ON MINIMAL SURACES IN A RIEMANNIAN MANIFOLD OF CONSTANT CURVATURE

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In the present paper, we shall study minimal surfaces in a Riemannian manifold $\hat{M}$ of (non-zero) constant curvature c. When $\hat{M}$ is a sphere $S(1)$ of constant curvature 1, we find the conditions such that the immersion is uniquely determined up to a rigid motion of $S(1)$. In §2, we define the $(n+1)$*-th shape operator* (n*-th torsion operator*) and the n*-th torsion index* (T_n*-index*) for minimal surfaces in $\hat{M}$. In §3, we study complete flat minimal surfaces in $\hat{M}$. When $\hat{M}$ is $S(1)$, we find the conditions such that the immersion is uniquely determined up to a rigid motion of $S(1)$. In §4, we give examples of minimal immersions of the Euclidean plane into a sphere. In the last section, we study compact minimal surfaces of non-negative curvature ($\not\equiv 0$) in $\hat{M}$ and prove that they are generalized Veronese surfaces if they are immersed in $\hat{M}=S(1)$ by the immersions with full torsion.

§ 1. **Preliminaries.** Let $\hat{M}$ be a $(2+\nu)$-dimensional Riemannian manifold of constant curvature c, and M a 2-dimensional Riemannian manifold which is isometrically immersed in $\hat{M}$ by an immersion x: $M \longrightarrow \hat{M}$. The *geodesic codimension* of M in $\hat{M}$ is defined to be the minimum of codimension of M in totally geodesic submanifolds of $\hat{M}$, see [7]. We denote by ∇(resp. $\hat{\nabla}$) the covariant differentiation on M (resp. $\hat{M}$). Then the (*first*) *shape operator* (*second fundamental form*) φ_1 of the immersion is given by $\varphi_1(X, Y)=\hat{\nabla}_XY-\nabla_XY$ for any tangent vector fields X and Y of M. There holds then $\varphi_1(X, Y)=\varphi_1(Y, X)$.

$F(\hat{M})$ and $F(M)$ denote the orthonormal frame bundles over $\hat{M}$ and M, respectively. Let B be the set of all elements $b=(p, e_1, e_2, \cdots, e_{2+\nu}) \in F(\hat{M})$ such that $(p, e_1, e_2)\in F(M)$, identifying $p\in M$ with $x(p)$ and e_i with $dx(e_i)$, $i=1, 2$. Then B is a smooth submanifold of $F(\hat{M})$. Let $\hat{\omega}_A$, $\hat{\omega}_{AB}=-\hat{\omega}_{BA}$, $A, B=1, 2, \cdots, 2+\nu$, be the basic and connection forms of $\hat{M}$ on $F(\hat{M})$ which satisfy the following structure equations

$$(1.1)\qquad \begin{aligned} d\hat{\omega}_A &= \sum_B \hat{\omega}_{AB}\wedge\hat{\omega}_B\,,\\ d\hat{\omega}_{AB} &= \sum_C \hat{\omega}_{AC}\wedge\hat{\omega}_{CB}-c\hat{\omega}_A\wedge\omega_B\,. \end{aligned}$$

In this paper, we use the following convention on the range of indices

$$i,\ j,\ \cdots = 1,\ 2, \qquad \alpha,\ \beta,\ \cdots = 3,\ 4,\ \cdots,\ 2+\nu.$$

Deleting the hats of $\hat{\omega}_A$, $\hat{\omega}_{AB}$ on B, as is well known, we have

$$
\begin{aligned}
&\omega_\alpha = 0\,,\\
&\omega_{i\alpha} = \sum_j h^\alpha_{ij}\omega_j\,, \qquad h^\alpha_{ij} = h^\alpha_{ji}\,,\\
&d\omega_i = \omega_{ij}\wedge\omega_j\,,\ i\neq j\,,\\
(1.2)\qquad &d\omega_{ij} = \sum_k \omega_{ik}\wedge\omega_{kj} - \Omega_{ij}\,,\ \Omega_{ij} = \frac{1}{2}\sum R_{ijkl}\,\omega_k\wedge\omega_l\,,\\
&R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk})\,,\\
&d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma}\wedge\omega_{\gamma\beta} - \Omega_{\alpha\beta} \qquad \Omega_{\alpha\beta} = \frac{1}{2}\sum R_{\alpha\beta ij}\omega_i\wedge\omega_j,\\
&R_{\alpha\beta ij} = \sum_k (h^\alpha_{ik}h^\beta_{jk} - h^\alpha_{jk}h^\beta_{ik})\,.
\end{aligned}
$$

M is said to be *minimal* if its mean curvature vector $\frac{1}{2}\sum_{j,\alpha} h^\alpha_{jj}e_\alpha$ vanishes identically, i. e., if trace $H_\alpha = 0$ for all α, $H_\alpha = (h^\alpha_{ij})$. The *minimal index* at $p\in M$ (m-index$_pM$) is defined to be the dimension of the linear space of all second fundamental forms corresponding to normal vectors at p with vanishing trace. We easily have m-index $_pM \leqq 2$ at every point $p\in M$. We denote the *square of the norm of the second fundamental form* by

$$S = \frac{1}{2}\sum h^\alpha_{ij}h^\alpha_{ij}$$

The *normal scalar curvature* K_N of M in $\hat{M}$ is defined by

$$K_N = \sum_{\substack{i<j\\ \alpha<\beta}} (R_{\alpha\beta ij})^2 = \sum_{\substack{i<j\\ \alpha<\beta}} \{\sum_k (h^\alpha_{ik}h^\beta_{jk} - h^\alpha_{jk}h^\beta_{ik})\}^2.$$

§ 2. **The $(n+1)$-th shape operator and the n-th torsion index (T_n-index).** In this section, we assume that *M is minimal in $\hat{M}$.* We define the $(n+1)$-th shape operator (the n-th torsion operator) and the n-th torsion index for minimal surfaces in $\hat{M}$ by induction on n.

The (first) shape operator φ_1 can be written as

$$\varphi_1(X,\ Y) = \sum_{i,j,\alpha} h^\alpha_{ij}\omega_i(X)\omega_j(Y)e_\alpha\,.$$

Since $\sum_{j=1}^{2} h_{jj}^{\alpha} = 0$, for $X = e_1\cos\theta + e_2\sin\theta$, we have

(2.1) $$\widetilde{\varphi}_1(X) := \varphi_1(X, X) = \cos 2\theta \cdot F_1 + \sin 2\theta \cdot G_1,$$

where $F_1 = \sum_\alpha h_{11}^{\alpha} e_\alpha$ and $G_1 = \sum_\alpha h_{12}^{\alpha} e_\alpha$. Let S_p^1 be the unit circle in a tangent space M_p to M at $p \in M$. It is clear from (2.1) that the image of S_p^1 under $\widetilde{\varphi}_1$ is a point, a segment, or an ellipse (a cirele) according as m-index$_pM = 0$, 1, or 2. We easily see that

(2.2) $$S^2 - K_N = (\|G_1\|^2 - \|F_1\|^2)^2 + 4 \langle G_1, F_1 \rangle^2,$$

which is geometrically stated as follows:

$S^2 - K_N = \{(\text{length of major axis})^2 - (\text{length of minor axis})^2\}^2$ if m-index$_pM = 2$,

$S^2 - K_N = (\text{length of the segment})^4$ if m-index$_pM = 1$,

$S^2 = K_N = 0$ if m-index$_pM = 0$.

If m-index$_pM = 0$ at every point $p \in M$, then M is totally geodesic in $\widehat{M}$. If m-index$_pM = 1$ at every point $p \in M$, then the geodesic codimension of M is 1.

Henceforth, we consider the case that m-index$_pM = 2$ at every point $p \in M$. Then, we can choose a local frame field $b \in B$ such that

(2.3) $$\omega_{j\alpha_1} \neq 0, \qquad \omega_{j\beta} = 0, \qquad \alpha_1 \in \{3, 4\}, \qquad \text{for} \quad \beta > 4.$$

From (1.1), (1.2) and (2.3) we may write

$$\sum_{\alpha_1} h_{ij}^{\alpha_1} \omega_{\alpha_1\beta} = \sum_k h_{ijk}^{\beta} \omega_k, \qquad \beta > 4,$$

where h_{ijk}^{β} are symmetric in the indices i, j, k and $\sum_j h_{jjk}^{\beta} = 0$. Then, we can consider the 3-linear mapping from $M_p \times M_p \times M_p$ into the normal space N_p at p as follows

$$\varphi_2(X_1, X_2, X_3) = \sum_{\substack{\beta>4 \\ i,j,k}} h_{ijk}^{\beta} \omega_i(X_1)\omega_j(X_2)\omega_k(X_3) e_\beta, \quad X_j \in M_p.$$

We call this mapping φ_2 the *second shape operator* (*first torsion operator*) of M in $\widehat{M}$. Putting $\widetilde{\varphi}_2(X) = \varphi_2(X, X, X)$ for $X \in M_p$, we get the mapping $\widetilde{\varphi}_2$ from M_p into N_p. For $X = e_1\cos\theta + e_2\sin\theta$, we have

(2.4) $$\widetilde{\varphi}_2(X) = \cos 3\theta \cdot F_2 + \sin 3\theta \cdot G_2,$$

where $F_2 = \sum_{\beta>4} h_{111}^{\beta} e_\beta$ and $G_2 = \sum_{\beta>4} h_{112}^{\beta} e_\beta$. We call the dimension of the image of M_p under $\widetilde{\varphi}_2$ the *first torsion index* of M in $\widehat{M}$ at $p\in M$ and denote it by T_1-index$_pM$. It is clear from (2. 4) that the image of S_p^1 under $\widetilde{\varphi}_2$ is a point, a segment, or an ellipse (a circle) according as T_1-index$_pM$ $= 0$, 1, or 2. We easily see that

$$(\|G_2\|^2-\|F_2\|^2)^2 + 4\langle G_2, F_2\rangle^2 = \begin{cases} \{(\text{length of major axis})^2-(\text{length of minor axis})^2\}^2 & \text{if } T_1\text{-index}_pM=2, \\ (\text{length of the segment})^4 & \text{if } T_1\text{-index}_pM=1, \\ 0 & \text{if } T_1\text{-index}_pM=0. \end{cases}$$

Thus, $(\|G_2\|^2-\|F_2\|^2)^2+4\langle G_2, F_2\rangle^2$ is invariant under the rotation of frames $\{e_1, e_2\}$ and $\{e_3, e_4\}$, so it is a differentiable function on M. If T_1-index$_pM = 0$ at every point $p\in M$, then we see that the geodesic codimension of M is 2. If T_1-index$_pM = 1$ at every point $p\in M$, then we see that the geodesic codimension of M is 3, this fact was proved in Theorem 4 in [7].

We consider the case that T_1-index$_pM = 2$ at every point $p\in M$. Then, we can choose a local frame field satisfying (2. 3) and

$$(2.5) \qquad \omega_{\alpha_1\alpha_2}\neq 0, \quad \omega_{\alpha_1\gamma}=0, \quad \alpha_1\in\{3, 4\}, \quad \alpha_2\in\{5, 6\}, \quad 6<\gamma.$$

It follows from (1. 1), (1. 2) and (2. 5) that we may write

$$\sum_{\alpha_2} h_{ijk}^{\alpha_2}\omega_{\alpha_2\gamma} = \sum_l h_{ijkl}^{\gamma}\omega_l, \qquad 6<\gamma,$$

where h_{ijkl}^{γ} are symmetric in the indices i, j, k, l and $\sum_i h_{iijk}^{\gamma}=0$. Then, we can consider the 4-linear mapping from $M_p\times\cdots\times M_p$ into N_p as follows: for $X_j\in M_p$, $j = 1, 2, 3, 4$,

$$\varphi_3(X_1, X_2, X_3, X_4) = \sum_{\substack{i,j,k,l \\ \gamma>6}} h_{ijkl}^{\gamma}\,\omega_i(X_1)\omega_j(X_2)\omega_k(X_3)\omega_l(X_4)e_\gamma.$$

We call this mapping φ_3 the *third shape operator* (*second torsion operator*) of M in $\widehat{M}$. Putting $\widetilde{\varphi}_3(X)=\varphi_3(X, X, X, X)$ for $X\in M_p$, we get the mapping $\widetilde{\varphi}_3$ from M_p into N_p. Particularly, for a unit tangent vector $X = e_1\cos\theta + e_2\sin\theta$, we have

$$(2.6) \qquad \widetilde{\varphi}_3(X) = \cos 4\theta\cdot F_3+\sin 4\theta\cdot G_3,$$

where $F_3=\sum_{\gamma>6} h_{1111}^{\gamma}e_\gamma$ and $G_3=\sum_{\gamma>6} h_{1112}^{\gamma}e_\gamma$. We call the dimension of the image of M_p under $\widetilde{\varphi}_3$ the *second torsion index* of M in $\widehat{M}$ at p and denote it by T_2-index$_pM$. From (2. 6), we see that the image of S_p^1 under

$\widetilde{\varphi}_3$ is a point, a segment, or an ellipse (a circle) according as T_2-index$_pM$ $= 0$, 1, or 2. By the same reason as the case of $\widetilde{\varphi}_2$, we easily see that $(\|G_3\|^2 - \|F_3\|^2)^2 + 4\langle F_3, G_3\rangle^2$ is a differentiable function on M.

Now, we assume that T_{n-1}-index$_pM = 2$ at every point $p \in M$, $n \geqq 2$. Then, we can choose a local frame field such that

$$(2.7)\qquad \begin{cases} \omega_{\alpha_t\alpha_{t+1}} \neq 0, & \omega_{\alpha_t\gamma} = 0, \quad \alpha_t \in I_t = \{2t+1,\ 2t+2\}, \\ \alpha_{t+1} \in I_{t+1}, & 2t+4 < \gamma, \quad t = 0, 1, 2, \cdots, n-1. \end{cases}$$

From (1.1), (1.2) and (2.7), we may write

$$\sum_{\alpha_{t+1}} h^{\alpha_{t+1}}_{j_1 j_2 \cdots j_{t+2}} \omega_{\alpha_{t+1}\gamma} = \sum_{j_{t+3}=1}^{2} h^{\gamma}_{j_1 j_2 \cdots j_{t+3}} \omega_{j_{t+3}}, \quad 2t+4 < \gamma,$$

where $h^{\gamma}_{j_1 j_2 \cdots j_{t+3}}$ are symmetric in the indices $j_1, j_2, \cdots, j_{t+3}$ and $\sum_j h^{\gamma}_{jjk_1k_2\cdots k_{t+1}} = 0$ for $t = 0, 1, 2, \cdots, n-1$. Hence, for $t = n-1$, we can consider the $(n+2)$-linear mapping φ_{n+1} from $M_p \times \cdots \times M_p$ into N_p as follows: for $X_j \in M_p$, $j = 1, 2, \cdots, n+2$,

$$\varphi_{n+1}(X_1, \cdots, X_{n+2}) = \sum_{\gamma > 2n+2} h^{\gamma}_{j_1 j_2 \cdots j_{n+2}} \omega_{j_1}(X_1) \cdots \omega_{j_{n+2}}(X_{n+2}) e_\gamma.$$

Putting $\widetilde{\varphi}_{n+1}(X) = \varphi_{n+1}(X, \cdots, X)$ for $X \in M_p$, we get the mapping $\widetilde{\varphi}_{n+1}$ from M_p into N_p. We call this mapping φ_{n+1} (or $\widetilde{\varphi}_{n+1}$) the *$(n+1)$-th shape operator* (*$n$-th torsion operator*) of M in $\widehat{M}$. Particularly, for a unit tangent vector $X = e_1\cos\theta + e_2\sin\theta$, we have

$$(2.8)\qquad \widetilde{\varphi}_{n+1}(X) = \cos(n+2)\theta \cdot F_{n+1} + \sin(n+2)\theta \cdot G_{n+1},$$

where $F_{n+1} = \sum_{\gamma > 2n+2} h^{\gamma}_{11\cdots 1} e_\gamma$ and $G_{n+1} = \sum_{\gamma > 2n+2} h^{\gamma}_{1\cdots 12} e_\gamma$. We call the dimension of the image of M_p under $\widetilde{\varphi}_{n+1}$ the *n-th torsion index* of M in $\widehat{M}$ at $p \in M$ and denote it by T_n-index$_pM$. It is clear from (2.8) that the image of S^1_p under $\widetilde{\varphi}_{n+1}$ is a point, a segment, or an ellipse (a circle) according as T_n-index$_pM = 0$, 1, or 2. By the same reason as the case of $\widetilde{\varphi}_2$, we see easily that $(\|G_{n+1}\|^2 - \|F_{n+1}\|^2)^2 + 4\langle F_{n+1}, G_{n+1}\rangle^2$ is a function on M.

§ 3. **Complete flat minimal surfaces in $\widehat{M}$.** In this section, we assume that *M is a complete, connected and oriented 2-dimensional Riemannian manifold which is minimally immersed in a Riemannian manifold $\widehat{M}$ of constant curvature c ($\neq 0$) and that the Gaussian curvature K of M is identically zero.* Then, as is well known, M may be considered as a *Riemann surface.* Since $K \equiv 0$ and M is complete, as is stated in [1], M is *parabolic*, i.e., a negative subharmonic function on M must be constant. We first have

Lemma 1. *On M we have only one of the following cases:*
S_0-case $S = c$ *and* $K_N = 0$,
C_0-case $S = c$ *and* $K_N = S^2 = c^2$,
E_0-case $S = c$, $K_N =$ *constant* > 0 *and* $S^2 - K_N > 0$.

Proof. Since $K = 0$ and $K = c - S$, we have

$$S = c = \text{constant} > 0 \text{ on } M.$$

We shall prove that K_N is constant on M. Since $K = 0$ on M, we can choose a neighborhood U of a point $p \in M$ in which there exist isothermal coordinates (u, v) and a frame field $b \in B$ such that

$$ds^2 = du^2 + dv^2, \qquad \omega_1 = du, \qquad \omega_2 = dv,$$

where ds is the line element of M. We may write $\omega_{i\alpha}$ ($i = 1, 2$ and $3 \leqq \alpha$) as follows

$$\omega_{1\alpha} = f_\alpha \omega_1 + g_\alpha \omega_2, \qquad \omega_{2\alpha} = g_\alpha \omega_1 - f_\alpha \omega_2,$$

where f_α and g_α are differentiable functions on U. Using the structure equations, we can see that the complex-valued function

$$w(z, \bar{z}) = \|G_1\|^2 - \|F_1\|^2 + 2i \langle F_1, G_1 \rangle$$

is holomorphic in $z = u + iv$, where $F_1 = \sum_{\alpha \geqq 3} f_\alpha e_\alpha$ and $G_1 = \sum_{\alpha \geqq 3} g_\alpha e_\alpha$. Hence, $|w(z, \bar{z})|^2$ is a subharmonic function on M. Since $S^2 - K_N = |w(z, z)|^2$ and $S = c$, we see that K_N is a non-negative superharmonic function on M, so it must be constant on M, because M is parabolic. Thus, we have only one of three cases in Lemma.

By Lemma 1, at every point $p \in M$, the image of S_p^1 under $\tilde{\varphi}_1$ is a segment of the constant length or an ellipse with axes (a circle with radius) of the constant lengths according as S_0-case or E_0-case (C_0-case).

From now on, by *S-case*, *C-case*, or *E-case* we mean the case where the image of S_p^1 under the t-th shape operator (the $(t-1)$-th torsion operator) $\tilde{\varphi}_t$ is a segment, a circle, or an ellipse respectively.

In the S_0-case, m-index$_p M = 1$ at every point $p \in M$, so the geodesic codimension of M is 1 by Theorem 1 in [6].

In the C_0-case and the E_0-case, m-index$_p M = 2$ at every point $p \in M$. In the E_0-case, we can choose a local frame field $b \in B$ such that

(3.1) $H_3 = \begin{pmatrix} k_1 & 0 \\ 0 & -k_1 \end{pmatrix}, \quad H_4 = \begin{pmatrix} 0 & \sigma_1 k_1 \\ \sigma_1 k_1 & 0, \end{pmatrix}, \quad H_\beta = 0, \quad 4 < \beta,$

where k_1 and σ_1 are real constants ($\neq 0$) on M, $\sigma_1^2 \neq 1$. From (3.1) we have

$$\omega_{12} = \omega_{34} = 0\,,$$

which implies that on a neighborhood U of a point $p \in M$ we can choose isothermal coordinates (u, v) such that

(3.2) $$ds^2 = du^2 + dv^2, \quad \omega_1 = du, \quad \omega_2 = dv\,.$$

In the C_0-case, since $K \equiv 0$, on a neighborhood of a point of M we can choose isothermal coordinates and a frame field satisfying (3.1) and (3.2), where $\sigma_1 = 1$. Thus, we may consider that the C_0-case is a special case of the E_0-case. It follows from (3.1) and (3.2) that we have

(3.3) $$\begin{aligned} \omega_{13} + i\omega_{23} &= k_1 d\bar{z}, \qquad z = u + iv\,, \\ \omega_{14} + i\omega_{24} &= i\sigma_1 k_1 d\bar{z}, \qquad \omega_1 + i\omega_2 = dz\,. \end{aligned}$$

Since $\omega_{i\beta} = 0$ $(4 < \beta)$, we may write

(3.4) $$\omega_{3\beta} + i\sigma_1\omega_{4\beta} = (f_\beta + ig_\beta)d\bar{z}, \qquad \beta < 4\,,$$

where f_β and g_β are differentiable functions on U. Then, for a unit tangent vector $X = e_1 \cos\theta + e_2 \sin\theta$, the second shape operator $\tilde{\varphi}_2$ is written as

$$\tilde{\varphi}_2(X) = \cos 3\theta \cdot F_2 + \sin 3\theta \cdot G_2,$$

where $F_2 = k_1 \sum_{\beta>4} f_\beta e_\beta$ and $G_2 = k_1 \sum_{\beta>4} g_\beta e_\beta$ are normal vector fields on U. Using the structure equations, from (3.2) and (3.4) we see that the complex-valued function

$$w_1(z, \bar{z}) = -k_1^2 \sum_\beta (f_\beta - ig_\beta)^2 = \|G_2\|^2 - \|F_2\|^2 + 2i \langle G_2, F_2 \rangle$$

is holomorphic in z, because k_1 is constant on M. Since, as be stated in § 2, $|w_1(z, \bar{z})|^2$ is a differentiable function on M, $|w_1(z, \bar{z})|^2$ is a subharmonic function on M. Since $\omega_{12} = \omega_{34} = 0$, $|w_1(z, \bar{z})|^2 \leqq k_1^4 \{ \sum_{\beta>4} (f_\beta^2 + g_\beta^2)\}^2 = 4\sigma_1^4 k_1^8 = \text{constant } (> 0)$. Hence $|w_1(z, \bar{z})|^2$ must be constant on M, because M is parabolic. Then, for the second shape operator we have S-case, C-case or E-case on M.

In the S-case for $\tilde{\varphi}_2$ on M, T_1-index$_p M = 1$ at every point $p \in M$, so the geodesic codimension of M is 3, see § 2.

We next consider the C-case and E-case for $\widetilde{\varphi}_2$. Since the image of S_p^1 under $\widetilde{\varphi}_2$ is an ellipse with axes (a circle with radius) of constant length at every point $p \in M$, we can choose a neighborhood U of a point $p \in M$ in which there exist isothermal coordinates (u, v) and a local frame field $b \in B$ satisfying (3. 3) and

$$\text{(3. 5)} \qquad \begin{cases} \omega_{35} + i\sigma_1\omega_{45} = k_2 d\bar{z}, & \omega_{3\gamma} = 0\,, \\ \omega_{36} + i\sigma_1\omega_{46} = i\sigma_2 k_2 d\bar{z}, & \omega_{4\gamma} = 0, \qquad 6 < \gamma, \end{cases}$$

where k_2 is a non-zero complex constant on M and σ_2 is a non-zero real constant on M. In the C-case for $\widetilde{\varphi}_2$, we may assume that k_2 is a non-zero real constant on M and $\sigma_2 = 1$. From (3. 3) and (3. 5), we have $\omega_{56} = 0$ and we may write

$$\omega_{5\gamma} + i\sigma_2\omega_{6\gamma} = (f_\gamma + ig_\gamma)d\bar{z}, \qquad 6 < \gamma.$$

Hence, for $X = e_1 \cos\theta + e_2 \sin\theta \in M_p$, we write the third shape operator $\widetilde{\varphi}_3$ as

$$\widetilde{\varphi}_3(X) = \cos 4\theta \cdot F_3 + \sin 4\theta \cdot G_3\,,$$

where F_3 and G_3 are real normal vector fields such that $F_3 + iG_3 = k_1 k_2 \sum_{\gamma > 6} (f_\gamma + ig_\gamma)\, e_\gamma$. Continuing this way, we have

Lemma 2. *If the image of S_p^1 under the t-th shape operator $\widetilde{\varphi}_t$ is an ellipse with axes (a circle with radius) of constant length at every point $p \in M$, for $t = 1, 2, \cdots, s$, $2 \leqq s$, then the image of S_p^1 under the $(s+1)$-th shape operator $\widetilde{\varphi}_{s+1}$ is a segment of constant length or an ellipse with axes (a circle with radius) of constant length at every point $p \in M$.*

Proof. By induction, from the assumption we can verify that on a neighborhood U of a point $p \in M$ there exist isothermal coordinates (u, v) and a frame field $b \in B$ such that

$$\text{(3. 6)}_s \quad \begin{cases} \omega_{\alpha_1\beta_1} + i\sigma_t\omega_{\alpha_2\beta_1} = k_{t+1}\, d\bar{z}\,, & dz = du + idv = \omega_1 + i\omega_2\,, \\ \omega_{\alpha_1\beta_2} + i\sigma_t\omega_{\alpha_2\beta_2} = i\sigma_{t+1}k_{t+1}d\bar{z}\,, & \omega_{\alpha_1\gamma} = \omega_{\alpha_2\gamma} = 0\,, \\ \alpha_1 = 2t+1\,, \quad \alpha_2 = 2t+2\,, & \beta_1 = 2t+3\,, \quad \beta_2 = 2t+4\,, \\ 2t+4 < \gamma\,, & t = 0, 1, 2, \cdots, s-1\,, \end{cases}$$

where k_t $(2 \leqq t \leqq s)$ are non-zero complex constant on M and k_1 and σ_t $(1 \leqq t \leqq s,\ \sigma_0 = 1)$ are non-zero real constant on M. Using the structure equations, from $(3.\ 6)_s$ we obtain

$(3.7)_s$ $\quad \omega_{\alpha_1\alpha_2}=0, \quad \alpha_1=2t+1, \quad \alpha_2=2t+2 \quad$ for $\quad t=0,1,2,\cdots, \quad s$

and we may write

$$(3.8) \qquad \omega_{\alpha_1\gamma}+i\sigma_s\omega_{\alpha_2\gamma}=(f_\gamma+ig_\gamma)d\bar{z}, \quad a_1=2s+1, \quad a_2=2s+2<\gamma.$$

Let F_{s+1} and G_{s+1} be real normal vector fields on U such that $F_{s+1}+iG_{s+1}=k_1k_2\cdots k_s\sum_{\gamma>2s+2}(f_\gamma+ig_\gamma)e_\gamma$, then for a unit tangent vector $X=e_1\cos\theta+e_2\sin\theta\in M_p$ the $(s+1)$-th shape operator $\tilde{\varphi}_{s+1}$ is written as

$$\tilde{\varphi}_{s+1}(X)=\cos(s+2)\theta\cdot F_{s+1}+\sin(s+2)\theta\cdot G_{s+1}.$$

From $(3.6)_s$, $(3.7)_s$ and (3.8), we see that the complex-valued function

$$w_s(z,\bar{z})=-(k_1\bar{k}_2\cdots\bar{k}_s)^2\sum_{\gamma>2s+2}(f_\gamma-ig_\gamma)^2=|G_{s+1}|^2-|F_{s+1}|^2+2i<G_{s+1},\ F_{s+1}>$$

is holomorphic in z, because k_t $(t=1,2,\cdots,s)$ are constant on M. Since, as be stated in § 2, $|w_s(z,\bar{z})|^2$ is a differentiable function on M, $|w_s(z,\bar{z})|^2$ is a subharmonic function on M. On the other hand, from $(3.7)_s$ we have $|w_s(z,\bar{z})|^2\leqq|k_1k_2\cdots k_s|^4\{\sum(f_\gamma^2+g_\gamma^2)\}^2=|k_1k_2\cdots k_s|^4\{(1+\sigma_{s-1}^2)\sigma_s^2\cdot|k_s|^2/\sigma_{s-1}^2\}^2=$ constant (>0) on M. Hence the subharmonic function $|w_s(z,\bar{z})|^2$ must be constant on M, because M is parabolic. Thus, at every point $p\in M$ the image of a unit tangent circle S_p^1 to M under $\tilde{\varphi}_{s+1}$ is a segment of constant length, a circle with constant length or an ellipse with axes of constant length on M.

If the image of S_p^1 under $\tilde{\varphi}_{s+1}$ is a segment of constant length at every point $p\in M$, then T_s-index$_pM=1$ at every point p so the geodesic codimension of M is $2s+1$.

If the geodesic codimension of M is even $2s$, using the structure equations, from $(3.6)_s$ and (3.8) we have contradiction. Hence, the geodesic codimension of M is odd $2m+1$ and the images of unit tangent circles to M under $\tilde{\varphi}_{m+1}$ are segements of constant length on M. Thus, we have proved the following

Theorem 1. *Let M be a 2-dimensional, connected, oriented and complete Riemannian manifold which is minimally immersed in a $(2+\nu)$-dimensional Riemannian manifold $\hat{M}$ of non-zero constant curvature c. If Gaussian curvature of M is identically zero and the image of M under the immersion is not contained in a totally geodesic submanifold of $\hat{M}$, i.e., ν is the geodesic codimension of M, then ν is odd $2m+1$. Furthermore, the images of unit tangent circles to M*

under the t-th shape operators $(1 \leqq t \leqq m)$ *are ellipses with axes (or circles with radius) of constant length and the images of unit tangent circles to M under the* $(m+1)$*-th shape operator are segments of constant length on M.*

In view of Theorem 1, we see that on a neighborhood of a point of M there exist isothermal coordinates (u, v) and a local frame field satisfying $(3.6)_{m+1}$, where $\sigma_{m+1}=0$ and k_{m+1} is a positive constant cn M. In general, however, the constants k_t $(1 \leqq t \leqq m+1)$ and σ_t $(1 \leqq t \leqq m)$ depend on the immersions. Using the structure equations, from $(3.7)_m$ we get

$$(3.9)_m \quad (1+\sigma_{t+1}^2)|k_{t+1}|^2 = 2c\sigma_t^2/(1+\sigma_t^2), \quad \sigma_0=1, \quad \sigma_{m+1}=0, \quad 0 \leqq t \leqq m.$$

Since $c>0$ from Lemma 1, we consider the case where $\hat{M}$ is a $(2m+3)$-dimensional sphere $S^{2m+3}(c)$ of constant curvature c. We may consider $S^{2m+3}(c) \subset E^{2m+4}$ and set $\sqrt{c}\,x = e_{2m+4}$. Let $E_t = e_{\alpha_1} + i\sigma_t e_{\alpha_2}$ and $E_t^* = \sigma_t e_{\alpha_1} - ie_{\alpha_2}$, $\alpha_1 = 2t+1$, $\alpha_2 = 2t+2$, where $t=0, 1, 2, \cdots, m$. Since $E_t^* = ((1+\sigma_t^2)/2\sigma_t)\bar{E}_t - ((1-\sigma_t^2)/2\sigma_t)E_t$, we have the following Frenet formulas of M

$$(3.10)_m \quad \begin{aligned}
dx &= \frac{1}{2}(\bar{E}_0 dz + E_0 d\bar{z}), \qquad z = u + iv,\\
dE_0 &= -cx\,dz + k_1 E_1 d\bar{z},\\
dE_1 &= -((1+\sigma_1^2)k_1/2)E_0 dz - \{((1-\sigma_1^2)k_1/2)\bar{E}_0 - k_2 E_2\}\, d\bar{z},\\
dE_2 &= -((1+\sigma_2^2)\bar{k}_2/2\sigma_1)\bar{E}_1^* dz - \{((1-\sigma_2^2)k_2/2\sigma_1)\, E_1^* - k_3 E_3\}\, d\bar{z},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
dE_t &= -((1+\sigma_t^2)\bar{k}_t/2\sigma_{t-1})\bar{E}_{t-1}^* dz - \{((1-\sigma_t^2)\, k_t/2\sigma_{t-1})\, E_{t-1}^* - k_{t+1}E_{t+1}\}\, d\bar{z},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
dE_{m-1} &= -((1+\sigma_{m-1}^2)\bar{k}_{m-1}/2\sigma_{m-2})\bar{E}_{m-2}^* dz\\
&\qquad - \{((1-\sigma_{m-1}^2)k_{m-1}/2\sigma_{m-2})E_{m-2}^* - k_m E_m\}\, d\bar{z},\\
dE_m &= -((1+\sigma_m^2)\bar{k}_m/2\sigma_{m-1})\bar{E}_{m-1}^* dz\\
&\qquad - \{((1-\sigma_m^2)k_m/2\sigma_{m-1})E_{m-1}^* - k_{m+1}e_{2m+3}\}\, d\bar{z},\\
de_{2m+3} &= -(k_{m+1}/2\sigma_m)\bar{E}_m^* dz - (k_{m+1}/2\sigma_m)E_m^* d\bar{z}\,.
\end{aligned}$$

From $(3.9)_m$ and $(3.10)_m$ we easily see that the vector fields $E_0, E_1, E_2, \cdots, E_m, e_{2m+3}$ and $x = e_{2m+4}$ satisfy the following equation

$$(3.11) \qquad \partial^2 Y/(\partial z \cdot \partial \bar{z}) = -(c/2)Y.$$

Remark. In Theorem 1, it seems to be difficult to prove the rigidity for the minimal immersion of the Euclidean plane into a sphere without any assumption.

Under the assumption of Theorem 1, if the images of unit tangent circles to M under the t-th shape operators $(1 \leqq t \leqq m)$ are circles with radius of constant length on M, then we can choose a neighborhood of a point of M in which there exist isothermal coordinates and a local frame field $b \in B$ satisfying $(3.6)_{m+1}$ such that $\sigma_t = 1$ $(1 \leqq t \leqq m)$, $\sigma_{m+1} = 0$ and k_t $(1 \leqq t \leqq m+1)$ are non-zero real constants on M. These constants are independent of the immersion. When $c=1$, in the same way as in §8 of [7], from $(3.10)_m$ we can verify that M is the surface given by

$$(3.12)\quad \begin{aligned} x = \frac{1}{\sqrt{2(m+2)}} \sum_{j=1}^{m+2} \{ & A_j \exp i\sqrt{2}\left(u \cdot \sin\frac{2j-1+\varepsilon}{2(m+2)}\pi + v \cdot \cos\frac{2j-1+\varepsilon}{2(m+2)}\pi\right) \\ & + \bar{A}_j \exp(-i\sqrt{2})\left(u \cdot \sin\frac{2j-1+\varepsilon}{2(m+2)}\pi + v \cdot \cos\frac{2j-1+\varepsilon}{2(m+2)}\pi\right)\}, \end{aligned}$$

where $A_1, A_2, \cdots, A_{m+2}$ are constant vectors in $C^{m+2} = E^{2m+4}$ such that

$$A_j \cdot A_j = A_j \cdot A_k = A_j \cdot \bar{A}_k = 0, \qquad A_j \cdot \bar{A}_j = 1, \qquad j \neq k,$$

and $\varepsilon = 0$ or 1 according as $m =$ odd or even. Thus, we have proved the following

Theorem 2. *Under the assumption of Theorem 1, if $\hat{M}$ is a unit sphere and the images of unit tangent circles to M under the t-th shape operators $(1 \leqq t \leqq m)$ are circles with radius of constant length on M, then the immersion is uniquely determined up to a rigid motion of a sphere.*

We are interested in examples of flat minimal surfaces other than (3.12). In the next section, we shall find examples other than (3.12).

§ 4. Examples of minimal immersions of the Euclidean plane into a sphere of constant curvature 1.

In this section, we give examples of flat minimal surfaces in $S^5(1)$ and $S^7(1)$ other than (3.12).

We first give examples in $S^5(1)$, that is, find solutions of $(3.10)_1$ in the case $c = 1$. Noticing (3.11), we choose three fixed constant vectors A_1, A_2, A_3 in $C^3 = E^6$ such that

$$(4.1)\quad \begin{gathered} A_j \cdot A_j = A_j \cdot A_k = A_j \cdot \bar{A}_k = 0, \quad \sum_{j=1}^{3} A_j \cdot \bar{A}_j = 1, \\ j, k = 1, 2, 3, \qquad j \neq k, \end{gathered}$$

and let

$$(4.2)\quad x = \frac{1}{\sqrt{2}} \sum_{j=1}^{3} \{A_j \exp \frac{1}{\sqrt{2}}(ze^{i\alpha_j} - \bar{z}e^{-i\alpha_j}) + \bar{A}_j \exp \frac{1}{\sqrt{2}}(-ze^{i\alpha_j} + \bar{z}e^{-i\alpha_j})\},$$

where the bar denotes the conjugate and α_j $(j=1, 2, 3)$ are real constant numbers. It is clear that $x=\bar{x}$ and $x\cdot x=1$ by (4. 1). In this case, we set

$$E_0=2\frac{\partial x}{\partial \bar{z}}=-\sum_{j=1}^{3}e^{-i\alpha_j}\{A_j\exp\frac{1}{\sqrt{2}}(ze^{i\alpha_j}-\bar{z}e^{-i\alpha_j})-\bar{A}_j\exp\frac{1}{\sqrt{2}}(\bar{z}e^{-i\alpha_j}-ze^{i\alpha_j})\},$$

$$E_1=\frac{\partial E_0}{k_1\partial \bar{z}}=\frac{1}{k_1\sqrt{2}}\sum_{j=1}^{3}e^{-2i\alpha_j}\{A_j\exp\frac{(ze^{i\alpha_j}-\bar{z}e^{-i\alpha_j})}{\sqrt{2}}+\bar{A}_j\exp\frac{(-ze^{i\alpha_j}+\bar{z}e^{-i\alpha_j})}{\sqrt{2}}\},$$

$$e_5=\frac{\partial E_1}{k_2\partial \bar{z}}+\frac{(1-\sigma_1^2)k_1\bar{E}_0}{2k_2}=\frac{1}{2k_1k_2}\sum_{j=1}^{3}\Big[\{\frac{1-\sigma_1^2}{1+\sigma_1^2}e^{i\alpha_j}-e^{-3i\alpha_j}\}\cdot$$

$$\cdot\{A_j\exp\frac{1}{\sqrt{2}}(ze^{i\alpha_j}-\bar{z}e^{-i\alpha_j})-\bar{A}_j\exp\frac{1}{\sqrt{2}}(-ze^{i\alpha_j}+\bar{z}e^{i\alpha_j})\}\Big],$$

where k_1, k_2 and σ_1 are non-zero real constant numbers satisfying $(3.9)_1$. Then, we easily see that these vectors satisfy the equation $(3.10)_1$ in the case $c=1$. From the above equations we have

$$x\cdot E_0=E_0\cdot E_1=E_0\cdot\bar{E}_1=E_1\cdot e_5=e_5\cdot x=0,$$
$$E_0\cdot\bar{E}_0=2,\ E_1\cdot\bar{E}_1=1+\sigma_1^2.$$

Hence, we see that (4. 2) is a solution of $(3.10)_1$ if and only if there hold the following equalities

$$x\cdot E_1=E_0\cdot E_0=E_0\cdot e_5=0,\quad E_1\cdot E_1=1-\sigma_1^2,$$
$$e_5=\bar{e}_5,\quad e_5\cdot e_5=1.$$

By the above definitions of x, E_0, E_1 and e_5, these equations are equivalent to the following

(4. 3) $\sum_{j=1}^{3}A_j\cdot\bar{A}_je^{-2i\alpha_j}=0,$

(4. 4) $\sum_{j=1}^{3}A_j\cdot\bar{A}_je^{-4i\alpha_j}=k_1^2(1-\sigma_1^2),$

(4. 5) $\sum_{j=1}^{3}A_j\cdot\bar{A}_je^{-6i\alpha_j}=-2k_1^2k_2^2,$

(4. 6) $\cos 3\alpha_j-((1-\sigma_1^2)/(1+\sigma_1^2))\cos\alpha_j=0$ for $j=1, 2, 3.$

Now, we shall find constant vectors A_j $(j=1, 2, 3)$ and real constant numbers α_j $(j=1, 2, 3)$ satisfying (4. 3) $\sim$ (4. 6) under the condition $(3.9)_1$. *We consider the following special case:*

(4. 7) $A_1\cdot\bar{A}_1=A_2\cdot\bar{A}_2,\quad -\alpha_1=\alpha=\alpha_2,\quad 2\alpha_3=\pi.$

Then, from (4.1), (4.3) and (4.7) we have

$$(4.8)\qquad A_1\cdot\bar{A}_1=A_2\cdot\bar{A}_2=\frac{1}{2(1+\cos 2\alpha)},\quad A_3\cdot\bar{A}_3=\frac{\cos 2\alpha}{1+\cos 2\alpha}.$$

Hence, in this special case, if there exist A_j and α_j $(j=1,2,3)$ satisfying (4.3)~(4.6), the solutions (4.2) are determined only by one parameter α. Using $(1+\sigma_1^2)k_1^2=1$, from (4.4) and (4.8) we have

$$(4.9)\qquad \cos 2\alpha=\{(1-\sigma_1^2)/(1+\sigma_1^2)+1\}/2=1/(1+\sigma_1^2)=k_1^2.$$

Since $k_1\neq 0$, $\sigma_1\neq 0$ and $(1+\sigma_1^2)k_1^2=1$, we have $0<k_1^2<1$, which implies $0<\cos 2\alpha<1$ by (4.9), so that we may assume that $0<\alpha<\pi/4$. Then, from (4.9) we have

$$\cos\alpha=\sqrt{(k_1^2+1)/2},\qquad \sin\alpha=\sqrt{(1-k_1^2)/2}.$$

and from (4.8) we have

$$(4.10)\qquad A_1\cdot\bar{A}_1=A_2\cdot\bar{A}_2=\frac{1}{2(1+k_1^2)},\quad A_3\cdot\bar{A}_3=\frac{k_1^2}{1+k_1^2}.$$

We can easily see that these constants satisfy (4.5) and (4.6). Thus, we obtain examples given by

$$(4.11)\qquad \begin{aligned} x=\frac{1}{\sqrt{2}}[&A_1\exp i(v\sqrt{k_1^2+1}-u\sqrt{1-k_1^2})+\bar{A}_1\exp i(u\sqrt{1-k_1^2}-v\sqrt{1+k_1^2})\\ &+A_2\exp i(v\sqrt{1+k_1^2}+u\sqrt{1-k_1^2})+\bar{A}_2\exp(-i(v\sqrt{1+k_1^2}+u\sqrt{1-k_1^2}))\\ &+A_3\exp(\sqrt{2}\,iu)+\bar{A}_3\exp(-\sqrt{2}\,iu)], \end{aligned}$$

where A_1, A_2, A_3 are fixed constant vectors in $C^3=E^6$ satisfying (4.1) and (4.10) and k_1 is a positive constant smaller than 1.

Next, we shall give examples in $S^7(1)$, that is, find solutions of $(3.10)_2$ in the case where $c=1$ and $k_2=$ real constant $\neq 0$. Noticing (3.11), we choose four fixed vectors A_1, A_2, A_3, A_4 in $C^4=E^8$ such that

$$(4.12)\qquad A_j\cdot A_j=A_j\cdot A_k=A_j\cdot\bar{A}_k=0,\quad \sum_{j=1}^{4}A_j\cdot\bar{A}_j=1,\quad j,k=1,2,3,4,\quad j\neq k.$$

Let

$$(4.13)\qquad x=\frac{1}{\sqrt{2}}\sum_{j=1}^{4}\{A_j\exp\frac{1}{\sqrt{2}}(ze^{i\alpha_j}-\bar{z}e^{-i\alpha_j})+\bar{A}_j\exp\frac{1}{\sqrt{2}}(-ze^{i\alpha_j}+\bar{z}e^{-i\alpha_j})\},$$

where α_j $(j=1,2,3,4)$ are real constant numbers. It is clear from (4.12) and (4.13) that $x=\bar{x}$ and $x\cdot x=1$. We define vectors E_0, E_1, E_2

and e_7 as follows

$$
(4.14)\begin{cases}
E_0 = 2\dfrac{\partial x}{\partial \bar{z}} = -\sum_{j=1}^{4} e^{-i\alpha_j}B_j^* , \\
E_1 = \dfrac{\partial E_0}{k_1\partial \bar{z}} = \dfrac{1}{k_1\sqrt{2}}\sum_{j=1}^{4} e^{-2i\alpha_j}B_j , \\
E_2 = \dfrac{\partial E_1}{k_2\partial \bar{z}} + \dfrac{(1-\sigma_1^2)k_1\bar{E}_0}{2k_2} = \dfrac{1}{2k_1k_2}\sum_{j=1}^{4}\left\{\dfrac{1-\sigma_1^2}{1+\sigma_1^2}e^{i\alpha_j}-e^{-3i\alpha_j}\right\} B_j^* , \\
e_7 = \dfrac{\partial E_2}{k_3\partial \bar{z}} + \dfrac{(1-\sigma_2^2)k_2}{2k_3\sigma_1}\left\{\dfrac{1+\sigma_1^2}{2\sigma_1}\bar{E}_1 - \dfrac{1-\sigma_1^2}{2\sigma_1}E_1\right\} \\
= \dfrac{1}{2\sqrt{2}\, k_1k_2k_3}\sum_{j=1}^{4}\left\{e^{-4i\alpha_j}-\dfrac{1-\sigma_1^2}{1+\sigma_1^2}+\dfrac{1-\sigma_2^2}{1+\sigma_2^2}e^{2i\alpha_j}-\dfrac{(1-\sigma_1^2)(1-\sigma_2^2)}{(1+\sigma_1^2)(1+\sigma_2^2)}e^{-2i\alpha_j}\right\}B_j ,
\end{cases}
$$

where $B_j = A_j\exp((ze^{i\alpha_j} - \bar{z}e^{-i\alpha_j})/\sqrt{2}) + \bar{A}_j \exp((-ze^{i\alpha_j} + \bar{z}e^{-i\alpha_j})/\sqrt{2})$ and $B_j^* = A_j\exp((ze^{i\alpha_j} - \bar{z}e^{-i\alpha_j})/\sqrt{2}) - \bar{A}_j\exp((-ze^{i\alpha_j} + \bar{z}e^{-i\alpha_j})/\sqrt{2})$ and k_1, k_2, k_3, σ_1 and σ_2 are real constant numbers satisfying $(3.9)_2$. From (4.14) we easily see that

$$
\begin{gathered}
x\cdot E_0 = E_0\cdot E_1 = E_0\cdot \bar{E}_1 = E_0\cdot e_7 = E_1\cdot E_2 = E_1\cdot \bar{E}_2 = E_2\cdot x = E_2\cdot e_7 = 0, \\
E_0\cdot \bar{E}_0 = 2, \qquad E_1\cdot \bar{E}_1 = 1+\sigma_1^2, \qquad E_2\cdot \bar{E}_2 = 1 + \sigma_2^2 .
\end{gathered}
$$

Hence, (4.13) is a solution of $(3.10)_2$ if and only if the following conditions are satisfied:

$$
(4.15)\quad \begin{gathered}
x\cdot E_1 = x\cdot e_7 = E_0\cdot E_0 = E_0\cdot E_2 = E_0\cdot \bar{E}_2 = E_1\cdot e_7 = 0 , \\
E_1\cdot E_1 = 1 - \sigma_1^2 , \qquad E_2 \cdot E_2 = 1 - \sigma_2^2 ,
\end{gathered}
$$

(4.16) $\qquad e_7$ is a unit real vector.

Using (4.12), we see that the conditions (4.15) are equivalent to the following equalities:

$$
(4.17)\quad \begin{cases}
\sum_{j=1}^{4} A_j\cdot \bar{A}_j\, e^{-2i\alpha_j} = 0 , \\
\sum_{j=1}^{4} A_j\cdot \bar{A}_j\, e^{-4i\alpha_j} = k_1^2(1 - \sigma_1^2) , \\
\sum_{j=1}^{4} A_j\cdot \bar{A}_j\, e^{-6i\alpha_j} = -2k_1^2k_2^2 (1 - \sigma_2^2) .
\end{cases}
$$

In order to find constant vectors A_j in C^4 and constant numbers α_j $(j = 1, 2, 3, 4)$ satisfying (4.16) and (4.17), *we consider the following special case*:

$$
(4.18)\quad A_1\cdot \bar{A}_1 = A_3\cdot \bar{A}_3 , \quad -\alpha_1= \alpha_3 = \alpha, \quad \alpha_2 = 0, \quad 2\alpha_4 = \pi .
$$

By means of $(3.9)_2$ and (4.12), from (4.17) we have

$$\cos 2\alpha = k_2^2(1-\sigma_2^2)/(2\sigma_1^2) = k_1^2(1-k_3^2) =: k\,, \tag{4.19}$$

$$\begin{aligned} &A_1\cdot\bar{A}_1 = A_3\cdot\bar{A}_3 = (k_1^2-1)/(2k^2-2)\,,\\ &A_2\cdot\bar{A}_2 = (k-k_1^2)/(2k-2),\ \ A_4\cdot\bar{A}_4 = (k+k_1^2)/(2k+2)\,. \end{aligned} \tag{4.20}$$

Therefore, we have seen that if there exist A_j and α_j $(j=1, 2, 3, 4)$ satisfying (4.12), (4.16) and (4.17), then (4.13) is determined only by two parameters k_1 and k_3. In this case, we can show that e_7 is a unit real vector. Since $k_3^2 = 2\sigma_2^2/(1+\sigma_2^2)$, $k_1^2(1+\sigma_1^2)=1$, $\sigma_1 \neq 0$ and $\sigma_2 \neq 0$, we have

$$0 < k_1^2 < 1 \qquad \text{and} \qquad 0 < k_3^2 < 2\,,$$

which implies $|\cos 2\alpha| < 1$ by (4.19). Hence, we may assume that $0 < \alpha < \pi/2$. Thus, we obtain examples of minimal immersions of the Euclidean plane into a sphere $S^7(1)$ given by

$$\begin{aligned} x = \frac{1}{\sqrt{2}}\,[&A_1\exp i(v\sqrt{1+k}-u\sqrt{1-k})+\bar{A}_1\exp i(u\sqrt{1-k}-v\sqrt{1+k})\\ &+A_2\exp(\sqrt{2}\,iv)+\bar{A}_2\exp(-\sqrt{2}\,iv)\\ &+A_3\exp i(v\sqrt{1+k}-u\sqrt{1-k})+\bar{A}_3\exp(-i(v\sqrt{1+k}-u\sqrt{1-k}))\\ &+A_4\exp(\sqrt{2}\,iu)+\bar{A}_4\exp(-\sqrt{2}\,iu)]\,, \end{aligned} \tag{4.21}$$

where A_1, A_2, A_3, A_4 are fixed constant vectors in $C^4 = E^8$ satisfying (4.12) and (4.20) and k is a constant real number such that

$$k := k_1^2(1-k_3^2),\quad 0 < k_1^2 = \text{const.} < 1,\quad 0 < k_3^2 = \text{const.} < 2\,.$$

Remark. We have obtained many examples of the Euclidean plane minimally immersed into the Euclidean unit spheres S^5 and S^7 other than Otsuki's surfaces. When $m=1$ and $m=2$, Otsuki's surfaces (3.12) are included in these examples as the special case where $\sigma_1^2=1$ and $\sigma_2^2=1$.

§ 5. **Compact minimal surfaces of non-negative ($\neq 0$) curvature in $\hat{M}$.** In this section, we shall consider *connected compact minimal surfaces of non-negative curvature* K ($\neq 0$) in a $(2+\nu)$-dimensional Riemannian manifold $\hat{M}$ of constant curvature c.

Let U be a neighborhood of a point $p \in M$ in which there exist isothermal coordinates (u, v) and a frame field $b \in B$ such that

$$ds^2 = E\{du^2+dv^2\},\qquad \omega_1 = \sqrt{E}\,du,\qquad \omega_2 = \sqrt{E}\,dv, \tag{5.1}$$

where ds is the line element of M and $E = E(u, v)$ is a positive function on U. In this case, we may write

$$\omega_{1\alpha} = f_\alpha\omega_1 + g_\alpha\omega_2, \quad \omega_{2\alpha} = g_\alpha\omega_1 - f_\alpha\omega_2, \quad 3 < \alpha,$$

where f_α and g_α are functions on U. Then, using the structure equations, we can verify that the complex valued function

$$(5.2)\quad w(z, \bar{z}) = E^2(\|G_1\|^2 - \|F_1\|^2) + 2iE^2\langle F_1, G_1\rangle, \; F_1 = \sum f_\alpha e_\alpha, \; G_1 = \sum g_\alpha e_\alpha,$$

is holomorphic in $z = u + iv$. Then, we have the following

Lemma 3. *We have* $S^2 - K_N = 0$ *on* M.

Proof. By an easy computation, we see that $S^2 - K_N = |w(z, \bar{z})|^2/E^4$ on M. If $S^2 - K_N = 0$ does not hold identically on M, $S^2 - K_N$ takes its positive maximum A at some $p_0 \in M$. Let U be a neighborhood of p_0 in which $S^2 - K_N > 0$ and there exist isothermal coordinates (u, v) and a frame field $b \in B$ satisfying (5.1). Then, from (5.2) we have

$$(5.3)\quad \Delta\log(S^2 - K_N) = -4\Delta\log E = 8EK, \quad \Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2,$$

because the Gaussian curvature K is given by $K = -(1/2E)\Delta\log E$. If $K \geqq 0$, the function $\log(S^2 - K_N)$ is a subharmonic function on U, so it must be constant A on U. Therefore, the closed set $\{p \in M \mid S^2 - K_N = A$ at $p\}$ of M is open in M. Since M is connected, $S^2 - K_N$ is identically a positive constant A on M. It follows from this fact and (5.3) that K is identically zero, which contradicts $K \not\equiv 0$ on M.

By Lemma 3, if $m\text{-index}_p M \neq 0$ at every point $p \in M$, then we can choose a neighborhood U of a point $p \in M$ in which there exist isothermal coordinates and frame fields satisfying (5.1) and

$$(5.4)\quad \begin{cases} \omega_{13} = k_1\omega_1 = \omega_{24}, & \omega_{1\beta} = \omega_{2\beta} = 0, \\ \omega_{23} = -k_1\omega_2 = -\omega_{14}, & 4 < \beta, \end{cases}$$

where k_1 is a positive differentiable function M. Using the structure equations, from (5.4) we have

$$\omega_{34} = 2\omega_{12} - (\log k_1)_2\omega_1 + (\log k_1)_1\omega_2,$$

where $d(\log k_1) = \sum_{j=1}^{2} (\log k_1)_j\omega_j$. Furthermore, from (5.4) we may write

$$(5.5)\quad \omega_{3\beta} = f_\beta\omega_1 + g_\beta\omega_2, \quad \omega_{4\beta} = g_\beta\omega_1 - f_\beta\omega_2, \quad 4 < \beta,$$

and define two normal vector fields $F_2 = \sum_{\beta>4} f_\beta e_\beta$ and $G_2 = \sum_{\beta>4} g_\beta e_\beta$ on U.

Then, we can write the second shape operator $\widetilde{\varphi}_2$ as

$$\widetilde{\varphi}_2(X) = k_1\{\cos 3\theta \cdot F_2 + \sin 3\theta \cdot G_2\}, \quad X = e_1\cos\theta + e_2\sin\theta \in M_p.$$

Using the structure equations, from (5. 5) and (5. 6) we see that the complex-valued function

$$w_1(z, \bar{z}) = E^3k_1^2(\|G_2\|^2 - \|F_2\|^2) + 2iE^3k_1^2\langle F_2, G_2\rangle$$

is holomorphic in z. As stated in § 2, we see that $|w_1(z, \bar{z})|^2/E^6 = k_1^4\{(\|G_2\|^2 - \|F_2\|^2)^2 + 4\langle G_2, F_2\rangle^2\}$ is a differentiable function on M. Hence, by the same reason as the proof of Lemma 3, we can prove

Lemma 4. *If m-index$_pM \neq 0$ at every point $p \in M$, then, at each point $p \in M$, the image of S_p^1 under the second shape operator is a point p or a circle according as T_1-index$_pM = 0$ or $\neq 0$.*

By Lemma 3 and Lemma 4, if m-index$_pM \neq 0$ and T_1-index$_pM \neq 0$ at every point $p \in M$, then we can choose a neighborhood U of a point $p \in M$ in which there exist isothermal coordinates (u, v) and a frame field $b \in B$ satisfying (5. 1), (5. 4) and

$$\begin{aligned} &\omega_{35} = k_2\omega_1 = \omega_{46}, \qquad &&\omega_{3\gamma} = \omega_{4\gamma} = 0, \\ &\omega_{36} = k_2\omega_2 = -\omega_{45}, \qquad &&6 < \gamma, \end{aligned} \tag{5. 7}$$

where k_2 is a positive differentiable function on M. Let $\lambda_2 = k_1k_2$ and $d(\log\lambda_2) = \sum_{j=1}^{2}(\log\lambda_2)_j\omega_j$, then from (5. 7) we have

$$\omega_{56} = 3\omega_{12} - (\log\lambda_2)_2\omega_1 + (\log\lambda_2)_1\omega_2 \tag{5. 8}$$

and we may write

$$\omega_{5\gamma} = f_\gamma\omega_1 + g_\gamma\omega_2, \quad \omega_{6\gamma} = g_\gamma\omega_1 - f_\gamma\omega_2, \quad 6 < \gamma.$$

Hence, for a unit tangent vector $X = e_1\cos\theta + e_2\sin\theta \in M_p$, the third shape operator $\widetilde{\varphi}_3$ is written as

$$\widetilde{\varphi}_3(X) = \lambda_2\{\cos 4\theta \cdot F_3 + \sin 4\theta \cdot G_3\},$$

where $F_3 = \sum_{\gamma>6} f_\gamma e_\gamma$ and $G_3 = \sum_{\gamma>6} g_\gamma e_\gamma$ are normal vector fields on U. Continuing this way, we have the following

Theorem 3. *Let M be a 2-dimensional, connected and compact Riemannian manifold of non-negative curvature ($\not\equiv 0$) which is minimally immersed in a $(2+\nu)$-dimensional Riemannian manifold $\widehat{M}$ of constant* *curvature c. If we have*

(A) *the image of M is not contained in a totally geodesic submanifold of $\widehat{M}$, i. e., ν is the geodesic codimension of M,*

(B) m-index$_pM \neq 0$ *at every point* $p \in M$,

(C) T_n-index$_pM$ $(n = 1, 2, \cdots)$ *are defined at every point* $p \in M$ *and* T_n-index$_pM \neq 0$ *at every point* $p \in M$ *for* $n = 1, 2, \cdots, [\frac{\nu}{2}] - 1$,

then ν must be even $2m$ and the image of unit tangent circles to M under the n-th shape operators $\widetilde{\varphi}_n$ $(1 \leqq n \leqq m)$ are circles.

Proof. For the (first) shape operator and the second shape operator, we have proved our latter assertion in Lemma 3 and Lemma 4. By the induction on n, we shall prove that the image of a unit tangent circle S^1_p under the n-th shape operators $(1 \leqq n \leqq [\frac{\nu}{2}])$ are circles for every $p \in M$. Now, we assume that the above assertion holds for all $t \leqq s - 1$. Then, we can choose a neighborhood U of a point $p \in M$ in which there exist isothermal coordinates (u, v) and a frame field $b \in B$ satisfying (5. 1) and

$$(5.9)\quad \begin{cases} \omega_{\alpha_1\beta_1} = k_t\omega_1 = \omega_{\alpha_2\beta_2}, & \omega_{\alpha_1\gamma} = \omega_{\alpha_2\gamma} = 0, \\ \omega_{\alpha_1\beta_2} = k_t\omega_2 = -\omega_{\alpha_2\beta_1}, & 2t + 2 < \gamma, \\ \alpha_1 = 2t - 1, \quad \alpha_2 = 2t, & \beta_1 = 2t + 1, \quad \beta_2 = 2t + 2, \\ t = 1, 2, \cdots, s - 1, & \end{cases}$$

where k_t $(1 \leqq t \leqq s - 1)$ are positive differentiable functions on M. Using the structure equations, from (5. 9) we have

$$(5.10)_{s-1}\qquad \omega_{\beta_1\beta_2} = (t + 1)\omega_{12} - (\log \lambda_t)_2\omega_1 + (\log \lambda_t)_1\omega_2,$$

where $\beta_1 = 2t + 1$, $\beta_2 = 2t + 2$, $\lambda_t := k_1 \cdot k_2 \cdots \cdot k_t$ and $d(\log \lambda_t) = \sum_{j=1}^{2} (\log \lambda_t)_j\omega_j$ for $t = 1, 2, \cdots, s - 1$. From (5. 9), we may write

$$\omega_{a_1\gamma} = f_\gamma\omega_1 + g_\gamma\omega_2, \quad a_1 = 2s - 1,$$
$$\omega_{a_2\gamma} = g_\gamma\omega_1 - f_\gamma\omega_2, \quad a_2 = 2s, \quad 2s < \gamma.$$

Hence, for a unit tangent vector $X = e_1\cos\theta + e_2\sin\theta \in M_p$, the s-th shape operator $\widetilde{\varphi}_s$ is written as

$$\widetilde{\varphi}_s(X) = \lambda_{s-1}\{\cos(s + 1)\theta \cdot F_s + \sin(s + 1)\theta \cdot G_s\},$$

where $F_s = \sum_{\gamma > 2s} f_\gamma e_\gamma$ and $G_s = \sum_{\gamma > 2s} g_\gamma e_\gamma$ are normal vector fields on a neighborhood U of $p \in M$. Using the structure equations, from (5. 9) and $(5. 10)_{s-1}$ we can verify the complex-valued function

$$(5.11)\qquad w_{s-1}(z, z) = E^{s+1}\lambda_{s-1}^2(\|G_s\|^2 - \|F_s\|^2) + 2iE^{s+1}\lambda_{s-1}^2 \langle G_s, F_s \rangle$$

is holomorphic in $z = u + iv$. Since λ_{s-1} is a differentiable function on M, as be stated in §2, $|w_{s-1}(z, \bar{z})|^2/E^{2s+2} = \lambda_{s-1}^4\{(\|G_s\|^2 - \|F_s\|^2)^2 + 4\langle G_s, F_s\rangle^2\}$ is a differentiable function on M. Therefore, by the same way as the proof of Lemma 3, we see that at each point $p \in M$ the image of a unit tangent circle S_p^1 to M under the s-th shape operator $\tilde{\varphi}_s$ is a point p or a circle according as T_{s-1}-index$_p M = 0$ or $\neq 0$. Since T_{s-1}-index$_p M \neq 0$ at every point $p \in M$ if $s \leqq \left[\frac{\nu}{2}\right]$ from (C), at each point $p \in M$ we can choose a neighborhood U of a point p in which there exist isothermal coordinates (u, v) and a frame field $b \in B$ satisfying(5.1), (5.4), (5.9) for every $t \leqq s$ and

$$\omega_{a_1 b_1} = k_s\omega_1 = \omega_{a_2 b_2}, \quad \omega_{a_1 r} = \omega_{a_2 r} = 0,$$
$$\omega_{a_1 b_2} = k_s\omega_2 = -\omega_{a_2 b_1}, \quad b_1 = 2s+1, \quad b_2 = 2s+2, \quad 2s+2 < r,$$

where k_s is a positive differentiable function on M.

Thus, it is clear that the geodesic codimension ν of M is even $2m$ (m a positive integer).

Using the structure equations, from (5.4), (5.9) and $(5.10)_m$ we obtain

$$(5.12)\qquad \begin{cases} \Delta(\log \lambda_t) = E\{(t+1)K - 2k_t^2 + 2k_{t+1}^2\}, & t = 1, 2, \cdots, m-1, \\ \Delta(\log \lambda_m) = E\{(m+1)K - 2k_m^2\}, & \end{cases}$$

where $\Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2$. From (5.12) we have

$$(5.13)\qquad \Delta\log(\lambda_1 \cdot \lambda_2 \cdots \lambda_m) = E\left\{\frac{m(m+3)}{2}K - 2k_1^2\right\} = E\left\{\frac{(\nu+2)(\nu+4)}{8}K - c\right\},$$

because $K = c - 2k_1^2$ and $\nu = 2m$. Therefore, if $(\nu+2)(\nu+4)K - 8c$ does not change its sign, $\log(\lambda_1 \cdot \lambda_2 \cdots \lambda_m)$ is a subharmonic or superharmonic function on M, so it must be constant, because M is compact. Hence, $K = 8c/(\nu+2)(\nu+4) = \text{constant} > 0$ and so k_t $(1 \leqq t \leqq m)$ are constant on M. Supposing $K = 1$, from (5.12) we get

$$k_t^2 = (m - t + 1)(m + t + 2)/4 \quad \text{for} \quad t = 1, 2, \cdots, m,$$
$$c = (m+1)(m+2)/2.$$

Let $E_t = e_{2t+1} + ie_{2t+2}$, $t = 0, 1, 2, \cdots, m$. Then, the Frenet formulas of M can be written as follows

$$dx = \frac{1}{h}(\bar{E}_0 dz + E_0 d\bar{z}), \qquad z = u + iv,$$

$$DE_0 = \frac{1}{h} E_0(\bar{z}dz - zd\bar{z}) + \frac{2k_1}{h} E_1 d\,\cdot,$$

$$DE_1 = -\frac{2k_1}{h} E_0 dz + \frac{2}{h} E_1(\bar{z}dz - zd\bar{z}) + \frac{2k_2}{h} E_2 d\bar{z},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$DE_t = -\frac{2k_t}{h} E_{t-1}dz + \frac{t+1}{h} E_t(\bar{z}dz - zd\bar{z}) + \frac{2k_{t+1}}{h} E_{t+1}dz,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$DE_m = -\frac{2k_m}{h} E_{m-1}dz + \frac{m+1}{h} E_m(\bar{z}dz - zd\bar{z}),$$

where D denotes the covariant differentiation of $\hat{M}$ and $h=1+z\bar{z}$. Now, let $\hat{M}$ be a $(2m+2)$-dimensional sphere $S^{2m+2}(R)$ with radius $R=1/\sqrt{c}$ $=\sqrt{2/(m+1)(m+2)}$. We may consider $S^{2m+2}(R)\subset E^{2m+3}$ and put $x=Re_{2m+3}$. By an computation analogous to the one in [7], we can verify that M is a generalized Veronese surface. Thus, we have proved

Theorem 4. *If the assumptions of Theorem* 3 *are satisfied and* $(\nu+2)(\nu+4)K-8c$ *does not change its sign, then* K *is a positive constant on* $\hat{M}$. *Let* $K=1$ *and* $\hat{M}$ *be a* $(\nu+2)$*-dimensional sphere of constant curvature* $(m+1)(m+2)/2$. *Then* M *is a generalized Veronese surface.*

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