

ON A PROBLEM IN ADDITIVE NUMBER THEORY

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Let $g > 1$ be a fixed integer. Any positive integer n can be uniquely written in the form

$$n = \epsilon_1 g^{k-1} + \epsilon_2 g^{k-2} + \dots + \epsilon_{k-1} g + \epsilon_k,$$

where $\epsilon_i = \epsilon_i(n)$, $1 \leq i \leq k$, are integers such that $0 \leq \epsilon_i \leq g - 1$. We put

$$\alpha(n) = \sum_{i=1}^k \epsilon_i.$$

R. Bellman and H. Shapiro [1] proved the relation

$$\sum_{n \leq x} \alpha(n) = \frac{x \log x}{2 \log 2} + O(x \log \log x)$$

in the case of $g = 2$. L. Mirsky [2] and S. C. Tang [4] independently extended this result to the general case of $g \geq 2$, by establishing

$$\sum_{n \leq x} \alpha(n) = \frac{(g-1)x \log x}{2 \log g} + O(x).$$

In this paper we shall make a refinement on this result for the particular case of $g = 2$. Indeed, we prove the following

Theorem. *We have*

$$(A) \quad \liminf_{x \rightarrow \infty} \frac{1}{x} \left(\frac{x \log x}{2 \log 2} - \sum_{n \leq x} \alpha(n) \right) = 0,$$

and

$$(B) \quad \limsup_{x \rightarrow \infty} \frac{1}{x} \left(\frac{x \log x}{2 \log 2} - \sum_{n \leq x} \alpha(n) \right) = 1 - \frac{\log 3}{2 \log 2}.$$

1. Preliminaries

For any positive integer x we define

$$k = k(x) = \left\lceil \frac{\log x}{\log 2} \right\rceil + 1,$$

where $[z]$ is the integral part of z . Then it is easy to see that

$$k = \frac{\log x}{\log 2} + \frac{1}{\log 2} \log \frac{2^k}{x}.$$

Put

$$\sigma(x) = \sum_{i=1}^k (1 - 2\varepsilon_i(x)),$$

then

$$\sum_{n \leq x} \sigma(n) = \sum_{n \leq x} k(n) - 2 \sum_{n \leq x} \alpha(n).$$

But clearly

$$\begin{aligned} \sum_{n \leq x} k(n) &= xk - 2^k + k + 1 \\ &= \frac{x \log x}{\log 2} + \frac{x}{\log 2} \log \frac{2^k}{x} - 2^k + k + 1, \end{aligned}$$

where $k = k(x)$. Hence we have

$$\begin{aligned} &\frac{x \log x}{\log 2} - 2 \sum_{n \leq x} \alpha(n) \\ &= 2^k - \frac{x}{\log 2} \log \frac{2^k}{x} + \sum_{n \leq x} \sigma(n) - k - 1 \\ &= x h(x) + S(x) - k - 1, \end{aligned}$$

where

$$h(x) = \frac{2^k}{x} - \frac{1}{\log 2} \log \frac{2^k}{x},$$

and

$$S(x) = \sum_{n \leq x} \sigma(n).$$

It is readily seen that $h(x)$ decreases for $2^{k-1} \leq x < 2^k \log 2$ and increases for $2^k \log 2 < x < 2^k$, and that

$$(1) \quad \min_{\substack{2^{k-1} \leq x < 2^k \\ x: \text{integer}}} h(x) > h(2^k \log 2) = \frac{1 + \log \log 2}{\log 2}$$

and

$$(2) \quad \max_{2^{k-1} < x < 2^k} h(x) < h(2^{k-1}) = 1.$$

In order to estimate $S(x)$ we need some notations and lemmas.

Let $k \geq 1$. Denote by $\{c_{k,j}: 1 \leq j \leq 2^k\}$ the set consisting of the 2^k possible arrangements $c_{k,j}$ of k digits formed with 0's and 1's and ranked in ascending order of magnitude; e. g. $c_{2,1} = 00$, $c_{2,2} = 01$, $c_{2,3} = 10$ and $c_{2,4} = 11$. For any 0-1 sequence $\Delta = \delta_1 \delta_2 \cdots \delta_k$ of length l we write

$$\sigma(\Delta) = \sum_{i=1}^l (1 - 2\delta_i).$$

Now we put

$$T_k(j) = \sum_{i=1}^j \sigma(c_{k,i})$$

and define

$$T_k^* = \max_j T_k(j).$$

Further we set

$$l_k = \min(j; T_k^* = T_k(j))$$

and

$$L_k = \max(j; T_k^* = T_k(j)).$$

Lemma 1. For $k \geq 1$ and j , $1 \leq j \leq 2^k$, we have

$$T_k(j) = T_k(2^k - j) \geq 0.$$

It will be convenient to understand that $T_k(0) = 0$ for any $k \geq 1$.

Lemma 2. For $k \geq 1$, we have

$$l_{k+1} = L_k = \frac{1}{3}(2^{k+1} - (-1)^{k+1}).$$

Lemma 3. For $k \geq 1$, we have

$$T_k^* = \frac{1}{3}(2^{k+1} - \theta(k)),$$

where

$$\theta(k) = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ 2 & \text{if } k \text{ is even.} \end{cases}$$

Suppose $k \geq 5$. We set

$$j_k = \begin{cases} \frac{k-3}{2} & \text{for } k \text{ odd,} \\ \frac{k-4}{2} & \text{for } k \text{ even,} \end{cases}$$

and define for $0 \leq j \leq j_k - 1$

$$\begin{aligned} A_{k,j} &= \frac{1}{3} (2^k - 2^{k-2j}) = \sum_{i=1}^j 2^{k-2i}, \\ B_{k,j} &= A_{k,j} + 2^{k-2j-3} = \frac{1}{3} (2^k - 5 \cdot 2^{k-2j-3}), \\ C_{k,j} &= B_{k,j} + 2^{k-2j-4} = \frac{1}{3} (2^k - 7 \cdot 2^{k-2j-4}) \end{aligned}$$

and further

$$D_k = A_{k,j_k} = \begin{cases} l_k - 3 & \text{if } k \text{ is odd,} \\ l_k - 5 & \text{if } k \text{ is even.} \end{cases}$$

Lemma 4. *If $A_{k,j} < l \leq B_{k,j}$,*

then

$$T_k(l) \leq \frac{1}{3} (2^{k+1} - 3 \cdot 2^{k-2j-2} - \theta(k));$$

if $B_{k,j} < l \leq C_{k,j}$, then

$$T_k(l) \leq \frac{1}{3} (2^{k+1} - 7 \cdot 2^{k-2j-4} - \theta(k+1));$$

and if $C_{k,j} < l < A_{k,j+1}$, then

$$T_k(l) \leq \frac{1}{3} (2^{k+1} \cdot 3 \cdot 2^{k-2j-3} - \theta(k)).$$

For proofs of the above four lemmas see [3].

Let x be any integer such that $2^{k-1} \leq x < 2^k$. Put $j = x - 2^{k-1} + 1$, $1 \leq j \leq 2^{k-1}$. Then it is easy to see that the dyadic expansion $\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k$ of x can be written as $lc_{k-1,j}$. Hence we have

$$\sigma(x) = -1 + \sigma(c_{k-1,j})$$

and so

$$S(x) = S(2^{k-1} - 1) - j + T_{k-1}(j).$$

Especially we have from Lemma 1

$$(3) \quad S(2^k - 1) = -2^k + 1.$$

Hence we have

$$(4) \quad S(x) = -2^{k-1} + 1 - j + T_{k-1}(j).$$

Note that $T_{k-1}(j) = j + T_{k-2}(j)$, provided $1 \leq j \leq 2^{k-2}$. Thus we have also

$$(5) \quad S(x) = -2^{k-1} + 1 + T_{k-2}(j);$$

we notice that $-x < S(x) < 0$ for all $x \geq 1$.

Throughout in the rest of this section we suppose that $k > 5$. Put

$$x_i = 2^{k-1} - 1 + i 2^{k-5}, \quad 0 \leq i \leq 2^4.$$

Let x be a positive integer such that $x_{i-1} < x \leq x_i$, for some i , $1 \leq i \leq 2^4$, and write $j = x - x_{i-1} + L$, $1 \leq j \leq 2^{k-5}$. Then the dyadic expansion $\epsilon_1 \epsilon_2 \dots \epsilon_k$ of x is $l c_{i,i} c_{k-5,j}$. Hence we have

$$(6) \quad S(x) = -x + T_4(i-1) 2^{k-5} + j \sigma(c_{i,i}) + T_{k-5,j}.$$

The following table give the values of the $\sigma(c_{i,i})$ and the $T_4(i)$ for $0 \leq i \leq 2^4$.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\sigma(c_{i,i})$	0	4	2	2	0	2	0	0	-2	2	0	0	-2	0	-2	-2	-4
$T_4(i)$	0	4	6	8	8	10	10	10	8	10	10	10	8	8	6	4	0

2. Proof of (A)

By (3) we have

$$(7) \quad 0 > h(2^k - 1) + \frac{S(2^k - 1)}{2^k - 1} > -\frac{1}{2^k}$$

and by (2) and (4)

$$(8) \quad h(2^k) + \frac{S(2^k)}{2^k} = \frac{k-1}{2^{k-1}}.$$

Now we wish to show that if $k > 5$ and $2^{k-1} + 1 \leq x \leq 2^k - 2$, then

$$(9) \quad h(x) + \frac{S(x)}{x} > 0.$$

Case 1. $2^{k-1} + 1 \leq x \leq x_1$. Let l , $1 \leq l \leq k - 5$, be an integer such that $2^{k-1} - 1 + 2^{l-1} < x \leq 2^{k-1} - 1 + 2^l$. Put $j = x - 2^{k-1} + 1$ then $2^{l-1} < j \leq 2^l$.

Then we have

$$(10) \quad \begin{aligned} T_{k-1}(j) &= T_{k-1}(2^{l-1}) + (k - l - 4)(j - 2^{l-1}) + T_{l-1}(j - 2^{l-1}) \\ &\geq T_{k-1}(2^{l-1}) = (k - l - 2)2^{l-1}. \end{aligned}$$

From this and (4) we get

$$\frac{S(x)}{x} \geq \frac{-2^{k-1} + 2^{l-1}}{2^{k-1} + 2^{l-1}}.$$

Since $h(x)$ is decreasing we have

$$h(x) > h(2^{k-1} + 2^l) > \frac{2^k}{2^{k-1} + 2} + \frac{2^l}{2^{k-1}}.$$

Accordingly

$$h(x) + \frac{S(x)}{x} > \frac{2^l}{2^{k-1} + 2} + \frac{2^l}{2^{k-1}} - \frac{2^{l+1}}{2^{k-1} + 2} > 0.$$

Case 2. $x_1 \leq x \leq x_{15}$. Let $x_{i-1} \leq x \leq x_i$, $2 \leq i \leq 15$. Then by (6) and the table given in the preceding section we have

$$\frac{S(x)}{x} > -1 + \frac{2^{k-i} \min(T_i(i-1), T_i(i))}{2^{k-1} + i 2^{k-i}} \geq -\frac{27}{31}.$$

From this and (1) we have

$$h(x) + \frac{S(x)}{x} > \frac{1 + \log \log 2}{\log 2} - \frac{27}{31} > 0.$$

Case 3. $x_{15} < x \leq 2^k - 2$. Let l , $1 \leq l \leq k - 5$, be an integer such that $2^k - 1 - 2^l < x < 2^k - 1 - 2^{l-1}$. Put $j = x - 2^{k-1} + 1$, then $2^{l-1} \leq 2^{k-1} - j < 2^l$. From (4) and (10) we have

$$\frac{S(x)}{x} \geq \frac{-2^k + (k - l - 1)2^{l-1}}{2^k - 2}.$$

Since $h(x)$ is increasing we have

$$h(x) \geq h(2^k - 2^l) > \frac{2^k - 2^{l+1}}{2^k - 2^l}.$$

Hence

$$h(x) + \frac{S(x)}{x} > \frac{2^{l-1}}{2^k - 2} (k - 5 - l) \geq 0.$$

From (7), (8) and (9) we obtain

$$\liminf_{x \rightarrow \infty} \left(h(x) + \frac{S(x)}{x} \right) = 0.$$

This proves (A)

3. Proof of (B)

We estimate the magnitude of the quantity $\max_{2^{k-1} \leq x < 2^k} \left(h(x) + \frac{S(x)}{x} \right)$.

In what follows we suppose that $k > 7$. If $x = x_{i-1} + j - 1$, $1 \leq j \leq 2^{k-i}$, then from (6) and Lemma 3 we have

$$(11) \quad \frac{S(x)}{x} \leq -1 + \frac{3 \max(T_4(i-1), T_3(i)) + 2}{3(15+i)}.$$

Case 1. $x_{i-1} < x \leq x_i$, $i \leq 2$ or $i \geq 11$. By (2), (11) and the table in § 2 we readily find

i	=	1	2	11	12	13	14	15	16
$h(x) + \frac{S(x)}{x}$	<	$\frac{14}{48}$	$\frac{20}{51}$	$\frac{32}{78}$	$\frac{32}{81}$	$\frac{32}{84}$	$\frac{26}{87}$	$\frac{20}{90}$	$\frac{14}{93}$

Hence we have

$$h(x) + \frac{S(x)}{x} < \frac{32}{78} < 0.411.$$

Case 2. $x_{i-1} < x \leq x_i$, $7 \leq i \leq 10$. Since $h(x)$ is increasing in the interval $x_7 < x \leq x_{10}$ we have

$$(12) \quad h(x) < h(x_i + 1), \quad x_{i-1} < x \leq x_i, \quad 8 \leq i \leq 10.$$

But if $x_i < 2^k \log 2 < x_7$ then

$$(13) \quad h(x) < \max(h(x_6 + 1), h(x_7 + 1)), \quad x_6 < x \leq x_7.$$

Here

$$(14) \quad h(x_i + 1) = \frac{32}{16+i} + \frac{\log(16+i)}{\log 2} - 5, \quad 0 \leq i \leq 2^4.$$

From (11), (12), (13) and (14) we get

$$\begin{aligned}
h(x) + \frac{S(x)}{x} &< \frac{32}{66} + \frac{32}{23} + \frac{\log 23}{\log 2} - 6 & (x_6 < x \leq x_7); \\
&< \frac{32}{69} + \frac{32}{24} + \frac{\log 24}{\log 2} - 6 & (x_7 < x \leq x_8); \\
&< \frac{32}{72} + \frac{32}{25} + \frac{\log 25}{\log 2} - 6 & (x_8 < x \leq x_9); \\
&< \frac{32}{75} + \frac{32}{26} + \frac{\log 26}{\log 2} - 6 & (x_9 < x \leq x_{10}).
\end{aligned}$$

Hence we have

$$h(x) + \frac{S(x)}{x} < \frac{32}{66} + \frac{32}{23} + \frac{\log 23}{\log 2} - 6 < 0.402.$$

Case 3. $x_2 < x < 2^{k-1} + l_{k-2}$. From (5) and Lemma 3 we have (noticing that $k > 7$)

$$\begin{aligned}
\frac{S(x)}{x} &\leq \frac{-2^{k-1} + 1 + T_{k-2}^*}{2^{k-1} + l_{k-2} - 1} \\
&< -\frac{4}{7} + \frac{1}{3 \cdot 2^{k-3} + 2^{k-3}} \\
&< -\frac{127}{224}.
\end{aligned}$$

Since $h(x)$ is decreasing in this interval we have (using (14))

$$h(x) + \frac{S(x)}{x} < \frac{7}{9} + \frac{2 \log 3}{\log 2} - 3 - \frac{127}{224} < 0.4.$$

Case 4. $2^{k-1} + l_{k-2} \leq x < 2^{k-1} + L_{k-2}$. We have from (5)

$$\begin{aligned}
h(x) + \frac{S(x)}{x} &\leq \frac{2^k}{x} - \frac{1}{\log 2} \log \frac{2^k}{x} + \frac{-2^{k-1} + 1 + T_{k-2}^*}{x} \\
&= w(x), \text{ say.}
\end{aligned}$$

Since $x \geq 2^{k-1} + l_{k-2}$ we have

$$w'(x) = \frac{1}{x} \left(\frac{1}{\log 2} - \frac{2^{k-1} + 1 + T_{k-2}^*}{x} \right) > 0$$

and so

$$\begin{aligned}
 h(x) + \frac{S(x)}{x} &\leq w(2^{k-1} - 1 + L_{k-2}) \\
 (15) \qquad &= h(2^{k-1} - 1 + L_{k-2}) + \frac{S(2^{k-1} - 1 + L_{k-2})}{2^{k-1} - 1 + L_{k-2}} \\
 &= 2 - \frac{\log 3}{\log 2} + O\left(\frac{1}{2^k}\right).
 \end{aligned}$$

Case 5. $2^{k-1} + L_{k-2} < x \leq x_{..}$. Put $l = 2^{k-1} - 1 + 2^{k-2} - x$, so that $2^{k-4} \leq l < l_{k-2}$. By (5) and Lemma 1 we have

$$\begin{aligned}
 S(x) &= -2^{k-1} + 1 + T_{k-2}(x - 2^{k-1} + 1) \\
 &= -2^{k-1} + 1 + T_{k-2}(l).
 \end{aligned}$$

Now we appeal to Lemma 4. For $A_{k-2,j} < l \leq B_{k-2,j}$ we have

$$\begin{aligned}
 \frac{S(x)}{x} &\leq \frac{-2^{k-1} + 1 + T_{k-2}(l)}{2^{k-1} + 2^{k-2} - A_{k-2,j}} \\
 &\leq \frac{-2^{k-1} + 1 + \frac{1}{3}(2^{k-1} - 3 \cdot 2^{k-2j-4})}{2^{k-1} + 2^{k-2} + \frac{1}{3}(2^{k-2} - 2^{k-2j-2})} \\
 &= -\frac{1}{2} + O\left(\frac{1}{2^k}\right).
 \end{aligned}$$

Similarly we can easily verify that the inequality

$$(16) \qquad \frac{S(x)}{x} \leq -\frac{1}{2} + O\left(\frac{1}{2^k}\right)$$

holds for $B_{k-2,j} < l \leq C_{k-2,j}$ and for $C_{k-2,j} < l \leq A_{k-2,j+1}$. It is also clear that (16) holds true for $D_{k-2} < l < l_{k-2}$, since $D_{k-2} = l_{k-2} + O(1)$. On the other hand, $h(x)$ is decreasing in this interval, and so

$$\begin{aligned}
 h(x) &< h(2^{k-1} - 1 + L_{k-2}) \\
 &< \frac{5}{2} - \frac{\log 3}{\log 2} + O\left(\frac{1}{2^k}\right).
 \end{aligned}$$

From this and (16) we obtain

$$h(x) + \frac{S(x)}{x} \leq 2 - \frac{\log 3}{\log 2} + O\left(\frac{1}{2^k}\right).$$

As the result we have

$$h(x) + \frac{S(x)}{x} \leq 2 - \frac{\log 3}{\log 2} + O\left(\frac{1}{2^k}\right)$$

for all x , $2^{k-1} < x < 2^k$, since $2 - \frac{\log 3}{\log 2} > 0.415$. But by (15) the equality holds here at least for $x = 2^{k-1} - 1 + L_{k-2}$. Therefore

$$\limsup_{x \rightarrow \infty} \left(h(x) + \frac{S(x)}{x} \right) = 2 - \frac{\log 3}{\log 2},$$

which proves (B).

This completes the proof of our theorem.

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