite subgroup of $G(S'/R')$ with $S'^{G'}=R'$. If for each $\sigma'\in G'$ there corresponds some $\sigma\in H$ such that $\sigma' f=f\sigma$ and $\sigma'\neq I$ implies $\sigma\neq I$, then f is an epimorphism and S' is a strongly Galois extension of R' with Galois group G'.*

Proof. Let $x_1,\cdots, x_n, y_1,\cdots, y_n$ be elements of S such that $\sum_i x_i\,\tau(y_i)=\begin{cases}1 & \text{if } \tau=I\\ 0 & \text{if } \tau\neq I\end{cases}$ $(\tau\in H)$. We have then $\sum_i f(x_i)\sigma'(f(y_i))=\sum_i f(x_i)\, f(\sigma(y_i))=f(\sum_i x_i\,\sigma(y_i))=\begin{cases}1 & \text{if } \sigma=I\\ 0 & \text{if } \sigma\neq I\end{cases}$ In particular, we see that $\sigma=I$ if and only if $\sigma'=I$. Accordingly, it is obtained that $\sum_i f(x_i)\sigma'(f(y_i))=\begin{cases}1 & \text{if } \sigma'=I\\ 0 & \text{if } \sigma'\neq I\end{cases}$ $(\sigma'\in G')$. This shows that S' is a strongly Galois extension of R' with Galois group G'. Furthermore, for any $x'\in S'$ we have $\sum_{\sigma'\in G'}\sigma'(f(y_i)x')\in R'$ $(i=1,\cdots, n)$. By the assumption on f, there exists some $r_i\in R$ such that $f(r_i)=\sum_{\sigma'\in G'}\sigma'(f(y_i)x')$. Hence, we have $f(\sum_i x_i r_i)=\sum_{\sigma'\in G'}\sum_i f(x_i)\,\sigma'(f(y_i))\sigma'(x')=x'$, that is, f is an epimorphism.

As a combination of Propositions 1 and 2, we readily obtain

Corollary 3. *Under the same assumption as in Proposition 2, if $\psi: H\longrightarrow G(S'/R')$ is a group homomorphism such that* Im $(\psi)\supset G'$ *and $\psi(\sigma)f=f\sigma$ for all $\sigma\in H$, then f is an epimorphism, ψ a monomorphism, and S' a strongly Galois extension of R' with Galois group G'. Furthermore, if $f|R$ is an isomorphism then f is an isomorphism.*

Now, let $S\supset R$ be rings such that S is an f. g. R-module, and G a subgroup of G^* such that $S^G=R$. If R has no idempotent other than 0 and 1, then there exist orthogonal primitive idempotents $e_1,\cdots, e_n\in S$ such that $1=\sum_i e_i$. We put $H_i=\{\sigma\in G;\ e_i=\sigma(e_i)\}$ $(i=1,\cdots, n)$. Then, there

exist $\sigma_1, \cdots, \sigma_n \in G$ such that $\sigma_i(e_1)=e_i$ $(i=1, \cdots, n)$. Since $G=\bigcup_i \sigma_i H$ and $H_i=\sigma_i H_1 \sigma_i^{-1}$, we have $(Se_i)^{H_i}=Re_i$. Furthermore, if S is a weakly Galois extension of R, then Se_i is a strongly Galois extension of Re_i with Galois group $H_i|Se_i=G(Se_i/Re_i)$ $(i=1, \cdots, n)$. Then, we have the following remark.

Remark 2. Let S be a weakly Galois extension of R. If S' is a commutative ring and $f: S \longrightarrow S'$ is a non-zero ring-homomorphism such that for each $\sigma \in G$ there exists an endomorphism σ' of S satisfying $f\sigma=\sigma' f$, then $f(e_1), \cdots, f(e_n)$ are non-zero orthogonal idempotents in S'.

Proof. If $f(e_i)=0$, then $f(e_j)=f(\sigma_j\sigma_i^{-1}(e_i))=(\sigma_j\sigma_i^{-1})'f(e_i)=0$, and so $f(1)=\sum_j f(e_j)=0$. Since f is a ring homomorphism, we obtain $f=0$, a contradiction.

We give here the following definition that is analogous to the one in [4]: Let $S \supset R$ be commutative rings, G a subgroup of G^* and $e_1, \cdots, e_n$ orthogonal idempotents in S with $\sum_i e_i=1$. We say that G is *fat* respect to $e_1, \cdots, e_n$, if every element $\sigma \in G^*$ verifying $\sigma|Se_i \in G|Se_i$ is contained in G.

Proposition 3. *Let S be a weakly Galois extension of R such that $S^G=R$ for a subgroup G of G^*. Suppose that R has no idempotents other than* 0 *and* 1 *and $e_1, \cdots, e_n$ are orthogonal primitive idempotents in S such that $\sum_i e_i=1$. Let $S' \supset R'$ be commutative rings such that $S'^{G'}=R'$ for a finite subgroup G' of $G(S'/R')$, and $f: S \longrightarrow S'$ a ring-homomorphism which induces an epimorphism $f|R: R \longrightarrow R'$. If* (a) *$G'$ is fat respect to $f(e_1), \cdots, f(e_n)$,* (b) *for each $\sigma \in G$ there exists some $\sigma' \in G$ such that $\sigma' f=f\sigma$, and* (c) *for each $\tau' \in G'$ there corresponds some $\tau \in G$ such that $\tau' f=f\tau$ and $\tau' \neq I$ implies $\tau \neq I$, then f is an epimorphism and S' is a weakly Galois extension of R'. Furthermore, if G verifies* (I) *and $f|R$ is an isomorphism, then f is an isomorphism.*

Proof. As is well-known, Se_i is strongly Galois over Re_i with Galois group $G_i=\{\sigma|Se_i;\ \sigma \in G,\ \sigma(e_i)=e_i\}$. Since $\sigma_i(e_1)=e_i$ for some $\sigma_i \in G$, there is some $\sigma_i' \in G'$ such that $\sigma_i'(f(e_i))=f(\sigma_i(e_1))=f(e_i)$ $(i=1, \cdots, n)$. Noting that $S'^{G'}=R'$ and $\sum_i f(e_i)=1$, the subgroup $G_i'=\{\sigma'|S'f(e_i);\ \sigma' \in G',\ \sigma'(f(e_i))=f(e_i)\}$ of $G(S'f(e_i)/R'f(e_i))$ satisfies $(S'f(e_i))^{G_i'}=R'f(e_i)$ $(i=1, \cdots, n)$. For each $\sigma'|S'f(e_i) \in G_i'$, by (c) we can find some $\sigma \in G$ such that $\sigma' f=f\sigma$. Then, $\sigma|Se_i$ is in G_i. Because, $f(e_i)=\sigma'(f(e_i))=f(\sigma(e_i))$ and $\sigma(e_i)$ is one of $e_1, \cdots, e_n$. However, by Remark 2 we have $0 \neq f(e_i)=f(e_i)f(e_i)=f(\sigma(e_i))f(e_i)=f(\sigma(e_i)e_i)$, therefore $\sigma(e_i)e_i \neq 0$, and so $\sigma(e_i)=e_i$. Next,

if $\sigma'|S'f(e_i)\neq I$ then $\sigma|Se_i\neq I$. To see this, we consider $\tau'\in G(S'/R')$ defined by $\tau'|S'f(e_i)=\sigma'|S'f(e_i)$ and $\tau'|S'f(e_j)=I$ $(j\neq i)$. Then, by (a) τ' is in G', and there exists some $\tau\in G$ such that $fI=\tau'f=f\tau$. Since $S\supset R$ is strongly Galois with Galois group G_0, there exist $u_1, \cdots, u_m, v_1, \cdots, v_m\in S$ such that $\sum_i \bar{\tau}(u_i)v_i=\begin{cases}1 & \text{if } \bar{\tau}=I\\ 0 & \text{if } \bar{\tau}\neq I\end{cases}$ $(\bar{\tau}\in G_0)$. As is well-known, $\tau=\sum_{\bar{\tau}\in G_0} e_{\bar{\tau}}\bar{\tau}$ with pairwise orthogonal idempotents $e_{\bar{\tau}}\in S$ such that $\sum_{\bar{\tau}} e_{\bar{\tau}}=1$. Noting that $f(\tau(x)y)=f(\tau(x))f(y)=f(x)f(y)=f(xy)$ for every $x, y\in S$, we obtain $f(\sum_{\bar{\tau}} e_{\bar{\tau}})=1=f(1)=\sum_i f(u_iv_i)=\sum_i f(\tau(u_i)v_i)=\sum_{i,\bar{\tau}} f(e_{\bar{\tau}}\bar{\tau}(u_i)v_i)=f(e_I)$, whence by Remark 2 it follows $e_I=1$ and $e_{\bar{\tau}}=0$ $(\bar{\tau}\neq I)$, which implies $\tau=I$. Therefore, by (c) $\tau'=I$, and of course $\tau'|S'f(e_i)=\tau'|S'f(e_i)\tau=I$, which is a contradiction. Accordingly, by Proposition 2 the ring-homomorphism $f|Se_i: Se_i\longrightarrow S'f(e_i)$ is an epimorphism and $S'f(e_i)$ is a Galois extension of $R'f(e_i)$, and therefore by [5; (3.15)], S' is a weakly Galois extension of R'. The remaining is clear by Proposition 1.

From the last proposition we readily obtain the following:

Corollary 4. *Under the same assumption as in Proposition* 3, *if* $\psi: G\longrightarrow G'$ *is a group-epimorphism such that* $\psi(\sigma)f=f\sigma$ *for all* $\sigma\in G$ *and if* G' *is fat respect to* $f(e_1), \cdots, f(e_n)$, *then* f *is an epimorphism. Furthermore, if* G *verifies* (I) *and* $f|R$ *is an isomorphism, then* f *and* ψ *are isomorphisms.*

Now, let $S\supset R$ and $S'\supset R'$ be commutative rings. Suppose that S is a weakly Galois extension of R and $f: S\longrightarrow S'$ is a ring-homomorphism such that $f|R$ is an isomorphism of R to R'. Then, for each $x\in\mathrm{Spec}\ B(R)=\mathrm{Spec}\ B(R')$ there exist orthogonal primitive idempotents $e(x)_1, \cdots, e(x)_{n_x}\in S_x$ with $\sum_i e(x)_i=1$, since R_x has no idempotents other than 0 and 1. For a subgroup G' of $G(S'/R')$, we say that G' verifies (II), if the following conditions are satisfied:

(II_1) Every $\sigma'\in G(S'/R')$ verifying $\sigma'|S'e'=\tau_1'|S'e'$ and $\sigma'|S'(1-e')=\tau_2'|S'(1-e')$ for some $\tau_1', \tau_2'\in G'$ and some idempotent $e'\in S'$ is in G'.

(II_2) For every $x\in\mathrm{Spec}\ B(R)$, G_x' is fat respect to $f(e(x)_1), \cdots, f_x(e(x)_{n_x})$.

We claim here that $G(S'/R')$ verifies (II_2) (Lemma 2), and hence (II).

Theorem 1. *Let* $S\supset R$ *and* $S'\supset R'$ *be commutative rings, and* $f: S\longrightarrow S'$ *a ring-homomorphism. Suppose that* S *is a weakly Galois extension of* R *and* f *induces an isomorphism* $f|R: R\longrightarrow R'$. *If* 1) *a subgroup* G *of* G^* *verifies* (I), 2) *a subgroup* G' *of* $G(S'/R')$ *verifies* (II), 3) *there is a finite subgroup* H' *of* G' *such that* $S'^{H'}=R'$, *and* 4) *there is a group-homomorphism* $\psi: G\longrightarrow G'$ *such that* $\psi(\sigma)f=f\sigma$ *for all* $\sigma\in G$, *then* f *is an isomorphism,* S' *is a weakly Galois extension of* R',

and ψ is an isomorphism. Furthermore, if $G' = G(S'/R')$ then the condition 2) is necessarily verified and G coincides with G^*.

Proof. From Proposition 1, f is a monomorphism and ψ is (a monomorphism and so) an isomorphism. For every $x \in \operatorname{Spec} B(R)$, S_x is a weakly Galois extension of R_x and R_x has no idempotents other than 0 and 1. By [5; (2.17)], we have $S_x^{G_x} = R_x$ and $S'^{G'_x}_x = R'_x$. Obviously, f_x: $S_x \longrightarrow S'_x$ is a ring-homomorphism and induces an isomorphism $f_x | R_x$: $R_x \longrightarrow R'_x$. Now, we shall show that the group-homomorphism ψ: $G \longrightarrow G'$ induces a group-homomorphism ψ_x: $G_x \longrightarrow G'_x$. To our end, it suffices to show that $\sigma_x \longmapsto (\psi(\sigma))_x$ is well-defined. We suppose $\sigma_x = I$. Then, for each $s \in S$ we have $(\sigma(s) - s)(1 - e_s) = 0$ for some $e_s \in x$. Since S is an f. g. R-module, we can find some $e \in x$ such that $(\sigma(s) - s)(1 - e) = 0$ for all $s \in S$. And then, $(\psi(\sigma)(f(s)) - f(s))(1 - f(e)) = 0$ for all $s \in S$. We can define now an R'-algebra-automorphism σ' of S' by $\sigma'(s') = s'f(1-e) + \psi(\sigma)(s')f(e)$. Since $\sigma' | S'f(1-e) = I | S'f(1-e)$ and $\sigma' | S'f(e) = \psi(\sigma) | S'f(e)$, σ' is in G' by the condition (II). If $\rho \in G$ is such that $\sigma' = \psi(\rho)$, then we have $f(\rho(s)) = \sigma'(f(s)) = f(s)f(1-e) + \psi(\sigma)(f(s))f(e) = \psi(\sigma)(f(s))f(1-e) - (\psi(\sigma)(f(s)) - f(s))$ and $(1 - f(e)) + \psi(\sigma)(f(s))f(e) = \psi(\sigma)(f(s))f(1-e) + \psi(\sigma)(f(s))f(e) = \psi(\sigma)f(s) = f(\sigma(s))$ for all $s \in S$. But, f being a monomorphism, $\rho = \sigma$. Since $e_x = 0$, it follows that $(\psi(\sigma))_x = (\psi(\rho))_x = \sigma'_x$ and $\sigma'_x(S'_x) = (\sigma'(s'))_x = s'_x f_x((1-e)_x) + (\psi(\sigma))_x(s'_x)f_x(e_x) = s'_x(f(1-e))_x = s'_x$ for all $s' \in S'$. Therefore, we have $(\psi(\sigma))_x = \sigma'_x = I$, i. e., $\sigma_x \longmapsto (\psi(\sigma))_x$ is well-defined. Since ψ is an epimorphism, ψ_x is a group-epimorphism such that $\psi_x(\rho)f_x = f_x\rho$ for all $\rho \in G_x$. From the condition (II) and Corollary 4, it follows then that f_x is an epimorphism for every $x \in \operatorname{Spec} B(R)$, and so f is an epimorphism. Finally, if $G' = G(S'/R')$ then G' verifies (II) and $G = G^*$. In fact, given an arbitrary $\sigma \in G^*$, $\sigma' = f\sigma f^{-1}$ is in G' and there exists some $\gamma \in G$ such that $\psi(\gamma) = \sigma'$ and $\psi(\gamma)f = f\gamma$, that is, $\sigma = f^{-1}\sigma' f = \gamma \in G$.

Remark 3. In the proof of Theorem 1, if we assume that S' is a weakly Galois extension of R' then the fact that f is an epimorphism will be seen very easily: Since S is R-separable, $f(S)$ is separable over $R' = f(R)$. By [5; (3.8)], there is a finite subgroup F' of G' such that $f(S) = S'^{F'}$. If we set $F = \psi^{-1}(F')$ then $f(S^F) = (f(S))^{F'} = f(S)$. Recalling that f is a monomorphism, we obtain $S^F = S$, whence $F = I$, $F' = I$ and $f(S) = S'$.

Proposition 4. *Let S be a weakly Galois extension of R, and A a commutative R-algebra. Then, $S \otimes_R A$ is a weakly Galois extension of A. If all the idempotents of A are contained in R, then the subgroup $G^* \otimes I$ of $G(S \otimes_R A/A)$ verifies* (I).

Proof. If R has no idempotents other than 0 and 1, S is a strongly Galois extension of R with Galois group $H \subset G^*$. And, by [1; Lemma 7], $S \otimes_R A$ is a strongly Galois extension of A with Galois group $H \otimes I \cong H$. Therefore, $G^* \otimes I$ verifies (I). If R is general, then by Corollary 2 we can choose a finite subgroup G of G^* such that G verifies (I). For each $x \in \operatorname{Spec} B(R)$, S_x is a strongly Galois extension of R_x with Galois group $H(x) \subset G_x$. From the first part of this proof, we have $(S_x \otimes_{R_x} A_x)^{G_x \otimes I} = A_x$, and so $A_x \subset ((S \otimes_R A)^{G \otimes I})_x = (S_x \otimes_{R_x} A_x)^{G_x \otimes I} \subset (S_x \otimes_{R_x} A_x)^{H(x) \otimes I} = A_x$ for every $x \in \operatorname{Spec} B(R)$. Therefore, we have $A = (S \otimes_R A)^G$ and $S \otimes_R A$ is a weakly Galois extension of A by [5; (3.6)]. Furthermore, if all the idempotents of A are contained in R then $B(A) = B(R)$, and so $G^* \otimes I$ verifies (I).

Finally, we consider commutative rings $A \supset B$ and $A \supset S$. Suppose that S is $G(A/B)$-invariant, i.e., $\sigma(S) = S$ for all $\sigma \in G(A/B)$, and that $S^{G(A/B)} = R$. We denote by $\varphi : S \otimes_R B \longrightarrow A$ the contraction homomorphism defined by $\varphi(x \otimes y) = xy$. Then, we consider the following conditions:

(A) Every R-algebra-automorphism of S can be uniquely extended to a B-algebra-automorphism of A.

(B) Every R-algebra-automorphism of S can be extended to a B-algebra-autorphism of A and $G(A/\mathrm{Im}(\varphi)) = I$.

(C) Every R-algebra-automorphism of S can be extended to a B-algebra-automorphism of A and the homomorphism $G(A/B) \longrightarrow G^* : \sigma \longmapsto \sigma | S$ is a monomorphism.

(D) The homomorphism $G(A/B) \longrightarrow G^* : \sigma \longrightarrow \sigma | S$ is an isomorphism.

(E) $\varphi : S \otimes_R B \longrightarrow A$ is an isomorphism.

Lemma 4. *Let $A \supset B$ and $A \supset S$ be as above. Then the conditions* (A), (B), (C) *and* (D) *are equivalent, and* (E) *implies* (A).

Proof. (E)$\Rightarrow$(B), (C)$\Rightarrow$(D) and (D)$\Rightarrow$(A) are obvious, and the proof of (A)$\Rightarrow$(B) and (B)$\Rightarrow$(C) will be completed in an obvious way.

Theorem 2. *Let $A \supset B$ and $A \supset S$ be as above. Suppose that a finite subgroup H of G^* verifies $A^H = B$ and $S^H = R$, all the idempotents of B are contained in R, and that S is a weakly Galois extension of R. Then, the conditions* (A), (B), (C), (D) *and* (E) *are equivalent. Furthermore, if one of the conditions* (A)-(E) *is satisfied then A is a weakly Galois extension of B and the correspondence $G^* \longrightarrow G(S \otimes_R B/B) : \sigma \longmapsto \sigma \otimes I$ is an isomorphism.*

Proof. Suppose the condition (D). Obviously, the contraction homomorphism φ is a B-algebra-homomorphism. By Proposition 4, $S \otimes_R B$ is a

weakly Galois extension of B, and the subgroup $G^* \otimes I$ of $G(S \otimes_R B/B)$ verifies (I). If ψ: $G^* \longrightarrow (A/B)$ is the inverse of the isomorphism given in (D), then we have $\psi(\sigma)\varphi = \varphi\sigma$ for all $\sigma \in G^*$. Accordingly, by Theorem 1 we obtain (E). Thus, by Lemma 4 we have the first part of the theorem. The other part is evident by (D) and (E).

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UNIVERSIDAD DE ROSARIO

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