ON THE IMMERSIONS OF MANIFOLDS IN ELLIPTIC SPACES AND A THEOREM OF S.S. CHERN-M. DO CARMO-S. KOBAYASHI AND T. OTSUKI

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Introduction. S. S. Chern, M. do Carmo and S. Kobayashi proved in [3] (see also [12]) that if V^n is a compact n-dimensional manifold minimally immersed in an $(n+p)$-dimensional space V^{n+p} of constant curvature c and the norm σ of the system of second fundamental forms of V^n in V^{n+p} satisfies $\sigma < nc\left(2-\frac{1}{p}\right)^{-1}$, then either $\sigma = 0$ or $\sigma = nc\left(2-\frac{1}{p}\right)^{-1}$. If $V^{n+p} = S^{n+1}$ (the $(n+1)$-dimensional unit sphere) and $\sigma = n$, then V^n is locally a Riemannian direct product of spaces V_1 and V_2 of constant curvatures $\frac{n}{m}$ and $\frac{n}{n-m}$, where dim $V_1 = m \geqslant 1$ and dim $V_2 = n-m \geqslant 1$. In the latter part of this statement the second fundamental form has but two eigenvalues with respective multiplicities dim V_1 and dim V_2. Moreover it is proved in Theorem 2 of the cited article that making an appropriate choice of the frame of reference the connexion form ω_B^A of S^{n+1} restricted to V^n, are given by

$$\left(\begin{array}{ccc|ccc|c}
\omega_1^1 & \cdots & \omega_m^1 & & & & \lambda\omega^1 \\
\vdots & & \vdots & & 0 & & \vdots \\
\omega_1^m & & \omega_m^m & & & & \lambda\omega^m \\
\hline
 & & & \omega_{m+1}^{m+1} & \cdots & \omega_n^{m+1} & \mu\omega^{m+1} \\
 & 0 & & \vdots & \cdots & \vdots & \vdots \\
 & & & \omega_{m+1}^n & \cdots & \omega_n^n & \mu\omega^n \\
\hline
-\lambda\omega^1 & \cdots & -\lambda\omega^m & -\mu\omega^{m+1} & \cdots & -\mu\omega^n & 0
\end{array}\right)$$

where $\lambda = \left(\frac{n-m}{m}\right)^{1/2}$ and $\mu = -\left(\frac{n}{n-m}\right)^{1/2}$.

T. Otsuki's main purpose in [8] is to treat the inverse problem for minimal hypersurfaces V^n in S^{n+1}. First of all he proves a local theorem on the integrability of the distributions of the spaces of principal vectors of a hypersurface in a Riemannian manifold of constant curvature. Making use of this theorem he investigates minimal hypersurfaces with constant multiplicities of the principal curvatures. He proves that if the minimal hypersurface has but two principal curvatures and their multiplicity is at least two, the minimal hypersurface V^n is locally a Riemannian direct product of two spheres of dimensions m and $n-m$ (i. e. the respective multiplicities of the principal curvatures) and radius $\left(\frac{m}{n}\right)^{1/2}$ and $\left(\frac{n-m}{n}\right)^{1/2}$. For such a minimal hypersurface $\sigma = n$. In addition T. Otsuki generalised certain theorems of [8] in [9] and [9′].

In [10] R. Rosca, L. Verstraelen and the author studied a similar minimal manifold V^4 in a 5-dimensional elliptic space P_e^5, especially with respect to the specific properties in such a space, and also considered related properties of symplectic manifolds and hamiltonian fields.

In the present contribution we again consider immersions of manifolds in more-dimensional elliptic spaces of curvature $+1$, which can always be realised by an appropriate preliminary homothetic transformation. We observe that certain results obtained in this work can be proved without difficulty to be valid in any Riemannian space with constant curvature.

The main purpose of our study is, just like T. Otsuki's in [8], to treat an inverse problem, dropping however the hypothesis that the immersion should be minimal and starting from another property of the higher matrix. Indeed, we observe that the connexion forms whose indices belong to the two different groups of indices corresponding to the two principal curvatures are trivial.

Let $\pi : V_{con}^{n+m}(n, m) \longrightarrow P_e^{n+m+1}$ be an isometric immersion of an $(n+m)$-dimensional manifold into an $(n+m+1)$-dimensional elliptic space of curvature $+1$, having but two principal curvatures of multiplicities n and m respectively and with tangential connexion forms belonging to both principal curvatures which are trivial as in the higher matrix. First, as did T. Otsuki, an integrability-property is proved (§ 4). This leads to the result that $V_{con}^{n+m}(n, m)$ is locally a Riemannian direct product of an n- and m-dimensional submanifold of constant curvature.

In § 2 an analogous problem is posed for a hypersurface $V^{2n} \subset P_e^{2n+1}$ having but n principal curvatures and whose connexion forms satisfy the same conditions. The results are similar, but in addition we find $V_{con}^4(2, 2) \subset P_e^5$ to be the only possible case. The existence of such hypersurfaces is proved. In § 3 a similar investigation is carried through in a special case of manifolds with codimension >1. Always special attention is devoted to the particular case of minimal manifolds and to the impact of the condition $\sigma = n$. It then turns out that 4-dimensional manifolds play an important role in the problem which is posed. Moreover it has to be mentioned the manifolds under consideration are always symplectic.

In §§ 5, 6 and 7 some special properties are indicated concerning the hypersurfaces obtained by dilatation, the rectilinear system of the normal and the sectional curvature of the hypersurfaces in question. This sectional curvature is determined by the plane section defined by the so-called curvature and distinguished fields of the manifold.

Finally in § 8 the generalisation of the theory of [10] is studied for non-minimally immersed hypersurfaces and this, amongst others, leads to a result about the field of curvature being hamiltonian.

1. Let P_e^{2n+1} be a $(2n+1)$-dimensional elliptic space of which the curvature is supposed to be reduced to unity by a previous homothetic transformation, and $\pi: V^{2n} \longrightarrow P_e^{2n+1}$ an isometric immersion of an orientable C^∞-hypersurface $(n \geqslant 2)$.

With the generating point X_0 of the hypersurface V^{2n} we associate an orthonormal simplex $S_{n+1} \equiv \{X_A\}$, $(A, B, C = 0, 1, \cdots, 2n+1)$ and suppose that the dual tangent space $T_{X_0}(V^{2n})$ of V^{2n} at X_0 is determined by the points X_i $(i, j, k = 1, 2, \cdots, 2n)$, the dual basis of $T_{X_0}(V^{2n})$ being $\omega^i(u^j \mid du^j)$. In these circumstances V^{2n} is structured by the connexion

$$\mathrm{d}X_A = \omega_A^B X_B, \tag{1}$$

where ω_A^B are the connexion 1-forms of the immersion π. The structural equations associated with π are:

$$\begin{gathered} \mathrm{d} \wedge \omega_B^A = \omega_B^C \wedge \omega_C^A, \\ (\omega_A^B + \omega_B^A = 0), \end{gathered} \tag{2}$$

with

$$\omega_A^B = \gamma_{Ai}^B \omega^i, \tag{3}$$

where γ_{Ai}^B are the connexion coefficients. Since V^{2n} is an integral manifold of $\omega^{2n+1} = 0$ we obtain by exterior differentiation:

$$\gamma_{ij}^{2n+1} = \gamma_{ji}^{2n+1}.$$

The second fundamental form φ of the hypersurface V^{2n} is defined by

$$\varphi = -\langle \mathrm{d}X_0, \mathrm{d}X_{2n+1} \rangle = \omega_i^{2n+1}\omega^i. \tag{4}$$

It is always possible to choose the simplex S_{2n+1} in such a way as to make φ diagonal, in which case

$$(5) \qquad \omega_i^{2n+1} = -\lambda_i \omega_i^0, \ (\lambda_i \neq 0),$$

where λ_i are the principal curvatures of V^{2n} at X_0. According to the definitions of [10] we call $K \in T_{X_0}(V^{2n})$ and $\omega \in \wedge^1 (V^{2n})$, given by

$$(6) \qquad \langle K, \ X_i \rangle = \lambda_i \ \text{and} \ \omega = \sum_i \omega^i,$$

respectively the *field of curvature* and the *distinguished* 1-*form* associated with π.

2. We now suppose that the $2n$-tuple of principal curvatures $(\lambda_1, \lambda_2, \cdots, \lambda_{2n})$ can be partitioned into n sets of two, say according to the index-groups $(i^*, n+i^*)$, $(i^*, j^* = 1, \cdots, n)$. Moreover we suppose that the tangential connexion forms ω_i^j satisfy

$$(7) \qquad \omega_i^j = 0, \ \ i \ \text{and} \ j \in \text{different index-groups},$$

and we denote such a hypersurface by V_{con}^{2n}. Exterior differentiation of these relations gives, with the help of (5):

$$(8) \quad 1 + \lambda_i \lambda_j = 0 \Rightarrow \lambda_{i^*} = \lambda_{n+i^*}; \ \ i \ \text{and} \ j \in \text{different index-groups}.$$

Thus we conclude that V^{2n} has but n different principal curvatures. In the following a hypersurface of this kind will be denoted by $V^{2n}(2, 2, \cdots, 2)$. From (8) then follows immediately $n = 2$, and consequently we can formulate the

Theorem. *Let $V^{2n}(2, 2, \cdots, 2)$ be a $2n$-dimensional hypersurface immersed in P_e^{2n+1} and having but n different and non-trivial principal curvatures. The only hypersurfaces of this kind whose tangential connexion forms ω_i^j are trivial if i and j belong to different principal curvatures (after having diagonalised the second fundamental form) are the $V_{con}^4(2, 2)$.*

For a $V_{con}^4(2, 2)$ it is obvious that the system of Pfaffian equations

$$(10) \qquad \omega^{i^*} = \omega^{2+i^*} = 0,$$

corresponding to the principal curvature λ_{j^*} $(j^* \neq i^*)$, is completely integrable. Furthermore, exterior differentiation of (5) gives:

$$(11) \qquad \lambda_{i^*} = \text{constant}.$$

Hence we can formulate the following:

Theorem. *The system of Pfaffian equations corresponding to a principal curvature of a hypersurface* $V^4_{con}(2, 2)$ *is completely integrable and the principal curvatures are constant.*

We now consider the integral submanifold V_1 of $V^4_{con}(2,2)$ corresponding to λ_1 and thus determined by $\omega^2 = \omega^4 = 0$. From the connexion (1) restricted to V_1 one obtains:

$$
(12) \qquad \begin{cases} dX_0 = \bar\omega^1 X_1 + \bar\omega^3 X_3, \\ dX_1 = -\bar\omega^1 X_0 + \bar\omega_1^3 X_3 + \lambda_1 \bar\omega^1 X_5, \\ dX_3 = -\omega^3 X_0 + \bar\omega_3^1 X_1 + \lambda_1 \bar\omega^3 X_5, \\ dX_5 = -\lambda_1(\bar\omega^1 X_1 + \bar\omega^3 X_3), \end{cases}
$$

$\bar\omega_A^B$ being the forms restricted to V_1. α_1, α_2 defining the dual base of this submanifold, the intrinsic curvature of V_1 is found with the help of the tangential connexion from α_1^2 from the formula

$$
(13) \qquad d \wedge \alpha_1^2 = -K_{in}\alpha^1 \wedge \alpha^2.
$$

From (12) we readily get:

$$
(14) \qquad \alpha^1 = \omega^1, \ \alpha^2 = \omega^3, \ \alpha_1^2 = \omega_1^3
$$

and so we obtain

$$
(15) \qquad K_{in} = 1 + \lambda_1^2.
$$

Thus the intrinsic curvature is constant. Furthermore we deduce from (12) that the submanifold V_1 belongs to the three-dimensional elliptic space P_e^3 spanned by the points (X_0, X_1, X_3, X_5).

For the submanifold V_2 corresponding to λ_2 and determined by $\omega^1 = \omega^3 = 0$ we find an analogous result. In this case

$$
(16) \qquad K_{in} = 1 + \lambda_2^2
$$

and V_2 belongs to $P_e^3 \equiv \{X_0, X_2, X_4, X_5\}$.

Moreover each point of $V^4_{con}(2,2)$ has a neighbourhood U which is the Riemannian product $V_1 \times V_2$. Thus we have the

Theorem. *Every* $V^4_{con}(2, 2) \subset P_e^5$ *is locally a Riemannian direct product of manifolds* V_1 *and* V_2 *of dimension two, of constant curvature and belonging to three-dimensional linear subspaces of* P_e^5. *The constant curvatures are respectively* $1 + \lambda_1^2$ *and* $1 + \lambda_2^2$, λ_1 *and* λ_2 *being the principal curvatures of* $V^4_{con}(2, 2)$.

$V^4_{con}(2,2)$ is a minimal hypersurface (or π is a minimal immersion) if

(17) $$\lambda_1 + \lambda_2 = 0 .$$

Under this hypothesis, it follows from (8):

(18) $$\lambda_1^2 = 1 .$$

On the other hand, as is known, the norm σ of φ is defined by

(19) $$\sigma = \sum_{i,j}' (\gamma_{ij}^{2n+1})^2 ,$$

and in the case under discussion one finds:

(20) $$\sigma = 2 (\lambda_1^2 + \lambda_2^2) .$$

We shall say in general that a manifold V is of the Kenmotsu type [6] if $\sigma = \dim V$. So we have the

Theorem. *A hypersurface $V_{con}^4 (2, 2)$ is minimal if and only if it is of the Kenmotsu type.*

Remark. Following Kenmotsu [6] a minimal $V_{con}^4 (2, 2)$ is an open submanifold of a Clifford hypersurface $M_{k,4-k}$.

A closed form Ω of the second degree and defined on a manifold V^{2n} for which $(\wedge \Omega)^n$ is a volume-form, is called a symplectic form and the manifold $V^{2n}(\Omega)$ itself a symplectic manifold [1], [5]. It is easy to verify that for $V_{con}^4 (2, 2)$

(21) $$\Omega = \sum_{i^*}' \omega^{i^*} \wedge \omega^{n+i^*}$$

is closed and $(\wedge \Omega)^2$ is a volume-form, and so we have the

Theorem. *The hypersurface $V_{con}^4 (2, 2)$ are symplectic with $\Omega = \sum_{i^*} \omega^{i^*} \wedge \omega^{n+i^*}$ as symplectic form.*

It isequally easy to prove the existence of the studied hypersurfaces. Indeed, they are determined by the following closed system Σ of Pfaffian equations:

(22) $$\begin{cases} \omega^5 = 0, \\ \omega_1^5 = \lambda_1 \omega^1, \ \omega_2^5 = -\lambda_1^{-1} \omega^2, \ \omega_3^5 = \lambda_1 \omega^3, \ \omega_4^5 = -\lambda_1^{-1} \omega^4, \\ \omega_1^2 = \omega_1^4 = \omega_3^2 = \omega_3^4 = 0, \\ d\lambda_1 = 0 . \end{cases}$$

From this it follows the

Theorem. *The hypersurfaces* $V^4_{con}(2, 2)$ *are determined by the system* Σ *which is in involution and whose general solution depends on* 10 *arbitrary constants. For a minimal* $V^4_{con}(2, 2)$ *this number of constants decreases with one.*

3. The preceding problems can be considered as particular cases of the following theory. Let $\pi: V^{2n} \longrightarrow P_e^{2n+N}$ now be an isometric immersion of an orientable C^∞-manifold with codimension N. At the generating point X_0 of V^{2n} we again associate an orthonormal simplex $S_{2n+N} \equiv \{X_A\}$, $(A, B, C = 0, 1, \cdots, 2n+N)$, such that V^{2n} is an integral manifold of

$$(23) \qquad \omega^r = 0, \; r, \; s = 2n+1, \cdots, 2n+N,$$

and $\omega^i(u^j \mid du^j)$, $(i, j = 1, 2, \cdots, 2n)$, is the dual basis of $T_{X_0}(V^n)$. The relations (1), (2) and (3) remain valid here, while the exterior differentiation of (23) gives, making use of E. Cartan's lemma:

$$(24) \qquad \gamma^r_{ij} = \gamma^r_{ji}.$$

A manifold V^{2n} with codimension N has N second fundamental forms φ_r, actually defined by

$$(25) \qquad \varphi_r = -\langle \mathrm{d}X_0, \mathrm{d}X_r \rangle = \omega^r_i \omega^i .$$

In the following we suppose that the normal connexion forms ω^s_r satisfy

$$(26) \qquad \omega^s_r = 0$$

(the normal connexion thus is trivial) and also suppose that it is possible to diagonalise all the forms φ_r simultaneously. We then say that V^{2n} possesses lines of curvature and we may write:

$$(27) \qquad \omega^r_i = -\lambda^r_i \omega^0_i .$$

To arrive at an analogous situation as in the previous section we suppose that it is possible to partition the indices into sets of two, say $(i^*, n+i^*)$, and that this partition is the same for all r. Moreover we make the hypotheses

$$(28) \qquad \begin{cases} \lambda_{i^*} = \lambda_{n+i^*}, \\ \omega^j_i = 0, \quad i \text{ and } j \in \text{different index-groups.} \end{cases}$$

In the following such a manifold will be denoted by

$$V^{2n}_{con}(2, 2, \cdots, 2).$$

One readily verifies that the system of Pfaffian equations

$$\omega^1 = \omega^2 = \cdots = \omega^{i^*-1} = \omega^{i^*+1} = \cdots = \omega^{n+i^*-1} = \omega^{n+i^*+1} = \cdots = \omega^{2n} = 0, \tag{29}$$

corresponding to $\lambda^r_{i^*}$, is completely integrable. Thus we again obtain integral submanifolds (surfaces) of $V^{2n}_{con}(2, 2, \cdots, 2)$ which we shall denote by $V_i{}^*$. Exterior differentiation of (27) directly leads to

$$\lambda^r{}_{i^*} = \text{constant}\,. \tag{30}$$

As in § 2 it is possible to find the intrinsic curvature for these surfaces. We obtain

$$K_{in} = 1 + \sum_r (\lambda^r{}_{i^*})^2\,, \tag{31}$$

which by (30) is always constant. Next from the connexion (1) we infer that $V_i{}^*$ belongs to the three-dimensional linear subspace P^3_e spanned by thn points X_0, X_{i^*}, X_{n+i^*}, $f\lambda^r{}_{i^*}X_r$ (f = factor of normalisation). We also notice that each point X of $V^{2n}_{con}(2, 2, \cdots, 2)$ has a neighbourhood U which is the Riemannian product $V_1 \times V_2 \times \cdots \times V_n$, and consequently we have proved the

Theorem. *Every manifold* $V^{2n}_{con}(2, 2, \cdots, 2) \subset P^{2n+N}_e$ *is locally a Riemannian direct product of* n *manifolds* V_{i^*} *with dimension* 2, *constant curvature* $K_{in} = 1 + \sum_r (\lambda^r{}_{i^*})^2$, *and belonging to three-dimensional subspaces of* P^{2n+N}_e. *The system of Pfaffian equations corresponding to any* $\lambda^r{}_{i^*}$ *is completely integrable and every* $\lambda^r{}_{i^*}$ *is constant.*

Exterior differentiation of the trivial tangential connexion forms (28) gives

$$1 + \sum_r \lambda^r{}_{i^*}\lambda^r{}_{j^*} = 0\,, \tag{32}$$

when i^* and j^* belong to different index-groups. If $V^{2n}_{con}(2, 2, \cdots, 2)$ is of the Kenmotsu type, i. e. if

$$\sigma = \sum_{i,r} (\lambda^r_i)^2 = 2n \tag{32}$$

or

$$n = \sum_{i^*,r} (\lambda^r{}_{i^*})^2\,, \tag{34}$$

it follows from (32) that

$$\sum_r (\gamma^r)^2 = 4n(2-n), \quad \gamma^r = \text{trace}\,[\gamma^r_{ij}]\,. \tag{32}$$

Confining ourselves to the real domain we thus notice that a $V^4_{con}(2, 2) \subset$

P_e^{4+N} of the Kenmotsu type is minimal. Conversely, let us suppose that $V_{con}^{2n}(2, 2, \cdots, 2)$ is a minimal manifold. Then

$$\gamma^r = 0,$$

and with the help of (32) we find

$$\sigma = 2n(n-1). \tag{36}$$

Consequently such a manifold is of the Kenmotsu type if $n=2$ or $V_{con}^{2n}(2, 2, \cdots, 2) \equiv V_{con}^4(2, 2) \subset P_e^{4+N}$, and so we obtain the

Theorem. *Let $V_{con}^{2n}(2, 2, \cdots, 2)$ be a manifold of the Kenmotsu type. Then we have $\sum_r (\gamma^r)^2 = 4n(2-n)$ and for a minimal manifold $V_{con}^{2n}(2, 2, \cdots, 2)$ the norm of the second fundamental form is given by $\sigma = 2n(n-1)$. Limiting ourselves to the real domain, a manifold $V_{con}^4(2, 2) \subset P_e^{4+N}$ is a minimal manifold if and only if it is of the Kenmotsu type.*

We remark that according to a theorem of Kenmotsu [6] we may say that a minimal manifold $V_{con}^4(2, 2)$ of P_e^{4+N} is a minimal hypersurface of a totally geodesic manifold $V^5 \subset P_e^{2n+N}$, i.e. of a P_e^5, and so we rediscover the manifolds of § 2. Indeed, we have $\sigma = 4$ and the normal connexion is trivial. The manifolds in question are thus open submanifolds of a Clifford minimal hypersurface $M_{k,4-k}$.

The following property can also be proved immediately:

Theorem. *The manifolds $V_{con}^{2n}(2, 2, \cdots, 2)$ are symplectic having*

$$\Omega = \sum_{i^*} \omega^{i^*} \wedge \omega^{n+i^*} \tag{37}$$

as symplectic form.

Besides the manifold $V_{con}^{2n}(2, 2, \cdots, 2)$ generated by X_0 we also consider the manifolds generated by the points X_r of the totally normal space. Thus we obtain a set of $N+1$ manifolds and from the formulae the following theorem concerning this configuration τ of manifolds is self-evident:

Theorem. *All manifolds of the configuration τ are $V_{con}^{2n}(2, 2, \cdots, 2)$.*

4. On the other hand it is also possible to consider the hypersurfaces $V_{con}^4(2, 2) \subset P_e^5$, treated in § 2, as a particular case of the following.

Let $\pi: V^{n+m} \longrightarrow P_e^{n+m+1}$ be an isometric immersion of an orientable C^∞-manifold with codimension 1. At the generating point X_0 of V^{n+m} we associate an orthonormal simplex $S_{n+m+1} \equiv \{X_A\}$, $(A, B, C = 0, 1, \cdots, n+m+1)$, such that V^{n+m} is an integral manifold of

$$\omega^{n+m+1} = 0.$$

$\omega^i(u^j|du^j)$, $(i,\ j = 1, \cdots, n+m)$, is the dual basis of $T_{X_0}(V^{n+m})$. The relations (1), (2) and (3) also apply to this situation, and exterior differentiation of (38) gives

$$\gamma_{ij}^{n+m+1} = \gamma_{ji}^{n+m+1}. \tag{39}$$

Next as it is possible to diagonalise the second fundamental form

$$\varphi = -\langle \mathrm{d}X_0,\ \mathrm{d}X_{n+m+1}\rangle = \omega_i^{n+m+1}\omega^i, \tag{40}$$

we get:

$$\omega_i^{n+m+1} = -\lambda_i\omega_i^0, \tag{41}$$

where λ_i are the principal curvatures of the hypersurface.

In addition we suppose that these principal curvatures can be ordered into two sets, say

$$\left\{\begin{array}{l}(\lambda_1, \cdots, \lambda_n),\\ (\lambda_{n+1}, \cdots, \lambda_{n+m})\end{array}\right. \qquad n,\ m > 1 \tag{42}$$

and again require the tangential connexion forms ω_i^j to satisfy

$$\omega_i^j = 0,\ i \text{ and } j \in \text{different index-groups.} \tag{43}$$

Exterior differentiation of these equations gives

$$1 + \lambda_i\lambda_j = 0,\ i \text{ and } j \in \text{different index-groups,} \tag{44}$$

which implies

$$\left\{\begin{array}{l}\lambda_1 = \lambda_2 = \cdots = \lambda_n,\\ \lambda_{n+1} = \lambda_{n+2} = \cdots = \lambda_{n+m}.\end{array}\right. \tag{45}$$

(We suppose the principal curvatures to be different from zero.) Hence such a hypersurface (denoted by $V_{con}^{n+m}(n, m)$) has only two principal curvatures.

A simple calculation shows that the system of Pfaffian equations

$$\omega^{n+1} = \cdots = \omega^{n+m} = 0, \tag{46}$$

corresponding to the principal curvature λ_1, and the system

$$\omega^1 = \cdots = \omega^n = 0, \tag{47}$$

corresponding to the second principal curvatuer λ_{n+1}, are both completely integrable. Consequently we may again consider integral submanifolds $V^n(\lambda_1)$ and $V^m(\lambda_{n+1})$ of V^{n+m} with respective dimensions n and m and we remark that exterior differentiation of (41) gives

(48) $$\lambda_i = \text{constant}.$$

First of all we consider the integral submanifold $V^n(\lambda_1)$ of $V^{n+m}_{con}(n, m)$. From

(49) $$d \wedge \omega^j_i = \omega^k_i \wedge \omega^j_k - (1 + \lambda_i^2)\omega^i \wedge \omega^j \quad (i, j, k = 1, 2, \cdots, n)$$

it follows that $V^n(\lambda_1)$ has a constant curvature $1 + \lambda_1^2$. Analogous considerations show that $V^m(\lambda_{n+1})$ has a constant curvature $1 + \lambda^2_{n+1}$. It also follows, with the help of the connexion (1), that $V^n(\lambda_1)$ belongs to the P^{n+1}_e spanned by $(X_0, X_1, \cdots, X_n, X_{n+m+1})$ and that $V^m(\lambda_{n+1})$ belongs to the P^{m+1}_e spanned by $(X, X_{n+1}, \cdots, X_{n+m}, X_{n+m+1})$. From this we conclude that each point of $V^{n+m}_{con}(n, m)$ has a neighbourhood U which is a Riemannian product $V^n(\lambda_1) \times V^m(\lambda_{n+1})$. We summarise results in the following

Theorem. *The system of Pfaffian equations* $\omega^{n+1} = \cdots = \omega^{n+m} = 0$, *respectively* $\omega^1 = \cdots = \omega^n = 0$, *corresponding to the principal curvature* λ_1, *respectively* λ_{n+1}, *of the hypersurface* $V^{n+m}_{con}(n, m)$ *is completely integrable, and the principal curvatures are constant on* $V^{n+m}_{con}(n, m)$. *The integral submanifolds* $V^n(\lambda_1)$, *respectively* $V^m(\lambda_{n+1})$, *of the systems have a constant curvature* $1 + \lambda_1^2$, *respectively* $1 + \lambda^2_{n+1}$, *and belong to a* $(n+1)$-, *respectively* $(m + 1)$-*dimensional linear subspace of* P^{n+m+1}_e. *Each* $V^{n+m}_{con}(n, m) \subset P^{n+m+1}_e$ *is locally a Riemannian direct product of the manifolds* $V^n(\lambda_1)$ *and* $V^m(\lambda_{n+1})$.

For a minimal hypersurface $V^{n+m}_{con}(n, m)$ we have

(50) $$n\lambda_1 + m\lambda_{n+1} = 0.$$

With the help of (44) we then find for the norm σ of the second fundamental form

(51) $$\sigma = n + m$$

and so again we obtain a hypersurface of the Kenmotsu type. Conversely, assuming that $V^{n+m}_{con}(n, m)$ is of the Kenmotsu type, we must have

(52) $$\sigma = n + m = \sum_{i,j} (\gamma^{n+m+1}_{ij})^2 = n\lambda_1^2 + m\lambda^2_{n+1},$$

which using (44) becomes:

(53) $$(n\lambda_1^2 - m)(\lambda_1^2 - 1) = 0.$$

We deduce that if the first factor of (53) is zero then $V_{con}^{n+m}(n, m)$ is minimal and if the second factor is zero $V_{con}^{n+m}(n, m)$ is minimal if $n = m$, and so we obtain the

Theorem. *Each minimal hypersurface $V_{con}^{n+m}(n, m)$ is of the Kenmotsu type. A $V_{con}^{2n}(n, n)$ is a minimal hypersurface if and only if it is of the Kenmotsu type.*

Remarks. a) Previously we supposed $n, m > 1$. However, the case $n > 1$, $m = 1$ can be included immediately into our considerations since then λ_1 remains constant, and in virtue of (44) also λ_{n+1} is constant.

b) Further we notice that the theory of the present section can be extended in a similar way as in § 3.

c) We refer to [6] to conclude that a minimal manifold $V_{con}^{n+m}(n, m)$ is an open submanifold of a Clifford minimal hypersurface $M_{k,n+m-k}$.

5. In the following three sections we shall prove some more properties relating to the hypersurfaces $V_{con}^{n+m}(n, m)$.

Consider the dilatation $\delta: X_0 \longrightarrow Y$ defined by

(54) $$Y = X_0 \cos c + X_{n+m+1} \sin c, \quad c \text{ constant.}$$

With the help of the connexion (1) and with (41) we find

(55) $$dY = (\cos c - \lambda_i \sin c)\,\omega^i X_i .$$

So the dual basis of Y is

$$\alpha^i = (\cos c - \lambda_i \sin c)\omega^i .$$

The factor between parentheses is constant, so the dilatation is a homothetic application. The second fundamental form φ_c is given by

(56) $$\varphi_c = -\langle dY, d(-X_0 \sin c + X_{n+m+1} \cos c)\rangle = \sum_i \frac{\sin c + \lambda_i \cos c}{\cos c - \lambda_i \sin c} (\alpha^i)^2.$$

The manifolds in question thus also have but two principal curvatures. With the help of (55) we find

(57) $$\alpha_i^j = 0, \quad i \text{ and } j \in \text{different index-groups},$$

which implies that the hypersurfaces Y are of the same type as the basic hypersurface $V_{con}^{n+m}(n, m)$. They are locally the Riemannian direct product of a submanifold V^n of constant curvature

(58) $$1+\left(\frac{\sin c+\lambda_1\cos c}{\cos c-\lambda_1\sin c}\right)^2$$

and a submanifold V^m of constant curvature

(59) $$1+\left(\frac{\sin c+\lambda_{n+1}\cos c}{\cos c-\lambda_{n+1}\sin c}\right)^2 .$$

The Lipschitz-Killing curvature $\bar{K}$ and the mean curvature $\bar{H}$ of Y are:

(60) $$\bar{K}=(-1)^{n+m}\left(\frac{\sin c+\lambda_1\cos c}{\cos c-\lambda_1\sin c}\right)^n\left(\frac{\sin c+\lambda_{n+1}\cos c}{\cos c-\lambda_{n+1}\sin c}\right)^m ,$$

(61) $$\bar{H}=\frac{1}{n+m}\left\{n\frac{\sin c+\lambda_1\cos c}{\cos c-\lambda_1\sin c}+m\frac{\sin c+\lambda_{n+1}\cos c}{\cos c-\lambda_{n+1}\sin c}\right\}$$

(62) $$=\frac{1}{n+m}\left\{\frac{(n\lambda_1+m\lambda_{n+1})\cos^2 c-(m\lambda_1+n\lambda_{n+1})\sin^2 c+2(m+n)\sin c\cos c}{\cos^2 c-\sin^2 c-(\lambda_1+\lambda_{n+1})\sin c\cos c}\right\}.$$

Both curvatures are of course constant.

Now suppose $V_{con}^{n+m}(n,\ m)$ is a minimal hypersurface. By virtue of (44) and (50) we obtain

(63) $$\lambda_1=\sqrt{\frac{m}{n}}\,,\qquad \lambda_{n+1}=-\sqrt{\frac{n}{m}}\,,$$

and so

(64) $$H=\frac{1}{n+m}\,\frac{2(n+m)\sin c\cos c+\left(n\sqrt{\frac{n}{m}}-m\sqrt{\frac{m}{n}}\right)\sin^2 c}{\cos^2 c-\sin^2 c-\left(\sqrt{\frac{m}{n}}-\sqrt{\frac{n}{m}}\right)\sin c\cos c} .$$

This proves that the hypersurface Y determined by

(65) $$\operatorname{tg} c=\frac{2(n+m)}{m\sqrt{\frac{m}{n}}-n\sqrt{\frac{n}{m}}}$$

is always a minimal hypersurface; more precisely it is the dual hypersurface of $V_{con}^{n+m}(n,\ m)$ if and only if $m=n$. Under this latter hypothesis we find

(66) $$\bar{K}=(-1)^n,\quad \bar{H}=\operatorname{tg} 2c.$$

If the basic hypersurface $V_{con}^{n+m}(n,\ m)$ is not minimal, (62) proves that there always exist two minimal hypersurfaces Y, determined by

(67) $$(n\lambda_1^2-m)\cos^2 c-(m\lambda_1^2-n)\sin^2 c+2(m+n)\lambda_1 \sin c\cos c = 0.$$

We thus obtain the

Theorem. *All hypersurfaces Y derived from $V_{con}^{n+m}(n, m) \subset P_e^{n+m+1}$ by dilatation are hypersurfaces of the same type as the basic hypersurface. Just like the latter one they have a constant mean and Lipschitz-Killing curvature. If $V_{con}^{n+m}(n, m)$ is minimal there exists exactly one hypersurface Y that also is minimal, and which is the dual hypersurface if and only if $n = m$. In this case the mean curvature of these hypersurfaces Y is $\overline{H} = \operatorname{tg} 2c$ and the Lipschitz-Killing curvature is $\overline{K} = (-1)^n$.*
If the basic hypersurface is non-minimal there still exist two minimal hypersurfaces Y.

6. The straight lines $L \equiv (X\, X_{n+m+1})$, which are associated with the former hypersurface (they are its normals), generate a rectilinear system $\mathscr{L}$ depending on $n+m$ parameters. The focal points F, given by

(68) $$F = X_0 \cos t + X_{n+m+1} \sin t,$$

are obtained by putting

(69) $$\mathrm{d}F = \rho X_i + \sigma X_{n+m+1}.$$

This implies

(70) $$\omega^i(\cos t - \lambda_i \sin t) = 0.$$

From (7) we can conclude that the rectilinear system $\mathscr{L}$ which is a normal system, has but two focal points:

(71) $$\begin{cases} F_1 = (\lambda_1 X_0 + X_{n+m+1})(1+\lambda_1^2)^{-\frac{1}{2}}, \\ F_2 = (\lambda_{n+1} X_0 + X_{n+m+1})(1+\lambda_{n+1}^2)^{-\frac{1}{2}}, \end{cases}$$

or from (44):

(72) $$\begin{cases} F_1 = f(\lambda_1 X_0 + X_{n+m+1}), \\ F_2 = f(X_0 - \lambda_1 X_{n+m+1}) \end{cases} \quad (f = \text{factor of normalisation}).$$

The focal points thus are rectilinear points on the line L. Further we immediately find:

(73) $$\mathrm{d}F_1 = f(\lambda_1 - \lambda_{n+1})(\omega^{n+1} X_{n+1} + \cdots + \omega^{n+m} X_{n+m}),$$

(74) $$\mathrm{d}F_2 = f(1+\lambda_1^2)(\omega^1 X_1 + \cdots + \omega^n X_n).$$

So the focal point F_1, respectively F_2, describes an m-dimensional, respectively an n-dimensional focal variety V^m, respectively V^n. One can derive without difficulty that all the second fundamental forms identically vanish, and so the focal manifolds are totally geodesic. We find that $(F_1)\equiv V^m$ is identical with the linear space spanned by the points F_1, $X_{n+1}, \cdots, X_{n+m}$ and $(F_2)\equiv V^n$ is identical with the linear space spanned by the points $F_2, X_1, \cdots, X_n$. These two linear spaces are disjoint and normal. The union of each of these subspaces with the normal L each time gives a linear space containing the submanifolds of constant curvatures $V^n(\lambda_1)$ and $V^m(\lambda_{n+1})$ of the hypersurface $V^{n+m}_{con}(n, m)$ under consideration.

We finally remark that the points $X_1, \cdots, X_n$, respectively $X_{n+1}, \cdots, X_{n+m}$ all generate the same linear space V^n, respectively V^m.

So we proved the following

Theorem. *The normals L of a hypersurface $V^{n+m}_{con}(n, m)\subset P_e^{n+m+1}$ generate a normal rectilinear system $\mathscr{L}$ depending on $n+m$ parameters. On each of its elements L there exist only two focal points F_1 and F_2, which are rectilinear points. They generate two focal manifolds V^m, respectively V^n, which are m-dimensional, respectively n-dimensional, linear subspaces. The spaces $L\cup V^m$ and $L\cup V^n$ are those spaces that contain the manifolds $V^m(\lambda_{n+1})$ and $V^n(\lambda_1)$. V^m and V^n are orthogonal and disjoint.*

Remark. Let us now consider a hypersurface $V^{n+m}(n, m)\subset P_e^{n+m+1}$ having but two principal curvatures with respective multiplicities n and m, dropping however the hypothesis $\omega_i^j=0$ if i and j belong to different index-groups. Also in this case the straight lines L generate a normal rectilinear system $\mathscr{L}$ depending on $n+m$ parameters. $\mathscr{L}$ has also two focal varieties (F_1) and (F_2) which now however in general are a V^{m+1} respectively a V^{n+1} because λ_1 and λ_{n+1} are no longer necessarily constant. For the directions

$$\omega^1=\cdots=\omega^n=0 \tag{75}$$

corresponding to the focal point F_2 the linear tangent space of the system $\mathscr{L}$ at any L is identical with the tangent space of the focal manifold generated by the other focal point F_1. This tangent space is identical with the 2-osculating space of curve described by F_2 if (75) is satisfied. An analogous result holds when we consider the system

$$\omega^{n+1}=\cdots=\omega^{n+m}=0 \tag{76}$$

corresponding to the focal point F_1.

Obviously it is possible to build a $V^{n+m}(n, m)$ starting from a normal rectilinear system $\mathscr{L} \subset P_e^{n+m+1}$ depending on $n+m$ parameters and having but two focal manifolds. The hypersurfaces in question are the orthogonal trajectories of $\mathscr{L}$.

7. We now return to the hypersurfaces $V_{con}^{n+m}(n, m)$ of P_e^{n+m+1}. We denote by $C(c_i)$ and $\overline{C}(\overline{c}_i)$ two orthonormal fields belonging to $T_{X_0}(V_{con}^{n+m}(n, m))$ and by $\theta(C)$ Otsuki's operator [7] which in the elliptic space may be defined by

$$\theta(C) = \sum_{i,j} \gamma_{ij}^{n+m+1} c_j X_i . \tag{77}$$

The *relative sectional curvature* $K(\pi) = K_{in}(\pi) - 1$ at $X_0 \in V_{con}^{n+m}(n, m)$ for the plane-element π determined by the fields C and $\overline{C}$ is then given by the expression

$$K(\pi) = \frac{P}{G} \tag{78}$$

where

$$P = \langle \theta(C), C \rangle \langle (\theta(\overline{C}), \overline{C} \rangle - \langle \theta(C), \overline{C} \rangle^2, \tag{79}$$

$$G = \|C\|^2 \|\overline{C}\|^2 - \|\langle C, \overline{C} \rangle\|^2 . \tag{80}$$

Now consider the plane-element π determined by the field of curvature $K(\lambda_i)$, defined in § 1, and the *distinguished field* $D(\mu_i = 1)$ corresponding to the distinguished 1-form of § 1. These fields are rectangular if and only if $n\lambda_1 + m\lambda_{n+1} = 0$, i. e. if and only if $V_{con}^{n+m}(n, m)$ is a minimal hypersurface. We then find for the corresponding relative sectional curvature:

$$K(\pi) = \frac{P}{G} = -\frac{n\lambda_1^2 + m\lambda_{n+1}^2}{n+m} = -\frac{n}{m}\lambda_1^2 . \tag{81}$$

We notice that this formula also holds for a $V^{n+m}(n, m)$ for which the condition on the connexion forms ω_i^j is dropped. However, in the case where the hypersurface is a $V_{con}^{n+m}(n, m)$ we find

$$\lambda_1^2 = \frac{m}{n} \tag{82}$$

and so $K(\pi) = -1$.

If the basic hypersurface $V^{n+m}(n, m)$ is not minimal we still can consider the same plane-element. In general we find

(83) $$K(\pi)=\lambda_1\lambda_{n+1}\,,$$

which for a $V^{n+m}_{con}(n, m)$ becomes:

(84) $$K(\pi)=-1\,.$$

Next let us suppose that $n=m$ and that $V^{2n}(n, n)$ is minimal. The dual hypersurface then also is minimal, and for the corresponding relative sectional curvature $K^*(\pi)$ we have

(85) $$K^*(\pi)=-\frac{1}{\lambda_1^2}\,,$$

and so

(86) $$\frac{K(\pi)}{K^*(\pi)}=\lambda_1^4$$

which gives an interpretation for the invariant λ_1.

We thus proved the following

Theorem. *We consider the plane-element* π *determined by the curvature field* K *and the distinguished field* D *of a hypersurface* $V^{n+m}(n, m)$. *The corresponding relative sectional curvature* $K(\pi)$ *is* $\lambda_1\lambda_{n+1}$, *if* λ_1 *and* λ_{n+1} *are the two principal curvatures of* $V^{n+m}(n, m)$. *For a hypersurface* $V^{n+m}_{con}(n, m)$ *we have* $K(\pi)=-1$, *while for a minimal* $V^{2n}(n, n)$ *we have* $\lambda_1^4=K(\pi)K^{*-1}(\pi)$, *if* $K^*(\pi)$ *is the corresponding relative sectional curvature of the dual hypersurface.*

8. In [10] we dealt with a special case of this theory, especially for minimal hypersurfaces. Now we shall drop the restriction of minimality and so obtain an extension of the problem we have considered in [10].

For this consider a hypersurface $V^{n+m}(n, m)\subset P_e^{n+m+1}$ having but two different principal curvatures with respective multiplicities n and m. To begin we do not suppose the manifold to be a $V^{n+m}_{con}(n, m)$-hypersurface.

According to [2] we call the submanifolds of curvature *totally holonomous* if each equation $\omega^i=0$ is completely integrable, i. e. if

(87) $$d\wedge\omega^i=\omega^i\wedge\alpha\,.$$

where α is a 1-form. It is easy to verify that a necessary and sufficient condition for (87) to be true is

(88) $$\gamma^i_{jk}=0\,,$$

where i, j and k are three different indices. In this case we may write (87) as:

(89) $$d \wedge \omega^i = \omega^i \wedge \sum_j \gamma^j_{ii} \omega^j .$$

Next we suppose the exterior derivatives of two dual forms ω^i, ω^j belonging to indices of a same index-group are *conformal* for that group. By virtue of (70) this implies

(90) $$\gamma^j_{ii} = 0, \quad i \text{ and } j \in \text{different index-groups} .$$

It then follows

(91) $$\omega^j_i = 0, \quad i \text{ and } j \in \text{different index-groups} ,$$

and so we obtain $V^{n+m}_{con}(n, m)$-hypersurfaces. Hence we have the

Theorem. *The hypersurfaces $V^{n+m}(n, m)$ having totally holonomous submanifolds of curvature and for which each pair of dual forms of a same index-group have conformal exterior derivatives for that group are $V^{n+m}_{con}(n, m)$-hypersurfaces.*

In § 2 we saw that the hypersurfaces $V^4_{con}(2, 2) \subset P^5_e$ are symplectic, with the symplectic form

(92) $$\Omega = \omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^4 .$$

A field $h(h_i) \in T_{X_0}(V_{con}(2, 2))$ is an *hamiltonian field* [1], [5] if

(93) $$d \wedge (h \lrcorner \Omega) = 0 .$$

We now consider the special case of the field of curvature K and the distinguished field D. Both fields are hamiltonian if and only if

(94) $$d \wedge (\omega^3 - \omega^1) + d \wedge (\omega^4 - \omega^2) = 0 .$$

For hypersurfaces $V^4_{con,h}(2, 2)$ this is equivalent with

(95) $$\begin{aligned} d \wedge (\omega^3 - \omega^1) = 0 , \\ d \wedge (\omega^4 - \omega^2) = 0 , \end{aligned}$$

and so we can formulate the following

Theorem. *A necessary and sufficient condition for the field of curvature (or resp. the distinguished field) of a symplectic hypersurface V^4_{con} (2, 2) to be hamiltonian is that the dual forms of a same index-group are homologous. For hypersurfaces $V^4_{con}(2, 2)$ this condition is only sufficient.*

9. In the whole preceding theory we supposed at least one of the multiplicties n, m to be greater than 1. It might be interesting however to mention also the case $m = n = 1$. We then obtain a surface $V^2 \subset P_e^3$. Such a manifold of course always is symplectic with the symplectic form

$$\Omega = \omega^1 \wedge \omega^2 . \tag{96}$$

The distinguished field is hamiltonian if and only if the dual form ω^1 and ω^2 are homologous. If

$$\omega_1^2 = 0 , \tag{97}$$

then the orthogonal field of the curvature field is always hamiltonian. If in addition V^2 is a minimal surface, also the curvature field itself is always hamiltonian. We remark that in the first case V^2 is a $\mathscr{B}$-surfacn (of Bianchi) and in the second one a Clifford surface.

In this respcet we return a moment to the hypersurfaces $V^4_{con,h}(2, 2)$ of P^5_e. The manifolds of curvature being totally holonomous, we can consider the integral submanifolds of

$$\omega^i = \omega^j = 0, \quad i \text{ and } j \in \text{different index-groups.} \tag{98}$$

Since the tangential connexion forms corresponding to two such indices are trivial, it follows that these manifolds are $\mathscr{B}$-surfaces. An easy calculation proves the

Theorem. *Each* $V^4_{con,h}(2, 2) \subset P^5_e$ *is locally a Riemannian direct product of two* $\mathscr{B}$*-surfaces.*

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(Received March 14, 1973)