ON THE BESSEL-SERIES EXPRESSION FOR $\sum \frac{1}{n} \sin \frac{x}{n}$

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0. In their research [1] after Tauberian theorems for Lambert series, Hardy and Littlewood made use of the unboundedness of the functions

$$P(x) = \sum_{n \le x} \frac{1}{n} \cos \frac{x}{n}, \qquad Q(x) = \sum_{n \le x} \frac{1}{n} \sin \frac{x}{n},$$

to show the Tauberian conditions being best possible. They could show by an ingenious method that as $x \to \infty$,

$$P(x) = \Omega(\log\log x), \qquad Q(x) = \Omega(\sqrt{\log\log x}).\ ^{1)}$$

Recently, S. L. Segal [2] studied $\widetilde{Q}(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{n}$, which differs only $O(1)$ from $Q(x)$, which led him above all to the 'Bessel-series' expression

$$(0) \qquad \int_0^y \widetilde{Q}(x)dx = \frac{\pi}{2} y - \frac{1}{2} + \left(\frac{\pi}{2} y\right)^{\frac{1}{2}} \sum_{n=1}^{\infty} n^{-\frac{1}{2}} J_1(2\sqrt{2\pi yn}).\ ^{2)}$$

He stated in the concluding remarks of [2] a question that if we had the expression

$$(1) \qquad \widetilde{Q}(x) = \frac{\pi}{2} + \pi \sum_{n=1}^{\infty} J_0(2\sqrt{2\pi xn}), \qquad (x > 0),$$

which was obtained by formal term-by-term differentiation of (0). In this paper we shall show that the series on the right-hand side of (1) diverges for all x, though it is summable $(C, 1)$ for all x.[3] We can prove in fact that if

$$(2) \qquad \widetilde{Q}_N(x) = \frac{\pi}{2} + \pi \sum_{n=1}^{N} J_0(2\sqrt{2\pi xn}),$$

then for all $x > 0$ we have

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1) $f(x) = \Omega(g(x))$ means $f(x) \neq o(g(x))$.

2) See [3] about all notations and properties for Bessel functions.

3) S. L. Segal communicated to us that he also had become aware of this fact after writing his paper [2], but (3) and (5) were unknown to him.

$$(3)\qquad \begin{cases} \limsup\limits_{N\to\infty} N^{-\frac{1}{4}} \widetilde{Q}_N(x) = \pi^{-\frac{1}{4}}(2x)^{-\frac{3}{4}}, \\ \liminf\limits_{N\to\infty} N^{-\frac{1}{4}} \widetilde{Q}_N(x) = -\pi^{-\frac{1}{4}}(2x)^{-\frac{3}{4}}. \end{cases}$$

Furthermore if we denote by $D(x)$ the $(C, 1)$ means of the series

$$(4)\qquad \sum_{n=1}^{\infty} \frac{\cos\left(2\sqrt{2\pi x n} - \frac{\pi}{4}\right)}{n^{\frac{1}{4}}},$$

then we obtain the formula

$$(5)\qquad \widetilde{Q}(x) = \frac{\pi}{2} + \left(\frac{\pi}{2}\right)^{\frac{1}{4}} x^{-\frac{1}{4}} D(x) + O(x^{-\frac{1}{4}}),$$

as $x \to \infty$. The methods we used to obtain (3) and (5) are elementary except (0) and Hankel's formulae (6) and (24).

1. It is known ([3] p. 198) that

$$(6)\qquad J_0(t) = \sqrt{\frac{2}{\pi t}}\left\{\cos\left(t - \frac{\pi}{4}\right) + \frac{1}{8t}\sin\left(t - \frac{\pi}{4}\right)\right\} + R(t),$$

where $R(t) = O(t^{-\frac{5}{2}})$ as $t \to \infty$. By substitution $t = x\sqrt{n}\,(x > 0)$ in (6), we have as $n \to \infty$,

$$(7)\qquad J_0(x\sqrt{n}) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{x}}\left\{\frac{\cos\left(x\sqrt{n} - \frac{\pi}{4}\right)}{n^{\frac{1}{4}}} + \frac{1}{8x}\cdot\frac{\sin\left(x\sqrt{n} - \frac{\pi}{4}\right)}{n^{\frac{3}{4}}}\right\} + O\left(x^{-\frac{5}{2}} n^{-\frac{5}{4}}\right).$$

The following simple lemmas are useful for our purpose.

Lemma 1. *For any real sequence* $\{a_n\}$ $(n = 1, 2, \cdots)$ *we have*

(i) $\sin a_n - \sin a_{n-1} = (a_n - a_{n-1})\cos a_n + \frac{1}{2}(a_n - a_{n-1})^2 \sin a_n + O(|a_n - a_{n-1}|^3),$

and

(ii) $\cos a_n - \cos a_{n-1} = -(a_n - a_{n-1})\sin a_n + \frac{1}{2}(a_n - a_{n-1})^2 \cos a_n + O(|a_n - a_{n-1}|^3),$

where both O's do not depend on a_n.

Lemma 2. $\sqrt{n}-\sqrt{n-1}=\frac{1}{2\sqrt{n}}+O(n^{-\frac{3}{2}})$, $(\sqrt{n}-\sqrt{n-1})^2=\frac{1}{4n}+O(n^{-2})$, $n^{\frac{1}{4}}-(n-1)^{\frac{1}{4}}=\frac{1}{4}n^{-\frac{3}{4}}+O(n^{-\frac{7}{4}})$.

Now, let us define

$$I_n=\int_{n-1}^{n}\frac{\sin\left(x\sqrt{u}-\frac{\pi}{4}\right)}{u^{\frac{3}{4}}}\,du\,.$$

By partial integration, we have

$$I_n=4\left\{n^{\frac{1}{4}}\sin\left(x\sqrt{n}-\frac{\pi}{4}\right)-(n-1)^{\frac{1}{4}}\sin\left(x\sqrt{n-1}-\frac{\pi}{4}\right)\right\}$$

$$-2x\int_{n-1}^{n}\frac{\cos\left(x\sqrt{u}-\frac{\pi}{4}\right)}{u^{\frac{1}{4}}}\,du$$

$$=4n^{\frac{1}{4}}\left\{\sin\left(x\sqrt{n}-\frac{\pi}{4}\right)-\sin\left(x\sqrt{n-1}-\frac{\pi}{4}\right)\right\}$$

$$+4\{n^{\frac{1}{4}}-(n-1)^{\frac{1}{4}}\}\sin\left(x\sqrt{n-1}-\frac{\pi}{4}\right)-2x\int_{n-1}^{n}\frac{\cos\left(x\sqrt{u}-\frac{\pi}{4}\right)}{u^{\frac{1}{4}}}\,du.$$

Since

$$\sin\left(x\sqrt{n}-\frac{\pi}{4}\right)-\sin\left(x\sqrt{n-1}-\frac{\pi}{4}\right)=\frac{x}{2\sqrt{n}}\cos\left(x\sqrt{n}-\frac{\pi}{4}\right)$$

$$+\frac{x^2}{8n}\sin\left(x\sqrt{n}-\frac{\pi}{4}\right)+O(x^3n^{-\frac{3}{2}})+O(xn^{-\frac{3}{2}})+O(x^2n^{-2})$$

and

$$4\{n^{\frac{1}{4}}-(n-1)^{\frac{1}{4}}\}\sin\left(x\sqrt{n-1}-\frac{\pi}{4}\right)=\frac{\sin\left(x\sqrt{n}-\frac{\pi}{4}\right)}{n^{\frac{3}{4}}}$$

$$+O(x^3n^{-\frac{5}{4}})+O(n^{-\frac{5}{4}}),$$

by Lemma 2, it follows that

$$I_n = 2x \frac{\cos\left(x\sqrt{n}-\frac{\pi}{4}\right)}{n^{\frac{1}{4}}} + \left(1+\frac{x^2}{2}\right)\frac{\sin\left(x\sqrt{n}-\frac{\pi}{4}\right)}{n^{\frac{3}{4}}} \tag{8}$$
$$- 2x\int_{n-1}^{n} \frac{\cos\left(x\sqrt{u}-\frac{\pi}{4}\right)}{u^{\frac{1}{4}}} du + O(x^3 n^{-\frac{5}{4}}) + O(n^{-\frac{5}{4}}) + O(xn^{-\frac{3}{2}}) + O(x^2 n^{-2}).$$

Hence,

$$\int_1^N \frac{\sin\left(x\sqrt{u}-\frac{\pi}{4}\right)}{u^{\frac{3}{4}}} du = 2x\sum_{n=2}^{N} \frac{\cos\left(x\sqrt{n}-\frac{\pi}{4}\right)}{n^{\frac{1}{4}}} \tag{9}$$
$$+ \left(1+\frac{x^2}{2}\right)\sum_{n=2}^{N} \frac{\sin\left(x\sqrt{n}-\frac{\pi}{4}\right)}{n^{\frac{3}{4}}} - 2x\int_1^N \frac{\cos\left(x\sqrt{u}-\frac{\pi}{4}\right)}{u^{\frac{1}{4}}} du$$
$$+ O(x^3) + O(1).$$

By partial integration, we have

$$\int_1^N \frac{\cos\left(x\sqrt{u}-\frac{\pi}{4}\right)}{u^{\frac{1}{4}}} du = \frac{2}{x} N^{\frac{1}{4}} \sin\left(x\sqrt{N}-\frac{\pi}{4}\right) - \frac{2}{x}\sin\left(x-\frac{\pi}{4}\right) \tag{10}$$
$$- 2x\int_1^N \frac{\sin\left(x\sqrt{u}-\frac{\pi}{4}\right)}{u^{\frac{3}{4}}} du.$$

Thus we obtain from (9) and (10),

$$\sum_{n=1}^{N} \frac{\cos\left(x\sqrt{n}-\frac{\pi}{4}\right)}{n^{\frac{1}{4}}} = \frac{2}{x} N^{\frac{1}{4}} \sin\left(x\sqrt{N}-\frac{\pi}{4}\right) + \frac{1-4x^2}{2x} \times \tag{11}$$
$$\int_1^N \frac{\sin\left(x\sqrt{u}-\frac{\pi}{4}\right)}{u^{\frac{3}{4}}} du - \frac{x^2+2}{4x}\sum_{n=1}^{N} \frac{\sin\left(x\sqrt{n}-\frac{\pi}{4}\right)}{n^{\frac{3}{4}}} + O(x^2) + O\left(\frac{1}{x}\right).$$

But it can easily be seen that

$$\int_1^N \frac{\sin\left(x\sqrt{u}-\frac{\pi}{4}\right)}{u^{\frac{3}{4}}} du = O\left(\frac{1}{x}\right),$$

uniformly in N. Therefore from (7) (for $n \geq n_0$, sufficiently large),

$$\pi\sum_{n=n_0}^{N}J_0(x\sqrt{n})=2\sqrt{2\pi}\,x^{-\frac{3}{2}}N^{\frac{1}{4}}\sin\left(x\sqrt{N}-\frac{\pi}{4}\right)$$

$$(12)\qquad -\frac{\pi}{8}x^{-\frac{3}{2}}(2x^2+3)\sum_{n=1}^{N}\frac{\sin\left(x\sqrt{n}-\frac{\pi}{4}\right)}{n^{\frac{3}{4}}}+O(x^{\frac{3}{2}})+O(x^{-\frac{5}{2}})$$

$$+O((\sqrt{x}+x^{-\frac{3}{2}})n_0^{\frac{1}{4}}).$$

Next, we shall show that the series

$$(13)\qquad \sum_{n=1}^{\infty}\frac{\sin\left(x\sqrt{n}-\frac{\pi}{4}\right)}{n^{\frac{3}{4}}},$$

converges for all $x>0$. In fact this is implicitly shown it [2]. But it seems interesting here to give its elementary proof. Let us put

$$S_N(x)=\sum_{n=1}^{N}\frac{\sin\left(x\sqrt{n}-\frac{\pi}{4}\right)}{n^{\frac{3}{4}}},$$

and

$$T_N(x)=\sum_{n=1}^{N}\sin\left(x\sqrt{n}-\frac{\pi}{4}\right).$$

Then partial summation shows

$$S_N(x)=\sum_{n=1}^{N-1}\{n^{-\frac{3}{4}}-(n+1)^{-\frac{3}{4}}\}\,T_n(x)+N^{-\frac{3}{4}}T_N(x).$$

On the other hand, Euler-Maclaurin summation formula shows

$$(14)\qquad T_N(x)=O(x\sqrt{N})+O\left(\frac{\sqrt{N}}{x}\right)+O\left(\frac{1}{x^2}\right).$$

Hence

$$(15)\qquad \begin{aligned}&\sum_{n=1}^{N-1}|n^{-\frac{3}{4}}-(n+1)^{-\frac{3}{4}}|\cdot|T_n(x)|\\&=O\left(\sum_{n=1}^{N}n^{-\frac{7}{4}}\left(x\sqrt{n}+\frac{\sqrt{n}}{x}+\frac{1}{x^2}\right)\right)=O(x)+O\left(\frac{1}{x}\right)+O\left(\frac{1}{x^2}\right),\end{aligned}$$

and

$$(16)\qquad N^{-\frac{3}{4}}|T_N(x)|=O(xN^{-\frac{1}{4}})+O(x^{-1}N^{-\frac{1}{4}})+O(x^{-2}N^{-\frac{3}{4}}).$$

In view of (15) and (16), we know that (13) converges uniformly in any finite positive interval. We are now in a position to obtain (3) on substituting $2\sqrt{2\pi x}$ for x in (12).

Next, we shall consider the $(C, 1)$ means of (12). Since (13) converges for all $x>0$ and the error term in (12) is in fact the sum of certain absolutely and uniformly (in any finite positive interval) convergent series, it is sufficient to prove that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} n^{\frac{1}{4}}\sin\left(x\sqrt{n}-\frac{\pi}{4}\right)=0, \tag{17}$$

for all $x>0$. But on applying partial summation and by (14), we have

$$\sum_{n=1}^{N} n^{\frac{1}{4}}\sin\left(x\sqrt{n}-\frac{\pi}{4}\right)=O(xN^{\frac{3}{4}})+O(x^{-1}N^{\frac{3}{4}})+O(x^{-2}N^{\frac{1}{4}}),$$

and this proves (17).

2. The idea of proving (5) is roughly as follows: First, in Segal's formula (0), we replace the series on the right-hand side by its $(C, 1)$ means and define

$$L_n(x)=\sum_{k=1}^{n} k^{-\frac{1}{2}} J_1(2\sqrt{2\pi xk}),$$

$$\sigma_n(x)=\frac{1}{n}\sum_{k=1}^{n} L_k(x).$$

Then we show that

$$\lim_{n\to\infty}\frac{d}{dx}\{\sqrt{x}\,\sigma_n(x)\} \tag{18}$$

converges uniformly in any finite positive interval so that we can differentiate

$$2\sum_{n=1}^{\infty}\sin^2\left(\frac{x}{2n}\right)=\frac{\pi}{2}x-\frac{1}{2}+\left(\frac{\pi x}{2}\right)^{\frac{1}{2}}\lim_{n\to\infty}\sigma_n(x), \tag{19}$$

trm-by-term to yie ld the formula (5).

Now, let us define

$$K_n(x)=\sum_{k=1}^{n} J_0(2\sqrt{2\pi xk}),$$

then we obtain

$$\frac{d}{dx}\{\sqrt{x}\,\sigma_N(x)\}=\left(\frac{1}{2}-\sqrt{\frac{\pi}{2}}\right)x^{-\frac{1}{2}}\sigma_N(x)+\sqrt{2\pi}\,\frac{1}{N}\sum_{n=1}^{N} K_n(x). \tag{20}$$

On the other hand we have from (6)

$$\begin{aligned}(21)\qquad \sqrt{2\pi}\,K_n(x) &= \left(\frac{2}{\pi}\right)^{\frac{1}{4}} x^{-\frac{1}{4}} \sum_{k=1}^{n} k^{-\frac{1}{4}} \cos\left(2\sqrt{2\pi xk}-\frac{\pi}{4}\right)\\ &+ \frac{1}{32}\left(\frac{2}{\pi}\right)^{\frac{3}{4}} x^{-\frac{3}{4}} \sum_{k=1}^{n} k^{-\frac{3}{4}} \sin\left(2\sqrt{2\pi xk}-\frac{\pi}{4}\right) + E_n(x),\end{aligned}$$

where $E_n(x) = \sum_{k=1}^{n} R(2\sqrt{2\pi xk})$. Since $R(t) = O(t^{-\frac{5}{2}})$ as $t \to \infty$, we find that

$$\sum_{k=1}^{n} |R(2\sqrt{2\pi xk})| = O(x^{-\frac{5}{4}} \sum_{k=1}^{n} k^{-\frac{5}{4}}) = O(x^{-\frac{5}{4}}), \quad (x \to \infty)$$

and that $\lim_{n\to\infty} E_n(x)$ converges uniformly in any finite positive interval $a \leq x \leq b$ with sufficiently large a. From (20) and (21) we obtain

$$\begin{aligned}(22)\qquad \frac{d}{dx}\{\sqrt{x}\,\sigma_N(x)\} &= \left(\frac{1}{2}-\sqrt{\frac{\pi}{2}}\right) x^{-\frac{1}{2}} \sigma_N(x)\\ &+ \left(\frac{2}{\pi}\right)^{\frac{1}{4}} x^{-\frac{1}{4}} \frac{1}{N} \sum_{n=1}^{N} \left\{\sum_{k=1}^{n} k^{-\frac{1}{4}} \cos\left(2\sqrt{2\pi xk}-\frac{\pi}{4}\right)\right\}\\ &+ \frac{1}{32}\left(\frac{2}{\pi}\right)^{\frac{3}{4}} x^{-\frac{3}{4}} \frac{1}{N} \sum_{n=1}^{N} \left\{\sum_{k=1}^{n} k^{-\frac{3}{4}} \sin\left(2\sqrt{2\pi xk}-\frac{\pi}{4}\right)\right\}\\ &+ \frac{1}{N}\sum_{n=1}^{N} E_n(x).\end{aligned}$$

But we have already proved in the preceding section that the series (4) is summable $(C, 1)$ to $D(x)$ uniformly in any finite positive interval as well as the series (13). On the other hand (15) and (16) show

$$(23)\qquad \sum_{n=1}^{\infty} n^{-\frac{3}{4}} \sin\left(2\sqrt{2\pi xn}-\frac{\pi}{4}\right) = O(\sqrt{x}), \quad (x\to\infty).$$

Finally, Hankel's asymptotic formula

$$(24)\qquad J_1(t) = \sqrt{\frac{2}{\pi t}}\left\{\sin\left(t-\frac{\pi}{4}\right) + O\left(\frac{1}{t}\right)\right\}, \qquad (t \to \infty)$$

clearly shows that

$$\lim_{n\to\infty} \sigma_n(x) = \sum_{n=1}^{\infty} n^{-\frac{1}{2}} J_1(2\sqrt{2\pi xn})$$

$$= \pi^{-\frac{3}{4}} (2x)^{-\frac{1}{4}} \sum_{n=1}^{\infty} \frac{\sin\left(2\sqrt{2\pi x n} - \frac{\pi}{4}\right)}{n^{\frac{3}{4}}} + O(x^{-\frac{3}{4}}), \quad (x \to \infty).$$

Hence the desired formula (5) follows from (22) and (23).

Remarks. 1) One would expect an improved Ω-result for $Q(x)$ from (5): e. g. one may imagine that $Q(x) = \Omega(\log\log x)$ or even $Q(x) = \Omega((\log x)^{\frac{1}{4}})$. However, we do not yet succeed in supporting the question.

2) We can obtain (5) from a certain formula, essentially the same as (0), of which the proof is elementary but complicated.

3) More precise estimates than (14) and (23) do in fact hold and similar trigonometrical sums appear in the investigation of some probelms related to 'divisor problem' or 'circle problem'.

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