ON TORSION FREE MODULES OVER REGULAR RINGS II

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Throughout this paper we assume that all rings considered are commutative rings with identity and all modules are unital. As to undefined terms, we follow [2] or [7].

For a ring R, we denote its maximal ring of quotients by $Q(R)$. If R is semi-prime, then R has a unique minimal Baer ring of quotints which is called by Mewborn the Baer hull of R, and denoted by $C(R)$. It is known that $C(R)$ coincides with the ring generated by the set of all idempotents of $Q(R)$ over R ([4, Proposition 2.5]).

An R-module M is said to be torsion free if $\{x \in M \mid \mathrm{Hom}_R(Rx, I(R)) = 0\} = 0$, where $I(R)$ is the injective hull of R as an R-module. For a torsion free R-module M, we shall consider again the following conditions which are cited in [6]:

(α) M is a direct sum of cyclic R-submodules.

(β) M is isomorphic to an essential submodule of a direct sum of cyclic (torsion free) R-modules.

(γ) M is isomorphic to a submodule of a direct sum of cyclic torsion free R-modules.

In case R is semi-prime and $C(R) = Q(R)$, a torsion free R-module M is finitely generated and injective if and only if $M \simeq R/J_1 \oplus \cdots \oplus R/J_n$ (as a module), where J_i is an ideal of R such that R/J_i is self-injective ([6, Theorem 3.10]). On the other hand, given a Boolean space X and a finite field F, the ring R of global sections of the simple F-sheaf over X determines completely every finitely generated injective R-module as above ([7, Theorem 23.5]). Therefore it is natural to ask if $C(R)$ coincides with $Q(R)$. In § 2, we shall answer the question in the affirmative (Theorem 2.4).

Let X be a topological space, x in X, and ξ an arbitrary ordinal number. Following Pierce, x is called a ξ-point if there is a collection $\{U_\eta \mid \eta < \xi\}$ of pairwise disjoint open subsets of X such that $x \in U_\eta^- - U_\eta$, where U_η^- means the closure of U_η in X. The following question has been asked by Pierce [7, p. 109]: What characterizes those Boolean rings whose corresponding Boolean spaces contain no 3-points? In § 3, we shall give several characterizations of such Boolean rings (Theorem 3.3). Finally, we shall present an example of a Boolean ring R such that every non-isolated point of Spec(R) is an n-point but not an $(n+1)$-point.

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1. Preliminaries

Let R be a ring. $B(R)$ will represent the Boolean ring consisting of all idempotents of R, and $X(R)$ the spectrum of $B(R)$ consisting of all prime ideals of $B(R)$. Let x be a point of $X(R)$. Then, for every element e in x, $U_e^R = \{y \in X(R) \mid e \in y\}$ is a neighborhood of x and these neighborhoods form a basis of open subsets in $X(R)$. With this topology, $X(R)$ becomes a Boolean space (that is, a totally disconnected compact Hausdorff space) (see [7]). Furthermore, we can define the Pierce sheaf $\mathfrak{R}(R)$ whose base space is $X(R)$ and whose stalks are R/xR for $x \in X(R)$. Then, R is isomorphic to the ring of global sections of $\mathfrak{R}(R)$, and moreover the category of all R-modules is equivalent to that of all sheaves of $\mathfrak{R}(R)$-modules over $X(R)$.

Let X be a topological space, and R a ring. Regarding R as a topological space with discrete topology, the product space $X \times R$ becomes a sheaf over X with $x \times R$ $(x \in X)$ as its stalks, which is called by Pierce the simple R-sheaf over X. It is easy to see that if $r \in R$ then the mapping X_r given by $x \longrightarrow x \times r$, $x \in X$, is a section of the simple R-sheaf $X \times R$. Let $\mathfrak{R}$ be a sheaf of rings over a space X. Then $\Gamma(X, \mathfrak{R})$ will represent the ring of global cross sections of $\mathfrak{R}$. One may remark here that if X is a Boolean space, and $\mathfrak{R}$ a sheaf of fields over X, then $B(\Gamma(X, \mathfrak{R})) = \{\sigma \in \Gamma(X, \mathfrak{R}) \mid \sigma(x) = 0_x \text{ or } 1_x \text{ for every } x \text{ in } X\} = \{X_M \mid M$ is an open-closed subset of $X\}$, where 0_x and 1_x are respectively the zero element and the identity of the stalk for x, and X_M is the section given by $X_M(x) = \begin{cases} 1_x & (x \in M) \\ 0_x & (x \notin M) \end{cases}$.

Lemma 1.1. ([5]). *Let R be a regular ring, and $x \in X(R)$. Then, x is an isolated point in $X(R)$ if and only if the maximal ideal xR of R is a direct summand of R.*

Lemma 1.2. *Let R be a regular ring. If M is a non-empty open subset of $X(Q(R))$, then there exists an idempotent r in R such that $U_r^{Q(R)}$ is a non-empty subset of M.*

Proof. Since M is a non-empty open subset, there exists an idempotent e in $Q(R)$ such that $U_e^{Q(R)}$ is non-empty and contained in M. Then, $e \neq 1$, and hence $0 \neq s(1-e) \in R$ for some idempotent s in R. To be easily seen, $r = 1 - s(1 - e)$ is the one requested.

Lemma 1.3. *Let R be a regular ring. If y is a non-isolated point of $X(Q(R))$ and contains an idempotent e, then there exists an idempotent r in R such that $r \notin y$ and $U_r^{Q(R)}$ is a non-empty subset of $U_e^{Q(R)}$.*

Proof. By Lemma 1.1, there exists an idempotent s in R such that $0 \neq s(1 - e) \in yQ(R) \cap R$. To be easily seen, $r = 1 - s(1 - e)$ is the one requested.

2. Simple F-sheaf

Proposition 2.1. *Let X be a Boolean space, and F a field. Then, every element σ in $\Gamma(X, X \times F)$ can be expressed in the form*

$$\sigma = \sigma_1 X_{f_1} + \cdots + \sigma_r X_{f_r}$$

where $\{\sigma_1, \cdots, \sigma_r\}$ is a set of orthogonal idempotents of $\Gamma(X, X \times F)$ and $\{f_1, \cdots, f_r\}$ is a subset of F.

Proof. Let $\sigma \in \Gamma(X, X \times F)$, and $x \in X$. Then, $\sigma(x) = X_f(x) = x \times f$ for some f in F. Hence, by [7, p. 11], there is a neighborhood M of x in X such that $\sigma(y) = X_f(y)$ for all $y \in M$. By making use of the partition property (see [7, p. 12]), we obtain a finite family $\{M_1, \cdots, M_r\}$ of open-closed subsets of X and a finite subset $\{f_1, \cdots, f_r\}$ of F such that

$$X = \cup_{i=1}^{r} M_i$$
$$M_j \cap M_k = \phi \quad \text{if} \quad j \neq k$$
$$\sigma(y) = X_{f_i}(y) \quad \text{for all} \quad y \in M_i, \ i = 1, 2, \cdots, r.$$

Then, $\{X_{M_1}, \cdots, X_{M_r}\}$ is a set of orthogonal idempotents of $\Gamma(X, X \times F)$ and $\sigma = X_{M_1} X_{f_1} + \cdots + X_{M_r} X_{f_r}$.

In case $F = \{0, f_1, \cdots, f_n\}$ is a finite field in the above proposition, every element σ in $\Gamma(X, X \times F)$ can be expressed uniquely in the form $\sigma = \sigma_1 X_{f_1} + \cdots + \sigma_n X_{f_n}$, where $\{\sigma_1, \cdots, \sigma_n\}$ is a set of orthogonal idempotents of $\Gamma(X, X \times F)$. Since any p-ring R in the sense of McCoy and Montgomery [3] is commutative regular and is isomorphic to $\Gamma(S(R), \mathrm{GF}(p))$ (see [7, p. 52]), we readily obtain the following:

Corollary 2.2. ([1], [8]). *Every element a in a p-ring R can be expressed uniquely in the form $a = e_1 + 2e_2 + \cdots + (p-1)e_{p-1}$,* where *$\{e_1, \cdots, e_{p-1}\}$ is a set of orthogonal idempotents of R*

We claim here that if R is a regular ring then the canonical mapping given by $x \longrightarrow x \cap R$, $x \in X(Q(R))$, is a continuous (and hence closed) mapping of $X(Q(R))$ onto $X(R)$.

Proposition 2.3. *Let X be a Boolean space, F a field, and $Y = X(Q(\Gamma(X, X \times F)))$. Then $\Gamma^* = \Gamma(Y, Y \times F)$ is a ring of quotients of $\Gamma = \Gamma(X, X \times F)$, and moreover coincides with the Baer hull $C(\Gamma)$.*

Proof. Let ν be the canonical mapping of Y onto $X(\Gamma)$. By [7, p. 20], there exists then a homeomorphism μ of $X(\Gamma)$ onto X. Now, let $\lambda = \mu\nu$, $\sigma \in \Gamma$, and $y \in Y$. Then there exists f_y in F such that $\sigma(\lambda(y)) = \lambda(y) \times f_y$. Here, it is easy to see that the mapping σ^* given by $y \longrightarrow y \times f_y$, $y \in Y$, is a section of $Y \times F$ over Y. Hence, identifying σ with σ^*, Γ may be regarded as a subring of Γ^*. Accordingly, if $f \in F$ then $Y_f = X_f$, and if r is an idempotent of Γ then $Y_{U_r^{Q(\Gamma)}} = X_{\mu(U_r^{\Gamma})}$. Now, let σ be an arbitrary element of Γ^*. Then, by Proposition 2.1, there is a family $\{M_1, \cdots, M_r\}$ of open-closed subsets of Y and a subset $\{f_1, \cdots, f_r\}$ of F such that $\sigma = Y_{M_1}Y_{f_1} + \cdots + Y_{M_r}Y_{f_r} = Y_{M_1}X_{f_1} + \cdots + Y_{M_r}X_{f_r}$. In particular, if $\sigma \neq 0$ then $Y_{M_i}X_{f_i} \neq 0$ with some i. By Lemma 1.2, there exists an idempotent r in Γ such that $U_r^{Q(\Gamma)}$ is a non-empty subset of M_i. Then, $0 \neq Y_{U_r^{Q(\Gamma)}} = X_{\mu(U_r^{\Gamma})}\sigma = X_{\mu(U_r^{\Gamma})}X_{f_i}$, which means that Γ^* is an essential extension of Γ as a Γ-module. On the other hand, since $Y \simeq X(\Gamma^*)$, Γ^* coincides with the Baer hull $C(\Gamma)$.

Now, combining Proposition 2.3 with [7, Corollary 24.5], we obtain at once the principal theorem of this section:

Theorem 2.4. *Let X be a Boolean space. If F is a finite field then $C(\Gamma(X, X \times F))$ coincides with $Q(\Gamma(X, X \times F))$, and the converse is true, provided X is infinite.*

Finally, the next is only a combination of Theorem 2.4 and [6, Theorems 3.7 and 3.10].

Corollary 2.5. *Let X be a Boolean space, F a finite field, and $R = \Gamma(X, X \times F)$. Then the following statements hold:*

(1) *Every finitely generated torsion free R-module satisfies the conditions (β) and (γ).*

(2) *A finitely generated torsion free R-module M is injective if and only if $M \simeq R/J_1 \oplus \cdots \oplus R/J_n$ (as a module), where each J_i is an ideal of R such that R/J_i is self-injective.*

Remark 1. Corollary 2.5 (2) has been given by Pierce [7, p. 102] without the assumption that M is torsion free. However, it is no longer valid for arbitrary commutative regular rings (see [6, Example C]).

3. n-point

Recently, in [6, Example A], the author has given a counter example to the Pierce's question (4) of [7, p. 109]. We claim first that [6, Theorem 3.7] together with [7, Proposition 20.1] provides a more severe result:

Proposiiton 3. 1. *Let R be a regular ring, and $C(R) = Q(R)$. If $X(R)$ contains a 3-point, then there exists a finitely generated torsion free R-module which satisfies the conditions (β) and (γ) but not (α).*

Proposition 3. 2. *Let R be a regular ring, and λ the canonical mapping of $X(Q(R))$ onto $X(R)$. Then the following conditions are equivalent:*

(a) *x is an n-point of $X(R)$.*

(b) *x is a non-isolated point of $X(R)$ such that $n \leq |\lambda^{-1}(x)|$ (the number of elements of $\lambda^{-1}(x)$).*

Proof. (a) $\Rightarrow$ (b). If x is an n-point of $X(R)$, then there exists a family $\{U_1, \cdots, U_n\}$ of pairwise disjoint open subsets of $X(R)$ such that $x \in U_i^- - U_i$, $i = 1, 2, \cdots, n$. Let $W_i = (\lambda^{-1}(U_i))^-$. Then, since $X(Q(R))$ is an extremely disconnected space (see [7, p. 102]), it is easy to see that $W, \cdots, W_n$ are pairwise disjoint. On the other hand, as λ is a closed mapping, we obtain $U_i^- \subseteq \lambda(W)$. Hence, we have $n \leq |\lambda^{-1}(x)|$

(b) $\Rightarrow$ (a). Let x be a non-isolated point of $X(R)$ such that $n \leq |\lambda^{-1}(x)|$. Then, we can choose idempoints $e_1, \cdots, e_n$ in $Q(R)$ such that $U_{e_1}^{Q(R)}, \cdots, U_{e_n}^{Q(R)}$ are pairwise disjoint and every $\lambda(U_{e_i}^{Q(R)})$ contains x. Put $A_i = \{r \in B(R) \mid U_r^{Q(R)} \subseteq U_{e_i}^{Q(R)},\ x \notin \lambda(U_r^{Q(R)})\}$, and $U_i = \bigcup_{r \in A_i} \lambda(U_r^{Q(R)})$. Then, Lemma 1.3 enables us to see that each U_i is a non-empty open subset of $X(R)$ such that $x \in U_i^- - U_i$. This means that x is an n-point of $X(R)$.

Concerning the Pierce's question (7) of [7, p. 109], we can state the following:

Theorem 3. 3. *Let R be a Boolean ring, and λ the canonical mapping of $X(Q(R))$ onto $X(R)$. Then, the following conditions are equivalent:*

(a) *$|\lambda^{-1}(x)| \leq 2$ for all x in $X(R)$.*

(b) *$X(R)$ contains no 3-points.*

(c) *Every finitely generated torsion free R-module satisfies the* *condition* (α).

(d) *Every finitely generated R-submodule of* $Q(R)$ *satisfies the condition* (α).

(e) *Every R-submodule of* $Q(R)$ *with two generators satisfies the condition* (α).

(f) $Re+Rf$ *contains* ef *for each* e, f *in* $Q(R)$.

Proof. Clearly, $|\lambda^{-1}(x)|=1$ for any isolated point x of $X(R)$ (Lemma 1.1). Hence the equivalence of (a) and (b) is contained in Proposition 3.2. The equivalence of (b) and (c) is shown by Pierce [7, Proposition 20.1 and Theorem 20.4], and that of (e) and (f) follows from [6, Lemma 3.1]. Since the implications (c) $\Rightarrow$ (d) $\Rightarrow$ (e) are trivial, it remains only to prove (f) $\Rightarrow$ (a). To see this, suppose that there exists a point x in $X(R)$ such that $3\leq|\lambda^{-1}(x)|$. Then, $X(Q(R))$ contains pairwise disjoint open-closed subsets W_1, W_2, W_3 such that every $\lambda(W_i)$ contains x. Here, one may remark that $R\simeq\Gamma(X(R), \mathfrak{R}(R))\simeq\Gamma(X(R), X(R)\times F)$ and $Q(R)$ $\simeq\Gamma(X(Q(R)), \mathfrak{R}(Q(R)))\simeq\Gamma(X(Q(R)), X(Q(R))\times F)$, where $F=\mathrm{GF}(2)$. Now, by the hypothesis, Y_{W_2} is contained in $\Gamma(X(R), \mathfrak{R}(R))(Y_{W_1}+Y_{W_2})$ $+\Gamma(X(R), \mathfrak{R}(R))(Y_{W_2}+Y_{W_3})$, that is, $Y_{W_2}=(X_P)^*(Y_{W_1}+Y_{W_2})+(X_Q)^*$ $(Y_{W_2}+Y_{W_3})$ with some open-closed subsets P and Q of $X(R)$, where $(X_P)^*$ and $(X_Q)^*$ are respectively the associated sections in $\Gamma(X(Q(R))$, $\mathfrak{R}(Q(R)))$ for X_P and X_Q mentioned in the proof of Proposition 2.3. Then, it is obvious that $\lambda(W_2)\subseteq P\cup Q$ and $(\lambda(W_1)\cap\lambda(W_3))\cap(P\cup Q)=\phi$, which contradicts $x\in\lambda(W_1)\cap\lambda(W_2)\cap\lambda(W_3)$.

As was shown in [7, p. 92], there exists a Boolean space which contains 2-points but no 3-points. In what follows, we shall show that there exists a Boolean ring R such that every non-isolated point in $X(R)$ is an n-point but not an $(n+1)$-point.

Example. Let $\{S_1, \cdots, S_n\}$ be arbitrary pairwise disjoint countably infinite sets: $S_i=\{a_{i1}, \cdots, a_{is}, \cdots\}$. We consider the set Q of all subsets of $S=S_1\cup S_2\cup\cdots\cup S_n$. Then, Q becomes a Boolean ring under the operations: $a+b=(a\cup b)\cap(a\cap b)^c$, $ab=a\cap \mathrm{b}$, where $(a\cap b)^c$ denotes the complement of $a\cap b$ in S. Now, let $\varphi_{ij}: S_i\longrightarrow S_j$ be the mapping given by $a_{is}\longrightarrow a_{js}$ $(s=1, 2, \cdots)$. Let R be the set of all x in Q such that $\varphi_{ij}((S_i\cap x)\cup F_{i,x})=(S_j\cap x)\cup F_{j,x}$ with some finite subsets $F_{i,x}$ of S_i $(i, j=1, 2, \cdots, n)$. Then, it is easy to see that R is a subring of Q and $Q(R)$ coincides with Q. Putting $U_i=\{x_{is}=Ra_{is}^c\mid s=1, 2, \cdots\}$, $\{U_1, \cdots, U_n\}$ is a set of pairwise disjoint open subsets of $X(R)$ and $U=U_1\cup\cdots\cup U_n$ is the set of all isolated points of $X(R)$ (Lemma 1.1). Now, let x be an arbitrary non-isolated point in $X(R)$. If

x contains r in R then it is easy to see that $S_i \cap r \neq S_i$, which provides $x \in \bar{U_i} - U_i$. Hence, x is an n-point. Next, suppose that x is an $(n+1)$-point. Then, there exist pairwise disjoint open subsets $V_1, \cdots, V_n, V_{n+1}$ of $X(R)$ such that $x \in \bar{V_i} - V_i$, $i = 1, \cdots, n+1$. Since U is a dense open subset of $X(R)$, without loss of generality, we may assume that every V_i is contained in U. Obviously, we can find some k and $p \neq q$ such that both $V_p \cap U_k$ and $V_q \cap U_k$ are infinite. There exists then some b in x such that $U_b^R \cap V_p = \phi$ or $U_b^R \cap V_q = \phi$, which is a contradiction. Thus, $X(R)$ contains no $(n+1)$-points.

Remark 2. In the last example, let S_i be denoted by e_i.

(1) $Q = Re_1 \oplus \cdots \oplus Re_n$, and $e_i \notin R$ for $n \geq 2$.

(2) If $n=2$, then $X(R)$ contains no 3-points, and then every finitely generated torsion free R-module satisfies the condition (α) (Theorem 3.3), but R is not self-injective.

(3) If $n \geq 3$, then there exists a finitely generated torsion free R-module which satisfies the conditions (β) and (γ) but not (α) (Proposition 3.1). In fact, $R(e_1 + e_2) + R(e_2 + e_3)$ is such a module. Obviously, $R(e_1 + e_2) + R(e_2 + e_3)$ is a submodule of $Re_1 + Re_2 + Re_3$ not containing e_1, and hence it does not satisfy the condition (α) (Theorem 3.3).

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