ON A GROUP OF CYCLIC EXTENSIONS OVER COMMUTATIVE RINGS II

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Let R be a commutative ring with identity and let G be a finite abelian group. Let $E(GR)$ (resp. $SE(GR)$) be the group of isomorphism classes of commutative abelian extensions (resp. strongly abelian extensions) of R with group G. In [8], the author computed the abelian group $E(GR)$ for G of prime order p and an algebra R over $GF(p)$, and obtained the following group isomorphisms for a connected ring R;

$$E(GR) \cong R^+/\{r^p - r\,;\ r \in R\} \cong \mathrm{Hom}_c(\Pi, G) \cong H^2(R, G),$$

and for strongly cyclic extensions of a connected ring R with group G of prime order p, we obtained the following group isomorphisms

$$E(GR) \supseteqq SE(GR) \cong U(R)/U(R)^p \cong \mathrm{Hom}_c(\Pi, G) \cong H^2(R, G),$$

where Π is the group of automorphisms of a separable closure of R [4, Def. 5], $H^2(R, G)$ is the second cohomology group in the sense of D. K. Harrison and $U(R)$ is the group of all invertible elements in R (see, [8, Cor. 2.8 and Th. 3.6]). In §1, if $J \longrightarrow G \longrightarrow G/J$ is an exact sequence of finite abelian groups, then we have an exact sequence $0 \longrightarrow E(JR) \longrightarrow E(GR) \longrightarrow E((G/J)R)$ which is essentially obtained in [3, Th. 3]. In § 2, we show that if G is a finite abelian group, then $E(GR)$ is isomorphic to $\mathrm{Hom}_c(\Pi, G)$ as sets, and if R is an algebra over $GF(p)$, G is a cyclic group of order p^m and J is an arbitrary subgroup of G, then $0 \longrightarrow E(JR) \longrightarrow E(GR) \longrightarrow E((G/J)R) \longrightarrow 0$ is exact. In the last section, we give an example of a connected ring R and a group G such that $SE(GR)$ is properly contained in $E(GR)$.

Throughout this paper R will denote a commutative ring with identity, and unadorned $\otimes$ will mean $\otimes_R$. Moreover, every ring has identity which preserved by every homomorphism, every module is unital, and all ring extensions of R will be assumed to be commutative and have identities coinciding with the identity of R. As to other notations and terminologies used in this paper, we follow [8].

1. An exact sequence. Let G be a finite abelian group. Then the group algebra RG of G over R is a finite Hopf R-algebra with the usual diagonal and counit maps [1, p. 59], and $GR = \mathrm{Hom}_R(RG, R)$ is also a finite Hopf R-algebra [1, p. 55]. For brevity, we shall denote by Δ the diagonal map of GR. When discussiong several diagonal maps simultaneously, we shall sometimes write $\Delta = \Delta_{GR}$, in order to avoid confusion. Let $\mathscr{A}$ be the category of commutative R-algebras and R-algebra homomorphisms, and $\mathscr{A}^{GR}$ the category of GR-objects and GR-morphisms in $\mathscr{A}$ [1, p. 33]. If J is a subgroup of G and $\rho_1 = 1$, $\rho_2, \cdots, \rho_k$ is the complete representative system of $G/J = \bar{G}$, then the canonical exact sequence $J \longrightarrow G \longrightarrow G/J$ induces the following sequence of finite Hopf R-algebras

$$(G/J)R \xrightarrow{\ f\ } GR \xrightarrow{\ g\ } JR\,.$$

Explicitly, f is defined by $f(u_i) = \sum_{\tau \in J} v_{\rho_i \tau}$, and g is defined by $v_\sigma \longmapsto \begin{cases} v_\sigma\ (\sigma \in J) \\ 0\ \ (\sigma \notin J) \end{cases}$, where $\{u_i\}$ and $\{v_\sigma\}$ are the dual bases of $\{\bar{\rho}_i\}$ $(\subset R\bar{G})$ and $\{\sigma\}$ $(\subset RG)$, respectively. (Note that the map f is independent of the choice of the representative system for G/J.) Let $E(*)$ be the group of $*$-isomorphism classes of commutative Galois $*$-objects [8, p. 164]. Then by [1, Th. 2.20], we obtain a sequence of abelian groups

$$(1) \qquad 0 \longrightarrow E(JR) \xrightarrow{\ \tilde{g}\ } E(GR) \xrightarrow{\ \tilde{f}\ } E((G/J)R).$$

Here for (A) in $E(JR)$ with the structure map $\alpha_A\colon A \longrightarrow A \otimes JR$ [8, §1], $\tilde{g}(A)$ is defined by the following equalizer diagram in $\mathscr{A}$

$$(2) \qquad (\tilde{g}A) \xrightarrow{\ i\ } A \otimes GR \ \underset{\alpha_A \otimes 1}{\overset{(1\otimes g\otimes 1)(1\otimes \Delta)}{\rightrightarrows}}\ A \otimes JR \otimes GR$$

with the structure map $\alpha_{\tilde{g}(A)}\colon \tilde{g}(A) \longrightarrow \tilde{g}(A) \otimes GR$ such that the diagram

$$(2) \qquad \begin{array}{ccc} \tilde{g}(A) & \xrightarrow{\quad i \quad} & A \otimes GR \\ {\scriptstyle \alpha_{\tilde{g}(A)}} \downarrow & & \downarrow {\scriptstyle 1 \otimes \Delta} \\ \tilde{g}(A) \otimes GR & \xrightarrow[\ i \otimes 1\ ]{} & A \otimes GR \otimes GR \end{array}$$

is commutative, and for (B) in $E(GR)$ with the structure map $\alpha_B\colon B \longrightarrow B \otimes GR$, $\tilde{f}(B)$ is defined by the following equalizer diagram in $\mathscr{A}$

$$(3)\qquad \tilde{f}(B) \xrightarrow{\ i'\ } B \otimes (G/J)R \ \underset{\alpha_B \otimes 1}{\overset{(1\otimes f\otimes 1)(1\otimes \Delta)}{\rightrightarrows}}\ B \otimes GR \otimes (G/J)R$$

with the structure map $\alpha_{\tilde{f}(B)}\colon \tilde{f}(B) \longrightarrow \tilde{f}(B) \otimes (G/J)R$ such that the diagram

$$(3')\qquad \begin{array}{ccc} \tilde{f}(B) & \xrightarrow{\quad i' \quad} & B \otimes (G/J)R \\ {\scriptstyle \alpha_{\tilde{f}(B)}}\downarrow & & \downarrow{\scriptstyle 1\otimes\Delta} \\ \tilde{f}(B) \otimes (G/J)R & \xrightarrow[\ i'\otimes 1\ ]{} & B \otimes (G/J)R \otimes (G/J)R \end{array}$$

is commutative, where i and i' are the canonical inclusions. Under these notations, we have the following

Theorem 1.1. (1) *is an exact sequence.*

Proof. First, we show that $\tilde{g}$ is a monomorphism. Let (A) be an element in $E(JR)$ such that $\tilde{g}(A)$ is isomorphic to GR in $\mathscr{A}^{GR}$. We define an R-algebra homomorphism $h_1\colon\ A \longrightarrow A \otimes GR$ by the formula $h_1(a) = \sum_{i=1}^{k} \sum_{\sigma\in J}\sigma(a) \otimes v_{\sigma\rho_i}$ (a in A). Then we have the following diagram in $\mathscr{A}$,

$$\begin{array}{ccc} \tilde{g}(A) \xrightarrow{\ i\ } & A \otimes GR & \underset{\alpha_A\otimes 1}{\overset{(1\otimes g\otimes 1)\,(1\otimes\Delta)}{\rightrightarrows}}\ A \otimes JR \otimes GR \\ & \uparrow{\scriptstyle h_1} & \\ & A & \end{array}$$

and $(\alpha_A \otimes 1)h_1 = (1 \otimes g \otimes 1)\,(1 \otimes \Delta)h_1$, where i is the canonical inclusion. Therefore there exists an R-algebra homomorphism $i''\colon A \rightarrow \tilde{g}(A)$ such that $i\cdot i'' = h_1$. Now let g' be the canonical inclusion from JR to GR, and consider the following diagram in

$$\begin{array}{ccccccc} A & \xrightarrow{\quad i'' \quad} & \tilde{g}(A) & \xrightarrow{\quad \varphi \quad} & GR & \xrightarrow{\quad g \quad} & JR \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_{\tilde{g}(A)}} & & \downarrow{\scriptstyle \Delta_{JR}} & & \downarrow{\scriptstyle \Delta_{GR}} \\ A \otimes JR & \xrightarrow[\ i''\otimes g'\ ]{} & \tilde{g}(A) \otimes GR & \xrightarrow[\ \varphi\otimes 1\ ]{} & GR \otimes GR & \xrightarrow[\ g\otimes g\ ]{} & JR \otimes JR \end{array}$$

where φ is an isomorphism in $\mathscr{A}^{GR}$. By (2′) and the definition of g, the middle and right squares are commutative, we have

$$(g\otimes g)(\varphi\otimes 1)(i''\otimes g')\,\alpha_A=(g\otimes g)(\varphi\otimes 1)\alpha_{\tilde{g}(A)}\cdot i''=\Delta_{JR}\cdot g\varphi i''.$$

Thus $g\varphi i''\colon A\longrightarrow JR$ is a homomorphism in $\mathscr{A}^{GR}$ and by [1, Th. 1. 12], $(A)=(JR)$, that is, $\tilde{g}$ is a monomorphism.

Next, we shall prove that $\mathrm{Ker}(\tilde{f})$ contains $\mathrm{Im}(\tilde{g})$. Noting that R is the zero object in the category of commutative Hopf R-algebras, we have the commutative diagram

$$\begin{array}{ccc} (G/J)R & \xrightarrow{gf} & JR \\ & {}_{\varepsilon}\searrow \quad \nearrow_{\eta} & \\ & R & \end{array}$$

and $\tilde{f}\cdot\tilde{g}(A)=\widetilde{gf}(A)=\widetilde{\eta\varepsilon}(A)$ by [1, p. 20] ($(A)\in E(JR)$), where $\varepsilon(u_1)=1$; $\varepsilon(u_i)=0$ $(i\neq 1)$ and $\eta(r)=r\cdot 1$ (1 is the identity of JR). Consider the diagram

$$\begin{array}{ccccc} \widetilde{\eta\varepsilon}(A) & \xrightarrow{j} & A\otimes(G/J)R & \overset{(1\otimes\eta\varepsilon\otimes 1)(1\otimes\Delta)}{\underset{\alpha_A\otimes 1}{\rightrightarrows}} & A\otimes JR\otimes(G/J)R \\ & & \uparrow{\scriptstyle h_2} & & \\ & & (G/J)R & & \end{array}$$

in $\mathscr{A}$, where $h_2(u_i)=1\otimes u_i$ for $u_i\in(G/J)R$ and j is the inclusion map. Since h_2 is an R-algebra homomorphism and $(\alpha_A\otimes 1)h_2=(1\otimes\eta\varepsilon\otimes 1)(1\otimes\Delta)h_2$, there exists a homomorphism $\psi\colon(G/J)R\longrightarrow\widetilde{\eta\varepsilon}(A)$ in $\mathscr{A}$ such that $j\cdot\psi=h_2$. Then we have $(j\otimes 1)\alpha_{\widetilde{\eta\varepsilon}(A)}\cdot\psi=(j\otimes 1)(\psi\otimes 1)\Delta\colon(G/J)R\longrightarrow A\otimes(G/J)R\otimes(G/J)R$. Since $(j\otimes 1)$ is a monomorphism, ψ is a homomorphism in $\mathscr{A}^{GR}$ and thus by [1, Th. 1.12] ψ is an isomorphism. This shows $\mathrm{Ker}(\tilde{f})\supseteqq\mathrm{Im}(\tilde{g})$.

Conversely, let (B) be an element in $\mathrm{Ker}(\tilde{f})$, that is, $\tilde{f}(B)\cong(G/J)R$ in $\mathscr{A}^{(G/J)R}$. Let $x=\sum_{i=1}^{k}b_i\otimes u_i$ be an arbitrary element in $B\otimes(G/J)R$. Then by (3), $(\alpha_B\otimes 1)(x)=(1\otimes f\otimes 1)(1\otimes\Delta)(x)$ if and only if $x\in\tilde{f}(B)$. Hence we have

$$\tilde{f}(B)=\{\textstyle\sum_{i=1}^{k}\rho_i(b)\otimes u_i;\quad b\in B^J\}\tag{4}$$

where B^J is the subring of B consisting of all elements of B left fixed by every element of J. Since $(G/J)R \cong \widetilde{f}(B) \cong B^J \subset B$, there exist orthogonal idempotents b_i in B such that $\sum_{i=1}^{k} b_i = 1$, $\tau(b_i) = b_j$ if $\bar{\rho}_j = \overline{\rho\tau}_i$ $(\tau \in G)$ and $Rb_i \cong R$. Let $C = Bb_1$. Then C is a JR-object and one can easily check that C is a Galois JR-object over $Rb_1 \cong R$. We now can define a map h_3 from B to $C \otimes GR$ by $h_3(b) = \sum_{\tau \in J} \tau(b) b_1 \otimes v_\tau$ (b in B). Consider the diagram

$$\begin{array}{ccccc} \widetilde{g}(C) & \longrightarrow & C \otimes GR & \overset{(1\otimes g\otimes 1)(1\otimes \Delta)}{\underset{\alpha_C \otimes 1}{\rightrightarrows}} & C \otimes JR \otimes GR \\ & & \uparrow {\scriptstyle h_3} & & \\ & & B & & \end{array}$$

in $\mathscr{A}$. Then $(\alpha_C \otimes 1)h_3 = (1 \otimes g \otimes 1)(1 \otimes \Delta)h_3$, so by making use of the same method as in the proof of $\mathrm{Ker}(\widetilde{f}) \supseteqq \mathrm{Im}(\widetilde{g})$, we have $B \cong \widetilde{g}(C)$ in $\mathscr{A}^{GR}$. This completes the proof.

Remark 1.2. Let $T(G, R)$ be the group defined on [3, p. 3]. Then by [8, Th. 1.2], the map f: $E(GR) \longrightarrow T(G, R)$ defined by $(A) \longmapsto \{A\}$ is an isomorphism. Thus Th. 1.1 is equivalent to [3, Th. 3].

2. A group of cyclic extensions. Let R be a connected ring, that is, R has no proper idempotents. Let Ω be a separable closure of R [4, Def. 5] and Π the group of R-algebra automorphisms of Ω. Then Π is a topological group with finite topology. Let $\mathscr{Ab}_f$ and $\mathscr{Ab}$ be the category of finite abelian groups and the category of abelian groups, respectively. Then we have two functors $\mathrm{Hom}_c(\Pi, *)$: $\mathscr{Ab}_f \longrightarrow \mathscr{Ab}$ and $E(*)$: $\mathscr{Ab}_f \longrightarrow \mathscr{Ab}$, where $\mathrm{Hom}_c(\Pi, *)$ is the group of continuous homomorphisms from Π to the discrete group $*$ [8, § 2]. We define a map h_G: $\mathrm{Hom}_c(\Pi, G) \longrightarrow E(GR)$ $(G \in \mathscr{Ab}_f)$ as follows; If φ is in $\mathrm{Hom}_c(\Pi, G)$, $\Omega^{\mathrm{Ker}(\varphi)}$ will denote the fixed subring of Ω corresponding to $\mathrm{Ker}(\varphi)$. $\Omega^{\mathrm{Ker}(\varphi)}(= \Omega_\varphi)$ is a Galois extension of R with Galois group naturally isomorphic to $J = \mathrm{Im}(\varphi)$, and is thus a Galois JR-object in view of [8, Th. 1.2 and Remark 1.3]. The element of $E(GR)$ corresponding to φ is then $(\widetilde{j^*}(\Omega_\varphi))$, where j^*: $GR \longrightarrow JR$ is the homomorphism of Hopf R-algebras induced by the inclusion j: $J \longrightarrow G$, and $\widetilde{j^*}$ is as in (2). Under these notations we have

Lemma 2.1. *Let R be a connected ring. Then h: $\mathrm{Hom}_c(\Pi, *) \longrightarrow E(*R)$ is a natural transformation.*

Proof. Let $f: G \longrightarrow H$ be a homomorphism in $\mathscr{Ab}_f$. Then we have homomorphisms $f_*: \mathrm{Hom}_c(\Pi, G) \longrightarrow \mathrm{Hom}_c(\Pi, H)$ with $f_*(\varphi) = f\varphi$ and $\widetilde{f^*}: E(GR) \longrightarrow E(HR)$ with $\widetilde{f^*}((A)) = (\widetilde{f^*}(A))$, where $f^*: HR \longrightarrow GR$ is induced by f. Consider the following diagram

$$\begin{array}{ccc} \mathrm{Hom}_c(\Pi, G) & \xrightarrow{h_G} & E(GR) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \widetilde{f^*}} \\ \mathrm{Hom}_c(\Pi, H) & \xrightarrow[h_H]{} & E(HR) \end{array}$$

Then $\widetilde{f^*}h_G(\varphi) = \widetilde{f^*}(\widetilde{j^*}(\Omega_\varphi)) = \widetilde{(fj)^*}(\Omega_\varphi)$ [1, p. 20] and $h_H f_*(\varphi) = h_H(f\varphi) = h_H(fj\,\varphi) = \widetilde{(fj)^*}(\Omega_\varphi)$, where $j: \mathrm{Im}(\varphi) \longrightarrow G$ is the canonical inclusion. Therefore $\widetilde{f^*}h_G = h_H f_*$, that is, h is a natural transformation, completing the proof.

Let S be a commutative Galois extension of R with finite abelian group G. Then by [3, Th. 7] there exist a subgroup H of G and a $T \in E(HR)$ such that T has no proper idempotents and $S \cong (T \otimes GR)^{(\sigma, \sigma^{-1})}$ $(\sigma \in H)$ as Galois G-extensions. Since T is connected, we may consider T as a subring of Ω [4, p. 464]. Define $\varphi: \Pi \longrightarrow H \subseteqq G$ with $\varphi(\sigma) = \sigma | T$. Then φ is in $\mathrm{Hom}_c(\Pi, G)$, $\mathrm{Im}(\varphi) = H$ and $\Omega_\varphi = T$. Let $i: H \longrightarrow G$ be the canonical inclusion. Then

(6) $\widetilde{i^*}(\Omega_\varphi) = \{\sum_{\tau \in G} x_\tau \otimes v_\tau \in T \otimes GR\,;\ \sigma(x_\tau) = x_{\sigma\tau}$ for all $\tau \in G,\ \sigma \in H\}$.

On the other hand, $(T \otimes GR)^{(\sigma, \sigma^{-1})}$ is a Galois $(H \times G)/\{(\sigma, \sigma^{-1});\ \sigma \in H\}$ $(\cong G)$-extension of R and $(T \otimes GR)^{(\sigma, \sigma^{-1})} = \{\sum_{\tau \in G} a_\tau \otimes v_\tau \in T \otimes GR\,;$ $\sigma^{-1}(a_\tau) = a_{\sigma\tau}$ for all $\tau \in G,\ \sigma \in H.\}$ Since the map $f: \widetilde{i^*}(\Omega_\varphi) \longrightarrow T \otimes GR)^{(\sigma, \sigma^{-1})}$ defined by $f(\sum_{\tau \in G} x_\tau \otimes v_\tau) = \sum_{\tau \in G} x_{\tau^{-1}} \otimes v_\tau (\sum_{\tau \in G} x_\tau \otimes v_\tau \in \widetilde{i^*}(\Omega_\varphi))$ is an R-algebra homomorphism and $(1, \rho)$ $(\rho \in G)$ is a representative set of $(H \times G)/\{(\sigma, \sigma^{-1});\ \sigma \in H\}$, we have $\beta f = (f \otimes 1)\alpha$, where α and β are the structure maps of $\widetilde{i^*}(\Omega_\varphi)$ and $(T \otimes GR)^{(\sigma, \sigma^{-1})}$ in $\mathscr{A}^{GR}$, respectively [8, §1]. Thus $\widetilde{i^*}(\Omega_\varphi) \cong (T \otimes GR)^{(\sigma, \sigma^{-1})} \cong S$ in $\mathscr{A}^{GR}$ [1, Th. 1.12]. Hence we have

Lemma 2.2. *Let R be a connected ring and G a finite abelian group. Then $h_G: \mathrm{Hom}_c(\Pi, G) \longrightarrow E(GR)$ is an epimorphism of sets.*

Now, let $G = (\sigma)$ be a finite cyclic group, φ in $\mathrm{Hom}_c(\Pi, G)$, and $m = |G/\mathrm{Im}(\varphi)|$ (the order of $G/\mathrm{Im}(\varphi)$). Then $\Omega^{\mathrm{Ker}(\varphi)} = \Omega_\varphi$ is a cyclic extension of R with Galois group $(\sigma^m) = \mathrm{Im}(\varphi)$. If we define an R-algebra automorphism σ_1 of Ω_φ^m(m-times direct sum of Ω_φ) by

(7) $\sigma_1(x_1, x_2, \cdots, x_m) = (\sigma^m(x_m), x_1, \cdots, x_{m-1})(x_i \in \Omega_\varphi)$, then Ω_φ^m is a cyclic extension of R with Galois group $G_1 = (\sigma_1)$.

We shall now prove the following

Theorem 2.3. *Let R be a connected ring and $G = (\sigma)$ a finite cyclic group. Then the map h_G: $\mathrm{Hom}_c(\Pi, G) \longrightarrow E(GR)$ is an isomorphism of sets, and $h_G(\varphi) \cong \Omega_\varphi^{|G/\mathrm{Im}(\varphi)|}$ ($\varphi \in \mathrm{Hom}_c(\Pi, G)$) as Galois GR-objects.*

Proof. Let φ be $\mathrm{Hom}_c(\Pi, G)$, $m = |G/\mathrm{Im}(\varphi)|$, and i: $\mathrm{Im}(\varphi) \longrightarrow G$ the canonical inclusion. Then by [1, p. 59], $h_G(\varphi) = \widetilde{i}^*(\Omega_\varphi)$ is a Galois extension of R with group G. Explicitly, G operates $\widetilde{i}^*(\Omega_\varphi)$ by

(8) $\sigma(\sum_{\tau\in G} x_\tau \otimes v_\tau) = \sum_{\tau\in G} x_{\sigma\tau} \otimes v_\tau$ $(\sigma \in G, \sum_{\tau\in G} x_\tau \otimes v_\tau \in \widetilde{i}^*(\Omega_\varphi))$.

By (6), $\sum_{i=0}^{m-1} \sum_{j=1}^{k} \sigma^{jm}(x_{m-i}) \otimes v_{i+jm}$ is in $\widetilde{i}^*(\Omega_\varphi)$, where k is the order of $\mathrm{Im}(\varphi)$ and $\{v_t\}$ is the dual bases of $\{\sigma^t\}$. Therefore we may define an R-algebra homomorphism f: $\Omega_\varphi^m \longrightarrow \widetilde{i}^*(\Omega_\varphi)$ by the formula

$$f(x_1, x_2, \cdots, x_m) = \sum_{i=0}^{m-1} \sum_{j=1}^{k} \sigma^{jm}(x_{m-i}) \otimes v_{i+jm} \qquad (x_i \in \Omega_\varphi).$$

By (7) and (8), one can easily check that $f\sigma_1 = \sigma f$ and thus f is an isomorphism of Galois GR-objects [8, Remark 1.3 (1)]. Now let φ, ψ be in $\mathrm{Hom}_c(\Pi, G)$ such that $h_G(\varphi) = h_G(\psi)$. Then by the above fact, Ω_φ^m is isomorphic to Ω_ψ^n as G-Galois extensions, where m and n are the orders of $G/\mathrm{Im}(\varphi)$ and $G/\mathrm{Im}(\psi)$ respectively, and by [8, Lemma 2.5 (2)] we have $\Omega_\varphi = \Omega_\psi$. Hence $\varphi = \psi$, that is, h_G is a monomorphism. Combining this with the result of Lemma 2.2, we obtain the theorem.

Corollary 2.4. *Let R be a connected ring and G a finite abelian group. Then $\mathrm{Hom}_c(\Pi, G)$ is isomorphic to $E(GR)$ as sets.*

Proof. Let $G = G_1 \times \cdots \times G_n$, where G_i is a cyclic group. Then by [1, Prop. 3.8] $((S_1), \cdots, (S_n)) \longrightarrow (S_1 \otimes \cdots \otimes S_n)$ is an isomorphism from $\Pi_{i=1}^n E(G_iR)$ to $E(GR)$. Thus we have isomorphisms $\mathrm{Hom}_c(\Pi, G)$ $\Pi_{i=1}^n \mathrm{Hom}_c(\Pi, G_i) \cong \Pi_{i=1}^n E(G_iR) \cong E(GR)$.

For cyclic p^m-extensions, we have the following

Theorem 2.5. *Let R be a commutative algebra over the prime field $GF(p \neq 0)$, $G = (\sigma)$ a cyclic group of order p^m, and J the subgroup* *of* G *of order* $p^k (k \leqq m)$. *Then the sequence*

$$(9) \qquad 0 \longrightarrow E(JR) \xrightarrow{\tilde{g}} E(GR) \xrightarrow{\tilde{f}} E((G/J)R) \longrightarrow 0$$

is exact, where $\tilde{g}$ *and* $\tilde{f}$ *are defined in* (1).

Proof. By Th. 1.1, it is enough to show that $\tilde{f}$ is an epimorphism. Let (A) be an element in $E((G/J)R)$. Then A is a cyclic p^{m-k}-extension of R with group $G/J=(\rho_i)$, where $\rho_1, \cdots, \rho_h$ is the complete representative system of G/J, so by using [6, Th. 1.3] repeatedly, there exists a cyclic p^m-extension B of R with group (τ) such that $(\tau|A)=(\rho_i)$, and through the homomorphism $(\tau) \longrightarrow (\sigma)=G$ defined by $\tau \longmapsto \sigma$, G operates on B by $\sigma(b)=\tau(b)$ $(b \in B)$. Thus B is a cyclic p^m-extension of R with group G. Consider the following equalizer diagram in $\mathscr{A}$;

$$\tilde{f}(B) \longrightarrow B \otimes (G/J)R \underset{\alpha_B \otimes 1}{\overset{(1 \otimes f \otimes 1)(1 \otimes \Delta)}{\rightrightarrows}} B \otimes GR \otimes (G/J)R.$$

Then by (4), we have $\tilde{f}(B)=\{\sum_{i=1}^h \rho_i(b) \otimes u_i \,;\ b \in B^J\}$ and $B^J=A$. One can easily check that the mapping $\theta: A \longrightarrow \tilde{f}(B)$ with $\theta(a)=\sum_{i=1}^h \rho_i(a) \otimes u_i$ is an isomorphism in $\mathscr{A}^{(G/J)R}$, and $\tilde{f}$ is an epimorphism, completing the proof.

The following theorem is a partial extension of [8, Th. 2.4].

Theorem 2.6. *Let* R *be a commutative algebra over the prime field* $GF(p)$ $(p \neq 0)$ *and* G *a cyclic group of order* p^m. *Then* $E(GR)$ *is of exponent* p^k $(1 \leqq k \leqq m)$.

Proof. By [8, Th. 2.3], if H is a cyclic group of order p, then $E(HR)$ is an abelian group of exponent p. Now by making use of the exact sequence (9), the theorem will be proved by the induction on m.

3. A group of strongly cyclic n-extensions and an example. Let R be a commutative ring which contains a primitive n-th root ζ of 1 such that n and $\{1-\zeta^i\,;\ i=1,2,\cdots,n-1\}$ are invertible in R. Let G be an abelian group of order n such that $G=\Pi_{i=1}^k G_i$, the direct product of cyclic groups G_i. A Galois GR-object A will be called a *strongly abelian Galois GR-object* if A is a strongly abelian extension of R with group G [7, Def. 2.1]. We denote $SE(GR)$ the set of GR-isomorphism classes of strongly abelian Galois GR-objects, which is a subset of $E(GR)$.

Lemma 3.1. *Let R and G be as above. Then $SE(GR)$ is a subgroup of $E(GR)$ and $SE(GR) \cong \Pi_{i=1}^{k} SE(G_iR)$.*

Proof. By [1, Prop. 3.8 and Th. 3.9], the mapping f: $\Pi_{i=1}^{k} E(G_iR) \longrightarrow E(GR)$ defined by $f((S_1), \cdots, (S_k)) = (S_1 \otimes \cdots \otimes S_k)$ is a group isomorphism and by [7, Th. 2.1 and Th. 2.2], $f(\Pi_{i=1}^{k} SE(GR)) = SE(GR)$. Thus by [8, Th. 3.4], $SE(GR)$ is a subgroup of $E(GR)$.

Next, we give a relation between $SE(GR)$ and the Picard group of R. The following theorem is given by L. N. Childs [2, Th. 9].

Theorem 3.2. *Let R and G be as above. Assume that R is a connected ring. Then we have group isomorphisms*

$$E(GR)/SE(GR) \cong \Pi_{i=1}^{k} E(G_iR)/SE(G_iR) \cong \Pi_{i=1}^{k} \mathrm{Pic}(R)(e_i),$$

where e_i is the order of G_i and $\mathrm{Pic}(R)(e_i)$ *is the elements of* $\mathrm{Pic}(R)$ *annihilated by e_i.*

Proof. By [8, Lemma 3.1], $SE(G_iR)$ is the set of isomorphism classes of Galois extensions which have normal basis and by [2, Th. 1 and Prop. 10], $E(G_iR)/SE(G_iR) \cong \mathrm{Pic}(R)(e_i)$. Thus the assertion is an immediate consequence of Lemma 3.1.

Now, we shall construct a quadratic Galois extension which has no normal basis. For further details, we refer the reader to [2, Prop. 10].

Proposition 3.3. *Let R be a commutative ring such that 2 is invertible in R, T a ring extension of R, and P a finitely generated projective rank 1 R-submodule of T such that $P \otimes P \ni a \otimes b \longmapsto ab \in R$ is an R-module isomorphism. If P is not free, then $S = R \oplus P$ is a Galois extension of R with group G of order 2 which is not a strongly cyclic extension of R in the sense of* [7, Def. 1.1], *that is, S has no normal basis* [8, Lemma 3.1].

Proof. By [5, Lemma 1 and Footnote 2(iii)], it is easy to see that S is a strongly cyclic 2-extension of R if and only if S is a free quadratic Galois extension of R. If $S = R \oplus P$ is a free R-module, then by [9, Lemma 2] P is a free R-module, which is a contradiction. Thus it suffices to that S is a Galois extension of R with group G of order 2. By [5, Prop. 2.3], a commutative R-algebra A is separable if and only if $A_{\mathfrak{m}} = R_{\mathfrak{m}} \otimes A$ is a separable $R_{\mathfrak{m}}$-algebra for every maximal ideal $\mathfrak{m}$ in R. Thus we may assume that R is a local ring. Since P is a finitely generated projective rank 1, P is a free R-module and $P = Rx$ where

$x^2 = u$ is invertible in R. Therefore by [4, Cor. 2.4] $S = R \oplus Rx \cong R[X]/(X^2 - u)$ is a separable R-algebra. Thus $S = R \oplus P$ is separable. Define $\sigma(r+p) = r - p$ $(r \in R,\ p \in P)$. Then σ is an R-algebra automorphism of S of order 2 and $S^{(\sigma)} = R$. Since 2 is invertible in R and the elements of P generate S, the elements of G are pairwise strongly distinct on S. Therefore S is a Galois extension of R with Galois group G.

Finally, we give an example of a connected ring R and a group G such that $E(GR) \supsetneqq SE(GR)$ and thus $\mathrm{Pic}(R)\,(2) \neq 0$ (see, Th. 3.2). The following example was given by R. G. Swan [10, Th. 4].

Example 3.4. Let $\boldsymbol{R}$ be the real number field and R' the subring of $\boldsymbol{R}[x_0,\ x_1]$ consisting of all polynomials all of whose terms have even degree. Let R be the canonical homomorphic image of R' in $\boldsymbol{R}[x_0,\ x_1]/(x_0^2 + x_1^2 - 1)$ and P the R-submodule of $\boldsymbol{R}[x_0,\ x_1]/(x_0^2 + x_1^2 - 1)$ generated by the canonical homomorphic images of x_0, x_1. Then R is connected, 2 is invertible in R, and P is a finitely generated projective rank 1 R-module which is not free [10, p. 271]. Since P is a projective rank 1 R-module, the epimorphism $P \otimes P \ni a \otimes b \longmapsto ab \in R$ is an isomorphism. Hence by Prop. 3.3, $\boldsymbol{R}[x_0,\ x\ ]/(x_0^2 + x_1^2 - 1)$ is a quadratic Galois extension of R which has no normal basis, that is, $E(GR) \supsetneqq SE(GR)$ and $\mathrm{Pic}(R)\,(2) \neq 0$.

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(*Received January 10, 1973*)