ON RICCI CURVATURES AND EQUIVALENCE OF RIEMANNIAN MANIFOLDS

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Introduction

Recently, R. S. Kulkarni has studied in [2] the converse problem of the so-called "theorema egregium" of Gauss, which asserts the sectional curvature is a metric invariant, and proved that a diffeomorphism f of two n-dimensional Riemannian manifolds (M, g) and $(\overline{M}, \overline{g})$ is conformal if it preserves the sectional curvature and the set of non-isotropic points is dense in M. In addition, he has asked whether f is an isometry or not. This formulation in terms of sectional-curvature-preserving diffeomorphisms is one type of the equivalence problem for Riemannian manifolds.

Kulkarni has shown that the answer for the above is affirmative if $n > 3$. When $n = 3$, it is known that if we assume furthermore some global restrictions then f is an isometry (cf. [2], [5], [6]). On the other hand, very recently, S. T. Yau has given in [6] a local counter example of Riemannian 3-manifold for this question.

Now, the Ricci curvature is a typical example of curvature structures of order one as well as the normal curvature of hypersurfaces in a space of constant curvature. Moreover, it plays the same role as the sectional curvature for the above equivalence problem for Riemannian 3-manifolds. In this point of view, we shall ask in the present paper whether a Ricci-curvature-preserving diffeomorphism f is an isometry or not, assuming that the set of non-isotropic points with respect to Ricci curvature is dense in M.

In section 1, we shall prepare some general facts about the conformal change of metrics. In section 2, starting with the Kulkarni's result, we shall obtain several formulas satisfied by any Ricci-curvature-preserving diffeomorphism. In section 3, we shall construct an example of two metrics g and g^* in $\boldsymbol{R}^n$ such that they have the same Ricci curvatures but are non-isometric to each other, which gives a generalization of Yau's one. In section 4, we shall prove a few of theorems which assert the Ricci-curvature-preserving diffeomorphism is an isometry under some further global assumptions.

We shall assume, throughout this paper, that each Riemannian manifold is connected and of dimension $n > 2$, its metric is positive definite, and all manifolds and all diffeomorphisms are of differentiability class C^∞. For the terminology and notation, we generally follow [2] and [5].

I should like to express my hearty gratitude to Prof. R. S. Kulkarni for his kind communications.

1. Preliminaries

In this section, we shall summarize the transformation formulas of some geometric objects under the conformal change of metrics, and recall some well-known facts in Riemannian geometry which will be needed in the proofs to follow (for the details see Eisenhart's book [1]).

Let (M, g) and $(\overline{M}, \overline{g})$ be n-dimensional Riemannian manifolds with metrics g and $\overline{g}$, respectively. A diffeomorphism $f: (M, g) \longrightarrow (\overline{M}, \overline{g})$ is called a conformal one of (M, g) to $(\overline{M}, \overline{g})$ if the induced metric $g^* = f^*\overline{g}$ of $\overline{g}$ by f is related to g by

$$g^* = \frac{1}{\rho^2} g, \tag{1.1}$$

where ρ is a positive-valued function on M and is called the *associated function* of f. If ρ is constant, then we say f is homothetic and if especially ρ is identically equal to unity, then f is called an isometry.

Let us denote by $\mathfrak{X}(M)$ the Lie algebra of differentiable vector fields on M. For the given symmetric bilinear form H, we shall indicate by H_0 the corresponding linear transformation of $\mathfrak{X}(M)$, the tangent bundle of M, defined by $\langle H_0(X), Y \rangle = H(X, Y)$ for any $X, Y \in \mathfrak{X}(M)$, where $\langle \, , \rangle$ denotes the inner product defined by g. Let ∇ be the Riemannian connection with respect to g and $R(X, Y) = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y]$ $(X, Y \in \mathfrak{X}(M))$ be the curvature operator of ∇. Then we recall that the Ricci form is the symmetric bilinear form defined by $\mathrm{Ric}(X, Y) = \mathrm{Trace}\ \{Z \longrightarrow R(X, Z)\, Y\}$ for any $X, Y \in T_m(M)$, the tangent space at a point $m \in M$, and the scalar curvature $\mathrm{Sc}: M \longrightarrow \boldsymbol{R}$ is defined as $\mathrm{Sc} = \mathrm{Trace}\ \mathrm{Ric}_0$. Also we indicate the corresponding quantities with respect to the metric g^* or $\overline{g}$ by asterisking or by bar ovehead, respectively. Then we know that the above quantities with respect to g^* coincide with the induced ones of the corresponding quantities with respect to $\overline{g}$ by f.

Let $G = \mathrm{grad}\ \rho$ be the gradient vector field of ρ with respect to g, and let Hess_ρ be the hessian of ρ with respect to g. It is given by

$$\text{Hess}_\rho(X, Y) = (\nabla_X d\rho)Y = \langle \nabla_X G, Y \rangle, \tag{1.2}$$

for any $X, Y \in \mathfrak{X}(M)$. The Laplacian $\Delta\rho$ of ρ is defined by

$$\Delta\rho = \text{Trace}\,(\text{Hess}_\rho)_0\,. \tag{1.3}$$

Then we know the following transformation formulas under the conformal change (1. 1) of metric g;

$$\text{Ric}^* = \text{Ric} + \frac{1}{\rho}(n-2)\text{Hess}_\rho + \frac{1}{\rho}\Delta\rho\, g - \frac{1}{\rho^2}(n-1)\|G\|^2 g\,, \tag{1.4}$$

$$\text{Sc}^* = \rho^2\text{Sc} + 2(n-1)\rho\Delta\rho - n(n-1)\|G\|^2, \tag{1.5}$$

where we have put $\|G\| = \langle G, G\rangle^{\frac{1}{2}}$.

Let T be the well-known symmetric bilinear form defined by

$$T = \text{Ric} - \frac{1}{n}\text{Sc}\, g, \tag{1.6}$$

which plays an important role in this paper. Evidently we have

$$\text{Trace}\ T_0 = 0\,. \tag{1.7}$$

The manifold (M, g) is called an Einstein manifold if $T \equiv 0$. The Weyl's 3-index tensor D is given by

$$D(X, Y, Z) = (\nabla_X T)(Y, Z) - (\nabla_Y T)(X, Z) + \frac{n-2}{2n(n-1)}\{\langle Y, Z\rangle X(\text{Sc}) - \langle X, Z\rangle Y(\text{Sc})\}\,,$$

for any $X, Y, Z \in \mathfrak{X}(M)$, which vanishes if (M, g) is conformally flat. Suppose $n = 3$, then it holds

$$\begin{aligned} R(X, Y)Z = {} & \text{Ric}(X, Z)Y - \text{Ric}(Y, Z)X + \langle X, Z\rangle \text{Ric}_0(Y) \\ & - \langle Y, Z\rangle \text{Ric}_0(X) - \frac{1}{2}\text{Sc}\{\langle X, Z\rangle Y - \langle Y, Z\rangle X\}, \end{aligned} \tag{1.8}$$

for any $X, Y, Z \in \mathfrak{X}(M)$, from which we know that a 3-manifold (M, g) is of constant curvature if and only if it is Einsteinian. Finally we remark that from the Bianchi's second identity we have

$$\text{Trace}\,\{X \longrightarrow (\nabla_X \text{Ric}_0)Y\} = \frac{1}{2}Y(\text{Sc})\,, \tag{1.9}$$

for any $X, Y \in \mathfrak{X}(M)$.

§ 2. Ricci-curvature-preserving diffeomorphisms

In this section, we shall prepare for later use some basic formulas and lemmas for any Ricci-curvature-preserving diffeomorphism.

First of all, let us recall the general theorem of R. S. Kulkarni on curvature-preserving diffeomorphisms, which is fundamental for this paper (in detail, see [3]). Let ω be a curvature structure of order p on the manifold (M, g). Then we may define the corresponding curvature function K_ω: $G_p \longrightarrow \boldsymbol{R}$, being G_p the Grassmann bundle of p-plane sections of M, by the equation

$$K_\omega(\sigma) = \frac{\omega(\sigma, \sigma)}{\|\sigma\|^2}, \tag{2.1}$$

for any $\sigma \in G_p$, where $\|\cdot\|$ denotes the norm of a p-vector induced by the metric g. We say that a point $m \in M$ is *isotropic* with respct to ω if $K_\omega(\sigma)$ is constant for every p-plane section $\sigma \subset T_m(M)$; otherwise, we call m *non-isotropic*. A diffeomorphism f: $(M, g) \longrightarrow (\overline{M}, \overline{g})$ of two Riemannian manifolds (M, g), $(\overline{M}, \overline{g})$ with curvature structures ω, $\overline{\omega}$ of same order p is called a curvature-preserving one (or simply K_ω-preserving one) if

$$\overline{K}_{\overline{\omega}}(f_* \sigma) = K_\omega(\sigma),$$

for every $\sigma \in G_p$. Then we have (see Part II, § 5 and § 6 in [3])

General Theorem. *Let f : $M \longrightarrow \overline{M}$ be a curvature-preserving diffeomorphism of n-dimensional Riemannian manifolds (M, g, ω), $(\overline{M}, \overline{g}, \overline{\omega})$ where ω, $\overline{\omega}$ are curvature structures of order p $(<n)$. Suppose*

$(*)_\omega$: *the set of non-isotropic points w. r. t. ω is dense.*

Then f is conformal. Furthermore, let ρ be its associated function. Then we have

$$f^*\overline{\omega} = \frac{1}{\rho^{2p}}\omega, \tag{2.2}$$

if ω, $\overline{\omega}$ satisfy the Bianchi's first identity.

Set $p = 1$ and $\omega = \mathrm{Ric}$ in the above theorem. Then the equation (2.1) becomes

$$K_{\mathrm{Ric}}([X]) = \frac{\mathrm{Ric}(X, X)}{\|X\|^2},$$

where $X \neq 0$ represents the line $[X] \in G_1$, being the bundle of lines.

$K_{\mathrm{Ric}}([X])$ is nothing else than the Ricci curvature in the direction X. We shall call (M, g) *nowhere Einsteinian* if there does not exist a non-empty open subset of M, which is an Einstein manifold in the inherited Riemannian metric. Then the condition $(*)_{\mathrm{Ric}}$ is equivalent to

$(*)'_{\mathrm{Ric}}$: *(M, g) is nowhere Einsteinian.*

And also the equation (2. 2) becomes

$$\mathrm{Ric}^* = \frac{1}{\rho^2}\mathrm{Ric}, \tag{2.3}$$

from which we obtain

$$\mathrm{Sc}^* = \mathrm{Sc}. \tag{2.4}$$

Lemma 1. *Let $f: (M, g) \longrightarrow (\overline{M}, \overline{g})$ be a K_{Ric}-preserving homothety and let us assume $(*)_{\mathrm{Ric}}$. Then f is an isometry.*

Proof. Since f is homothetic, we have by the equations (1. 1) and (1. 4)

$$\rho^2 \overline{K}_{\overline{\mathrm{Ric}}}([f_*X]) = K_{\mathrm{Ric}}([X]),$$

so that we get $(\rho^2-1)K_{\mathrm{Ric}}([X])=0$, because f is K_{Ric}-preserving. There exists a non-vanishing $X\in T_m(M)$ such that $K_{\mathrm{Ric}}([X])\neq 0$ at any non-isotropic point $m\in M$ with respect to Ric, hence we have $\rho(m)=1$. By the assumption $(*)_{\mathrm{Ric}}$ and the continuity of ρ we find $\rho\equiv 1$ on M. Thus f is an isometry. q. e. d.

In a previous paper, the present author has obtained from the equations (1. 1) and (2. 3) the following identities (see Lemma 2 in [5]):

$$T_0(G) = 0, \tag{2.5}$$

$$\rho^3 D^*(X, Y, Z) - \rho D(X, Y, Z) = (Y\rho)T(X, Z) - (X\rho)\, T(Y, Z), \tag{2.6}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

Now we shall show that the equation (2. 5) implies the following two important identities. First, we find easily from the equations (1. 1), (1. 4), (1. 5) and (1. 6) that the tensor T is transformed under the conformal change (1. 1) of metric g as follows:

$$T^* = T + \frac{1}{\rho}(n-2)\left(\mathrm{Hess}_\rho - \frac{1}{n}\Delta\rho\, g\right).$$

On the other hand, we obtain by the equations (2. 3) and (2. 4)

$$T^* = \frac{1}{\rho^2} T.$$

Eliminating T^* from these two equations, we have by (1.2)

$$\nabla_X G = \frac{(1-\rho^2)}{\rho(n-2)} T_0(X) + \frac{1}{n} \Delta\rho\, X, \tag{2.7}$$

for any $X \in \mathfrak{X}(M)$. Put $X = G$ in the above, then we find by the equation (2.5) that the associated function ρ of any K_{Ric}-preserving conformal diffeomorphism satisfies the identity

$$\nabla_G G = \frac{1}{n} \Delta\rho\, G, \tag{2.8}$$

which means the trajectories of the gradient vector field G of ρ are geodesic arcs in a neighborhood of an ordinary point of ρ.

Next, owing to the equation (2.5) we have

$$(\nabla_X T_0)G = -T_0(\nabla_X G),$$

for any $X \in \mathfrak{X}(M)$. In each hand side of the above, it holds

$$\begin{aligned}
\mathrm{Trace}\ \{X \longrightarrow (\nabla_X T_0)G\} &= \mathrm{Trace}\ \{X \longrightarrow (\nabla_X \mathrm{Ric}_0)G\} \\
&\quad - \frac{1}{n} \mathrm{Trace}\ \{X \longrightarrow X(\mathrm{Sc})G\} && \text{(by (1.6))} \\
&= \frac{1}{2} G(\mathrm{Sc}) - \frac{1}{n} G(\mathrm{Sc}) && \text{(by (1.9))} \\
&= \frac{n-2}{2n} G(\mathrm{Sc})
\end{aligned}$$

and

$$\mathrm{Trace}\ \{X \longrightarrow -T_0(\nabla_X G)\} = \frac{(\rho^2-1)}{\rho(n-2)} \mathrm{Trace}\ (T_0^2) \qquad \text{(by (2.7))},$$

so that, equating these, we have

$$\rho(n-2)^2 \langle G, \mathrm{grad(Sc)} \rangle - 2n(\rho^2-1)\|T\|^2 = 0, \tag{2.9}$$

because $\mathrm{Trace}\ (T_0^2) = \|T\|^2$, where $\|\cdot\|$ denotes the canonical norm induced by the metric g in the tensor algebra of (M, g).

Finally, set $p = 2$ and $\omega = \widetilde{R}$ in General Theorem, where $\widetilde{R}$ denotes the tensor field of type (0, 4) defined by $\widetilde{R}(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle$ for any $X, Y, Z, W \in \mathfrak{X}(M)$. Then the equation (2.1) becomes

$$K_{\tilde{R}}(\sigma) = \frac{\langle R(X, Y)X, Y\rangle}{\|X\wedge Y\|^2},$$

for any $\sigma = \{X, Y\} \in G_2$, which is nothing else than the sectional curvature of the 2-plane section σ. We say (M, g) *nowhere of constant curvature* if there does not exist a non-empty open subset of M, which is a manifold of constant curvature in the inherited Riemannian metric. Then the condition $(*)_{\tilde{R}}$ is equivalent to

$(*)'_{\tilde{R}}$: (M, g) *is nowhere of constant curvature.*

Lemma 2. *Suppose* $n = 3$. *Then it holds*

(α) $(*)_{\mathrm{Ric}} \Longleftrightarrow (*)_{\tilde{R}}$, *and*

(β) K_{Ric}*-preserving* $\Longleftrightarrow$ $K_{\tilde{R}}$*-preserving, for any conformal diffeomorphism.*

Proof. Any Riemannian 3-manifold is of constant curvature if and only if it is Einsteinian, hence we find that $(*)'_{\mathrm{Ric}} \Longleftrightarrow (*)'_{\tilde{R}}$, from which we have ($\alpha$). To check ($\beta$), let $\{e_1, e_2, e_3\}$ be any orthogonal basis of the tangent space $T_m(M)$ at any point $m \in M$. Then, for the 2-plane sections $\sigma_k = \{e_i, e_j\}$ $(k = 1, 2, 3)$ where i, j and k are distinct, we have by the equation (1. 8)

$$\langle R(e_i, e_j)e_i, e_j\rangle = \mathrm{Ric}(e_i, e_i)\|e_j\|^2 + \mathrm{Ric}(e_j, e_j)\|e_i\|^2 - \frac{1}{2}\mathrm{Sc}\|e_i\|^2\|e_j\|^2,$$

so that we get

$$K_{\tilde{R}}(\sigma_k) = \frac{1}{2}\{K_{\mathrm{Ric}}([e_i]) + K_{\mathrm{Ric}}([e_j]) - K_{\mathrm{Ric}}([e_k])\},$$

or equivalently we have

$$K_{\mathrm{Ric}}([e_k]) = K_{\tilde{R}}(\sigma_i) + K_{\tilde{R}}(\sigma_j).$$

Since the orthogonality of vectors is preserved under any conformal change of metrics, we have the relation (β). q. e. d.

Remark. In Lemma 2, we can prove (β) more easily by making use of the equation $\tilde{R}^* = \frac{1}{\rho^4}\tilde{R}$, which is obtained by setting $p = 2$ and $\omega = \tilde{R}$ in the equation (2. 2). However, the above relation between the sectional curvatures and the Ricci curvatures in Riemannian 3-manifolds is helpful for the next section.

3. A generalization of Yau's example

S. T. Yau has given in [6] a local example of Riemannian 3-manifolds with only non-isotropic points with respect to $\widetilde{R}$ where a sectional-curvature-preserving diffeomorphism is not an isometry. On account of Lemma 2, we may regard it as a local counter example for our equivalence problem in terms of the Ricci curvature. From this point of view, let us generalize it to n-dimensional manifolds.

Following Yau, let the metric g in R^n be given in the form

$$ds^2 = \sum_{i=1}^{n-1} e^{2h_i}(dx^i)^2 + (dx^n)^2\,, \tag{3.1}$$

where $(x^1, x^2, \ldots, x^n)$ is the natural coordinate system of R^n and each $h_i\ (i=1, 2, \ldots, n-1)$ is a function of x^n alone. Let g^* be an another metric in R^n defined by the equation (1.1) in which ρ is also a function of x^n alone. We denote by prime ordinary differentiation with respect to x^n. In the following we shall show that we can choose, in the local, the functions $h_1, h_2, \ldots, h_{n-1}$ and ρ, such that

$$\begin{cases} \text{(a)} & \rho' \neq 0 \ \text{ and } \ \rho \neq 1, \\ \text{(b)} & K^*_{\mathrm{Ric}^*} = K_{\mathrm{Ric}}\,, \\ \text{(c)} & \text{all points are non-isotropic with respect to Ric.} \end{cases}$$

First, let us compute the Ricci form, the hessian of ρ and others with respect to g. We assume that the indices i and j run over the range $\{1, 2, \ldots, n-1\}$ and the indices λ, μ, ν and κ run over the range $\{1, 2, \ldots, n\}$. Let $\{E_\lambda\}$ be an orthonormal frame field over R^n defined by

$$E_i = e^{-h_i}\,\partial/\partial x^i \ \text{ and } \ E_n = \partial/\partial x^n\,.$$

The Christoffel's symbols of the Riemannian connection ∇ of g with respect to the natural frame field $\{\partial/\partial^\lambda\}$ which are defined by

$$\nabla_{\partial/\partial x^\nu}(\partial/\partial x^\mu) = \sum_{\lambda=1}^{n} \left\{{\lambda \atop \mu\nu}\right\} \partial/\partial x^\lambda\,,$$

are given by

$$\left\{{n \atop i\ i}\right\} = -e^{2h_i}\, h_i' \ \text{ and } \left\{{i \atop i\ n}\right\} = h_i'\,, \tag{3.2}$$

the other symbols being zero. Hence the components of curvature tensor

R with respect to $\{\partial/\partial x^\lambda\}$ which are defined by

$$R(\partial/\partial x^\nu, \partial/\partial^\mu)\partial/\partial x^\lambda = \sum_{\kappa=1}^{n} R_{\nu\mu\lambda}{}^\kappa \partial/\partial x^\kappa ,$$

are given by

$$\begin{aligned} R_{iji}{}^j &= -e^{2h_i} h_i' h_j' \qquad (i \neq j), \\ R_{ini}{}^n &= -e^{2h_i}\{h_i'' + (h_i')^2\} , \\ R_{nin}{}^i &= -\{h_i''+(h_i')^2\} , \end{aligned} \tag{3.3}$$

the components of another types being zero. Thus we find that the Ricci form is given by

$$\begin{aligned} \text{Ric}\,(E_\lambda,\ E_\mu) &= 0 \qquad (\lambda \neq \mu), \\ \text{Ric}\,(E_i,\ E_i) &= -\{h_i'' + h_i' \sum_{j=1}^{n-1} h_j'\}, \\ \text{Ric}\,(E_n,\ E_n) &= -\sum_{i=1}^{n-1} \{h_i'' + (h_i')^2\}. \end{aligned} \tag{3.4}$$

As for the gradient vector field G and the hessian of ρ with respect to the metric g, we have

$$G = \rho' E_n , \tag{3.5}$$

and by the equation (3. 2)

$$\text{Hess}_\rho\,(E_\lambda,\ E_\mu) = \begin{cases} 0 & (\lambda \neq \mu) \\ h_i'\rho' & (\lambda = \mu = i) \\ \rho'' & (\lambda = \mu = n) \end{cases} \tag{3.6}$$

respectively, so that we obtain by the equation (1. 3)

$$\Delta\rho = \rho'' + \rho' \sum_{i=1}^{n-1} h_i'. \tag{3.7}$$

Now the conditions (b) and (c) can be expressed in terms of the functions $\{h_i\}$ and ρ as follows. First, owing to the equation (1. 1) we see that (b) $\Longleftrightarrow$ (2. 3), and furthermore by (1. 4) the latter is equivalent to the equation

$$\begin{aligned} (1-\rho^2)\,\text{Ric}(X,\ Y) = \rho(n-2)\,\text{Hess}_\rho(X,\ Y) + \rho\Delta\rho\langle X,\ Y\rangle \\ -(n-1)\|G\|^2 \langle X,\ Y\rangle, \end{aligned}$$

for any vector fields $X,\ Y \in \mathfrak{X}(M)$. By the equations (3. 5), (3. 6) and (3. 7) this can be rewritten

$$(3.8)\quad \begin{aligned} &(1-\rho^2)\mathrm{Ric}(E_i, E_i) = \rho\rho'' + \rho\rho'\{(n-2)h_i' + \sum_{j=1}^{n-1} h_j'\} - (n-1)\rho'^2 \\ &(1-\rho^2)\mathrm{Ric}(E_n, E_n) = (n-1)\rho\rho'' + \rho\rho' \sum_{j=1}^{n-1} h_j' - (n-1)\rho'^2. \end{aligned}$$

By substituting (3.4) into (3.8) we get

$$(3.9)\quad \begin{aligned} \mathfrak{F}_i \equiv (1-\rho^2)\ &\{h_i'' + h_i' \sum_{j=1}^{n-1} h_j'\} + \rho\rho'' \\ &+ \rho\rho' \{ \sum_{j=1}^{n-1} h_j' + (n-2)h_i'\} - (n-1)\rho'^2 = 0, \\ \mathfrak{F}_n \equiv (1-\rho^2) &\sum_{j=1}^{n-1}\{h_j'' + h_j'^2\} + (n-1)\rho\rho'' \\ &+ \rho\rho' \sum_{j=1}^{n-1} h_j' - (n-1)\rho'^2 = 0. \end{aligned}$$

Next, suppose that a point $m \in \boldsymbol{R}^n$ is isotropic with respect to Ric, that is, $K_{\mathrm{Ric}}([X]) = \mathrm{const.}$ for every non-vanishing vector $X \in T_m(M)$. Then we have easily by the equation (3.8)

$$h_1' = h_2' = \cdots = h_{n-1}' = \frac{\rho''}{\rho'},$$

at m, if $\rho'(m) \neq 0$. Accordingly, in order to be valid (c) it suffices that there exist some indices i and j such that

$$(3.10)\qquad h_i' \neq h_j', \quad \text{everywhere}.$$

Thus, to construct our example it suffices to show that we can choose n functions $\{h_i\}$ and ρ which satisfy simultaneously the conditions (a), (3.9) and (3.10).

Beforehand, let us recall that the associated function ρ of any K_{Ric}-preserving conformal diffeomorphism satisfies the equation (2.8), which can be rewritten in the present case

$$(3.11)\qquad (n-1)\rho'' = \rho' \sum_{i=1}^{n-1} h_i',$$

by the equations (3.2), (3.5) and (3.7), if $\rho' \neq 0$. Assume (3.11), then we obtain by (3.9)

$$\begin{aligned} \sum_{i=1}^{n-1} \mathfrak{F}_i = (1-\rho^2)\ &\{ \sum_{i=1}^{n-1} h_i'' + (\sum_{i=1}^{n-1} h_i')^2\} + (n-1)\rho\rho'' \\ &+ (2n-3)\rho\rho' \sum_{i=1}^{n-1} h_i' - (n-1)^2\rho'^2 \end{aligned}$$

$$= (n-1)[(1-\rho^2)\rho'^{-2}\{\rho'''\rho' + (n-2)\rho''^2\} + (n-1)(2\rho\rho''-\rho'^2)] \qquad \text{(by (3.11))}.$$

Hence, when constructing our example, we may suppose the function ρ satisfies the differential equation

$$\rho'''\rho' + (n-2)\rho''^2 + \frac{n-1}{1-\rho^2}\rho'^2(2\rho\rho'' - \rho'^2) = 0\,. \tag{3.12}$$

To simplify our argument, let us set especially

$$h = h_1, \quad \text{and} \quad k = h_2 = \cdots = h_{n-1}\,,$$

where the functions h and k, of course, depend on x^n alone. Then the equations (3.9), (3.10) and (3.11) are transformed as follows, respectively. First, the equation (3.9) becomes

$$(3.9)' \quad \begin{aligned} \mathfrak{A} &\equiv (1-\rho^2)\{h'' + h'^2 + (n-2)h'k'\} + \rho\rho'' \\ &\quad + \rho\rho'\{(n-1)h' + (n-2)k'\} - (n-1)\rho'^2 = 0\,, \\ \mathfrak{B} &\equiv (1-\rho^2)\{k'' + h'k' + (n-2)k'^2\} + \rho\rho'' \\ &\quad + \rho\rho'\{h' + 2(n-2)k'\} - (n-1)\rho'^2 = 0\,, \\ \mathfrak{C} &\equiv (1-\rho^2)\{h'' + (n-2)k'' + h'^2 + (n-2)k'^2\} + (n-1)\rho\rho'' \\ &\quad + \rho\rho'\{h' + (n-2)k'\} - (n-1)\rho'^2 = 0\,, \end{aligned}$$

where we have put $\mathfrak{A} = \mathfrak{F}_1$, $\mathfrak{B} = \mathfrak{F}_2 = \cdots = \mathfrak{F}_{n-1}$ and $\mathfrak{C} = \mathfrak{F}_n$. With reference to Remark in section 2, we put

$$\begin{aligned} (n-2)\mathfrak{P} &= \mathfrak{A} + (n-2)\mathfrak{B} - \mathfrak{C}, \\ (n-2)\mathfrak{Q} &= -\mathfrak{A} + (n-2)\mathfrak{B} + \mathfrak{C}, \\ \mathfrak{R} &= \mathfrak{A} - (n-2)\mathfrak{B} + \mathfrak{C}\,. \end{aligned}$$

Then we find that the equation (3.9)′ is equivalent to

$$(3.9)'' \quad \begin{aligned} \mathfrak{P} &\equiv (1-\rho^2)\{2h'k' + (n-3)k'^2\} + 2\rho\rho'\{h' + (n-2)k'\} - (n-1)\rho'^2 = 0, \\ \mathfrak{Q} &\equiv (1-\rho^2)\{2k'' + (n-1)k'^2\} + 2\rho\rho'' + 2(n-2)\rho\rho'k' - (n-1)\rho'^2 = 0, \\ \mathfrak{R} &\equiv (1-\rho^2)\{2h'' + 2h'^2 - (n-2)(n-3)k'^2\} + 2\rho\rho'' \\ &\quad + 2\rho\rho'\{h' - (n-2)(n-3)k'\} + (n-1)(n-4)\rho'^2 = 0. \end{aligned}$$

The equations (3.10) and (3.11) become evidently

$$h' \neq k', \quad \text{everywhere}, \tag{3.10$'$}$$

$$(n-1)\rho'' = \rho'\{h' + (n-2)k'\}, \tag{3.11$'$}$$

respectively. Moreovere, suppose (a) then we can solve h' and k' from the equations $(3.11)'$ and $\mathfrak{P} = 0$. An elementary calculation shows that a necessary and sufficient condition in order to be valid $(3.10)'$ is

$$(3.13) \qquad \left(\frac{\rho''}{\rho'}\right)^2 + \frac{2\rho\rho''-\rho'^2}{1-\rho^2} > 0 .$$

Thus our problem has been reduced finally to showing that we can choose three functions h, k and ρ under the condition (a) such that the equations $(3.9)''$, $(3.11)'$ and (3.13) hold simultaneously.

Now, let us solve the equation $(3.9)''$. We reverse the argument and let ρ be a solution of the differential equation (3.12) satisfying

$$0 < \rho < 1, \quad \rho' \neq 0 \ \text{ and } \ (3.13),$$

at some initial point m_0. The existence theorem of the ordinary differential equations assures such a solution ρ exists in a sufficiently small neighborhood of $x^n(m_0)$. Having chosen ρ, let ϕ be be a solution of the differential equation, which is obtained from $\mathfrak{Q} = 0$ by setting $k' = \phi$,

$$(3.14) \qquad \begin{aligned} &(1-\rho^2)\{2\frac{d\phi}{dx^n} + (n-1)\phi^2\} + 2\rho\rho'' \\ &\qquad + 2(n-2)\rho\rho'\phi - (n-1)\rho'^2 = 0, \end{aligned}$$

satisfying the initial condition

$$(3.15) \qquad \phi^2 - \frac{2\rho''}{\rho'}\phi - \frac{1}{1-\rho^2}(2\rho\rho'' - \rho'^2) = 0 \quad \text{at } m_0 .$$

Since we have the inequality (3.13) at m_0, we know that such a solution ϕ exists in a sufficiently small neighborhood of $x^n(m_0)$ in consequence of the existence theorem of ordinary differential equations. Let k be a function of x^n such that $k' = \phi$. Finally, let h be any function of x^n such that

$$(3.16) \qquad h' = \frac{(n-1)\rho''}{\rho'} - (n-2)k' .$$

We remark that the initial condition (3.15) means now

$$(3.17) \qquad \mathfrak{P} = 0 \ \text{ at } m_0,$$

because by substituting (3.16) into the expression $\mathfrak{P}$ we have

$$\mathfrak{P} = (n-1)(\rho^2-1)\{k'^2 - \frac{2\rho''}{\rho'}k' - \frac{1}{1-\rho^2}(2\rho\rho''-\rho'^2)\} .$$

Therefore, we see that $(3.10)'$ holds in a neighborhood of $x^n(m_0)$ from our choice of h, k and ρ.

In the following, we shall prove that the functions h, k and ρ defined in the above satisfy identically the equation (3. 9)''. Since $\mathfrak{Q} \equiv 0$ in (3. 9)'' is automatically satisfied, it suffices to prove that $\mathfrak{P} \equiv \mathfrak{R} \equiv 0$. First, we have

$$\mathfrak{R} = -2(n-2)\mathfrak{P}. \tag{3.18}$$

In fact, eliminate two functions ρ''' and k'' from the equation which is obtained by differentiating (3. 16), by making use of two equations (3. 12) and $\mathfrak{Q} = 0$, respectively. And then substitute the resulting relation into the expression $\mathfrak{R}$. Then we find (3. 18) by straightforward calculations. On the other hand, we get directly

$$\frac{d\mathfrak{P}}{dx^n} = 2(1-\rho^2)h''k' + 2(1-\rho^2)\{h'+(n-3)k'\}k'' - 2\rho\rho'\{2h'k'+(n-3)k'^2\}$$
$$+2\rho\rho'\{h''+(n-2)k''\} + 2(\rho'^2+\rho\rho'')\{h'+(n-2)k'\} - 2(n-1)\rho'\rho''.$$

Eliminating h'' and k'' from the above by using the expression $\mathfrak{R}$ and the equation $\mathfrak{Q} = 0$, respectively, we find by (3. 16)

$$\frac{d\mathfrak{P}}{dx^n} = k'\mathfrak{R} + \frac{\rho\rho'}{1-\rho^2}\mathfrak{R} - (1-\rho^2)(h'+k')\{2h'k'+(n-3)k'^2\}$$
$$-2\rho\rho'\{(n+1)h'k' + h'^2 + (2n-5)k'^2\} + (n-1)\rho'^2(h'+k')$$
$$-\frac{2(n-1)}{1-\rho^2}\rho\rho'(2\rho\rho'' - \rho'^2),$$

from which we obtain furthermore by using the expression $\mathfrak{P}$

$$\frac{d\mathfrak{P}}{dx^n} = k'\mathfrak{R} + \frac{\rho\rho'}{1-\rho^2}\mathfrak{R} - (h'+k')\mathfrak{P} - 2\rho\rho'\{2h'k'+(n-3)k'^2\}$$
$$-\frac{2(n-1)}{1-\rho^2}\rho\rho'(2\rho\rho'' - \rho'^2)$$
$$= k'\mathfrak{R} + \frac{\rho\rho'}{1-\rho^2}\mathfrak{R} - (h'+k')\mathfrak{P} - \frac{2\rho\rho'}{1-\rho^2}\mathfrak{P}.$$

Substituting (3. 18) into the above, we get

$$\frac{d\mathfrak{P}}{dx^n} = -\{h' + (2n-3)k' + \frac{2(n-1)}{1-\rho^2}\rho\rho'\}\mathfrak{P}. \tag{3.19}$$

The uniqueness theorem of the ordinary differential equations and the initial condition (3. 17) imply $\mathfrak{P} \equiv 0$ from (3. 19), hence it holds $\mathfrak{R} \equiv 0$ by the equation (3. 18).

Thus we have proved our three functions ρ, h and k defined in a sufficiently small neighborhood of $x^n(m_0)$ satisfy identically the conditions (a), (3. 9)$''$ and (3. 10)$'$. Hence we conclude

Theorem 1. *There exist, in the local, two metrics g and g^* in $\boldsymbol{R}^n$ $(n>2)$ such that*

$$\begin{cases} (1) & g^* = \dfrac{1}{\rho^2} g, \text{ where } d\rho \neq 0 \text{ and } \rho \neq 1, \\ (2) & K^*_{\mathrm{Ric}^*} = K_{\mathrm{Ric}}, \\ (3) & \text{all points are non-isotropic with respect to Ric}. \end{cases}$$

4. Theorems

In this section, we shall give several affirmative answers under some further global hypothesis for the equivalence problem of Riemannian manifolds in terms of K_{Ric}-preserving diffeomorphisms. Throughout this section, we assume:

(A) *Let $f:(M, g) \longrightarrow (\overline{M}, g)$ be a K_{Ric}-preserving diffeomorphism and the condition $(*)_{\mathrm{Ric}}$ be satisfied.*

First, we have

Theorem 2. *Under the assumption* (A), *f is an isometry if (M, g) satisfies any one of the following conditions:*

(i) *(M, g) is conformally flat.*

(ii) *The scalar curvature* Sc *of (M, g) is constant.*

Proof. Suppose (i) holds. Then the manifold (M, g^*) is also conformally flat, so that we have $D^* \equiv D \equiv 0$. Hence, the equation (2. 6) gives

$$(X\rho)T(Y, Z) - (Y\rho)T(X, Z) = 0,$$

for any $X, Y, Z \in \mathfrak{X}(M)$. Set $X = G$ in the above, then we have by the equation (2. 5)

$$\|G\|^2\, T(Y, Z) = 0,$$

for any $Y, Z \in \mathfrak{X}(M)$. This implies $T = 0$ in a neighborhood of an ordinary point of ρ, if there is any. This fact contradicts the assumption $(*)_{\mathrm{Ric}}$. Thus $G \equiv 0$ on M, that is to say, f is homothetic. Hence f is an isometry by Lemma 1.

Next, suppose (ii) holds. Then the equation (2. 9) implies $T = 0$ in a neighborhood of a point $m\in M$ such that $\rho(m)\neq 1$, if there is any. This contradicts the assumption $(*)_{\mathrm{Ric}}'$. Thus we have $\rho\equiv 1$ on M.

q. e. d.

R. S. Kulkarni has proved independently the above theorem in the case (i) (see Theorem 2 in [4]).

Next, we have

Theorem 3. *Under the assumption* (A), *suppose that* M *is compact and all the points of* M *are non-isotropic with respect to* Ric. *Then* f *is an isometry.*

Proof. On the contrary, assume $\rho\not\equiv 1$ on M. Then, considering the maximum or minimum point of ρ, we may suppose that there exists a point $m\in M$ such that $\rho(m)\neq 1$ and $\|G\|_m=0$. From the equation (2. 9), we find at once $T=0$ at m. This says that the point m is isotropic. This fact contradicts the assumption of Theorem 3. Thus $\rho\equiv 1$ on M.

q. e. d.

Suppose $n=3$ in Theorem 3. Then, we may replace the Ricci curvature by the sectional curvature in consequence of Lemma 2. In this meaning, Theorem 3 generalizes S. T. Yau's result (see Theorem 3 in [6]).

Finally, modifying the technique developed in the proof of Theorem 6 in [2], we obtain the following

Theorem 4. *Under the assumption* (A), f *is an isometry if the manifold* M *is compact and for every point* $m\in M$ *it holds*

$$r\,\mathrm{Sc}-s\,K_{\mathrm{Ric}}([X])<0, \tag{4. 1}$$

for some non-vanishing $X\in T_m(M)$, *where* r *and* s *are arbitrary constants such that*

$$\mathrm{Min}\,\{r,\,r\,(n-1)\}\geqq\frac{s}{2}\,. \tag{4. 2}$$

Proof. To prove Theorem 4, we introduce symmetric bilinear forms S and $\overline{S}$ defined by

$$S=r\mathrm{Sc}\,g-s\,\mathrm{Ric}\ \text{ and }\ \overline{S}=r\overline{\mathrm{Sc}}\,\overline{g}-s\,\overline{\mathrm{Ric}},$$

on $(M,\,g)$ and $(\overline{M},\,\overline{g})$, respectively, where r and s are the arbitrary constants satisfying (4. 2). Then we have $K_S([X])=r\,\mathrm{Sc}-sK_{\mathrm{Ric}}([X])$ for any non-vanishing $X\in T_m(M)$. It follows from the equations (1. 1), (2. 3) and (2. 4) that

$$S^* = \frac{1}{\rho^2} S, \tag{4.3}$$

which implies f is also K_S-preserving. On the other hand, by a straightforward computation we can derive from the equations (1. 1), (1. 4) and (1. 5)

$$\begin{aligned} S^* = S - \frac{1}{\rho^2} [\rho s(n-2)\mathrm{Hess}_\rho + \rho\Delta\rho\{s - 2r(n-1)\} g \\ + (n-1)(nr - s)\|G\|^2 g]. \end{aligned} \tag{4.4}$$

Eliminating S^* from the equations (4. 3) and (4. 4), we have

$$\begin{aligned} (\rho^2 - 1) K_S([X]) = \rho s(n-2)\, \mathrm{Hess}_\rho \left(\frac{X}{\|X\|}, \frac{X}{\|X\|} \right) \\ + \rho\Delta\rho\{s - 2r(n-1)\} + (n-1)(nr - s)\|G\|^2 \end{aligned} \tag{4.5}$$

for any non-vanishing $X \in T_m(M)$.

On the contrary, suppose f is non-isometric, that is, $\rho \not\equiv 1$ on M. Then we may assume that the maximum value $\rho(m_0)$ of ρ is greater than unity. In fact, since f is K_S-preserving, we can consider f^{-1} which has the associated function $\frac{1}{\rho}$, if necessary. Let $X \in T_{m_0}(M)$ be the tangent vector satisfying (4. 1) or equivalently $K_S([X]) < 0$. Let $\{e_1 = \frac{X}{\|X\|}, e_2, \ldots\ldots, e_n\}$ be an orthonormal basis of $T_{m_0}(M)$. Since $\|G\|_{m_0} = 0$, we have from the equations (1. 3) and (4. 5)

$$\begin{aligned} (\rho^2 - 1) K_S([X]) = \rho(n-1)(s-2r)\, \mathrm{Hess}_\rho(e_1, e_1) \\ + \rho\{s - 2r(n-1)\} \sum_{i=2}^{n} \mathrm{Hess}_\rho(e_i, e_i). \end{aligned} \tag{4.6}$$

The hessian Hess_ρ is equal to the usual one at any critical point of ρ, so the form Hess_ρ is negative semi-definite at the maximum point m_0 of ρ. Thus, it follows from (4. 2) that the right hand side of (4. 6) is non-negative. On the other hand, owing to the equation (4. 1) the left hand side of (4. 6) is negative, because $\rho(m_0) > 1$. This is a contradiction. Hence we have $\rho \equiv 1$ on M. q. e. d.

Setting $r = 0$ and $s = -1$ in Theorem 4, we have

Corollary. *Under the assumption* (A), *f is an isometry if M is compact and for every point $m \in M$ it holds* $\mathrm{Ric}(X, X) < 0$ *for some* $X \in T_m(M)$.

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(*Received April 28, 1973*)