PSEUDO-UMBILICAL AND MINIMAL MANIFOLDS WITH CONSTANT RIEMANNIAN CURVATURE

L. VANHECKE

Introduction. In a series of recent papers [1]-[9] Bang-yen Chen published a great diversity of interesting studies concerning pseudo-umbilical and minimal submanifolds immersed in a Riemannian manifold of constant curvature. In two of these [1], [2] he investigated such surfaces which also have constant Gauss curvature. Among others he proves the following remarkable results:

1. *Let M be a pseudo-umbilical surface of constant Gauss curvature in a Riemannian space $R^m(c)$ of curvature c. If the mean curvature is constant and the normal curvature K_N is trivial, then M is either totally umbilical in $R^m(c)$ or isometric with a Euclidean plane. If in particular $c \geqslant 0$, then M is either totally umbilical or contained in a Clifford torus.*

2. *Let M be a minimal surface of a 3-sphere S^3 with constant Gauss curvature. Then M is either totally geodesic or contained in a Clifford torus in S^3.*

3. *If M is a minimal surface of $R^m(c)$ with constant Gauss curvature and trivial normal curvature, then M is either totally geodesic or isometric with a Euclidean plane.*

In the underlying paper (§§ 2 and 3) these properties are generalised to an n-dimensional manifold of an elliptic space $P_e^{n+N}(\mathbf{R})$ of dimension $n+N$ and of curvature $+1$. It is plausible however that these extensions are equally valid for an $R^m(c)$.

The study of such manifolds in elliptic spaces makes it possible, as shown in §§ 4, 5 and 6, to obtain so-called supplementary or dual manifolds and to study their properties. Some properties of the configurations thus obtained are proved in [15] and [16].

1. Let P_e^{n+N} be an $(n+N)$-dimensional elliptic space whose curvature is supposed to be reduced to unity by an appropriate homothetic transformation; π: $V^n \rightarrow P_e^{n+N}$ is an isometric immersion of an orientable n-dimensional C^∞-manifold.

With the general point X_0 of the manifold V^n we associate an orthonormal simplex $S_{X_0} \equiv \{X_A\}$, $(A, B, C = 0, 1, \cdots, n+N)$, and we suppose that the dual tangent space $T_{X_0}(V^n)$ of V^n at X_0 is determined by the points X_i $(i, j, k, l = 1, 2, \cdots, n)$. The dual basis of $T_{X_0}(V^n)$ is then $\omega^i(u^j|du^j)$. As is well known the manifold V^n is structured by the connexion

$$(1) \qquad \mathrm{d}X_A = \omega_A^B X_B ,$$

where ω_A^B are the connexion 1-forms associated with π. As such we have the structural equations

$$(2) \qquad \mathrm{d} \wedge \omega_A^B = \omega_A^C \wedge \omega_C^B, \quad \omega_A^B + \omega_B^A = 0.$$

We put

$$(3) \qquad \omega_A^B = \gamma_{Ai}^B \omega^i ,$$

where γ_{Ai}^B are the connexion coefficients and $\omega_0^A = \omega^A$. Since V^n is an integral manifold of

$$\omega^\alpha = 0, \quad \alpha, \beta, \gamma = n+1, \cdots, n+N,$$

we find by exterior differentiation:

$$(4) \qquad \gamma_{ij}^\alpha = \gamma_{ji}^\alpha .$$

From the above formula we obtain:

$$(5) \qquad \left\{ \begin{array}{l} \mathrm{d} \wedge \omega^i = \omega^j \wedge \omega_j^i, \quad \omega_i^j + \omega_j^i = 0 , \\ \mathrm{d} \wedge \omega_i^j = \omega_i^k \wedge \omega_k^j + \Omega_i^j, \quad \Omega_i^j = \dfrac{1}{2} R_i{}^j{}_{kl}\, \omega^k \wedge \omega^l , \\ \mathrm{d} \wedge \omega_\alpha^\beta = \omega_\alpha^\gamma \wedge \omega_\gamma^\beta + \Omega_\alpha^\beta, \quad \Omega_\alpha^\beta = \dfrac{1}{2} R_\alpha{}^\beta{}_{kl}\, \omega^k \wedge \omega^l , \\ \qquad R_i{}^j{}_{kl} + R_i{}^j{}_{lk} = 0 , \\ \qquad R_\alpha{}^\beta{}_{kl} + R_\alpha{}^\beta{}_{lk} = 0 . \end{array} \right.$$

This implies

$$(6) \qquad R_i{}^j{}_{kl} = \delta_{il}\delta_{jk} - \delta_{ik}\,\delta_{jl} + \sum_\alpha (\gamma_{il}^\alpha \gamma_{jk}^\alpha - \gamma_{ik}^\alpha \gamma_{jl}^\alpha) ,$$

$$(7) \qquad R_\alpha{}^\beta{}_{kl} = \sum_i (\gamma_{il}^\alpha \gamma_{ik}^\beta - \gamma_{ik}^\alpha \gamma_{il}^\beta) .$$

The second fundamental forms φ_α of the manifold V^n associated with π are

$$(8) \qquad \varphi_\alpha = - \langle \mathrm{d}X_0, \mathrm{d}X_\alpha \rangle = \omega_i^\alpha \omega^i = \gamma_{ij}^\alpha \omega^i \omega^j$$

and the immersion (or the manifold) is *minimal* if

(9) $$\gamma^{\alpha} = \mathrm{tr}\ [\gamma^{\alpha}_{ji}] = 0 .$$

If there exists at least one normal index α such that $\gamma^{\alpha} \neq 0$, then

(10) $$\boldsymbol{H} = f\gamma^{\alpha}\boldsymbol{X}_{\alpha}, \quad (f = \text{factor of normalization}),$$

is the *mean curvature point* in $\boldsymbol{X}_0$ of the manifold ($\boldsymbol{H}$ corresponds to the unitary mean curvature vector of Bompiani). The manifold having $\boldsymbol{H}$ as general point will be called the *mean curvature manifold* associated with π, and will be denoted by $V_{\boldsymbol{H}}$. Consequently

(11) $$\frac{1}{n}\{\sum_{\alpha} (\gamma^{\alpha})^2\}^{1/2}$$

is the *scalar mean curvature*.

If $\boldsymbol{H} \neq 0$ it is possible to choose the basic frame such that

(12) $$\boldsymbol{H} = \boldsymbol{X}_{n+1} .$$

This implies

(13) $$\gamma^{r} = 0, \quad r, s, t = n+2, \cdots, n+N,$$

and in this case the mean curvature of V^n is $\frac{1}{n}\gamma^{n+1}$. Following Otsuki we shall say that the immersion π or the manifold V^n is *pseudo-umbilical* if and only if

(14) $$\gamma^{n+1}_{ij} = \alpha\delta_{ij} .$$

If

(15) $$R_{\alpha}{}^{\beta}{}_{kl} = 0,$$

V^n has a *trivial normal connexion* [13]. If

(16) $$R_i{}^j{}_{kl} = -K(\delta_{il}\,\delta_{jk} - \delta_{ik}\,\delta_{jl}), \quad K = \text{constant} ,$$

the manifold V^n has a *constant Riemannian curvature.* Here K is the *Gauss curvature* of such a manifold and its *scalar curvature* R is given by $R = n(1-n)K$.

2. Suppose now that V^n is a *minimal manifold with a trivial normal connexion and constant Riemannian curvature.* By an appropriate choice of the basic simplex it is always possible to diagonalise one of the second fundamental forms φ_{α}, say φ_{n+1}. We then have

(17) $$\omega_i^{n+1} = -\gamma_{ii}^{n+1}\,\omega_i^0 .$$

From $R_r^{n+1}{}_{kl} = 0$ it follows immediately that

(18) $$\gamma_{kl}^r(\gamma_{kk}^{n+1} - \gamma_{ll}^{n+1}) = 0.$$

We assume that

(19) $$\gamma_{kk}^{n+1} - \gamma_{ll}^{n+1} \neq 0 \quad \text{for} \quad k \neq l \quad \text{when} \quad N > 1 .$$

Hence $\gamma_{kl}^r = 0$ for $k \neq l$, which means that all other second fundamental forms φ_α are diagonal. Thus we obtain

(20) $$\omega_i^\alpha = -\gamma_{ii}^\alpha\,\omega_i^0 .$$

By exterior differentiation of (20) we get:

(21) $$d\gamma_{ii}^\alpha \wedge \omega_i^0 + \gamma_{ii}^\alpha d \wedge \omega_i^0 + \omega_i^\beta \wedge \omega_\beta^\alpha - \gamma_{jj}^\alpha\,\omega_j^i \wedge \omega^j = 0 .$$

Multiplying (21) by γ_{ii}^α and summing over α we obtain:

(22) $$\sum_\alpha \gamma_{ii}^\alpha d\gamma_{ii}^\alpha \wedge \omega_i^0 + \sum_\alpha (\gamma_{ii}^\alpha)^2 d \wedge \omega_i^0 - \sum_\alpha \gamma_{ii}^\alpha\gamma_{jj}^\alpha\omega_i^j \wedge \omega_j^0 = \sum_\alpha \gamma_{ii}^\alpha\gamma_{ii}^\beta\omega_i^0 \wedge \omega_\beta^\alpha .$$

With the help of (6) and (16) it follows:

(23) $$\begin{cases} \sum_\alpha (\gamma_{ii}^\alpha)^2 = (K+1)(n-1) , \\ \sum_\alpha \gamma_{ii}^\alpha\gamma_{jj}^\alpha + (K+1) = 0 \quad \text{for} \quad i \neq j, \end{cases}$$

so that (22) becomes:

(24) $$(K+1)(n-1)d \wedge \omega_i^0 + (K+1)\omega_i^j \wedge \omega_j^0 = \sum_\alpha \gamma_{ii}^\alpha\gamma_{ii}^\beta\,\omega_i^0 \wedge \omega_\beta^\alpha = 0 ,$$

since $\omega_\alpha^\beta + \omega_\beta^\alpha = 0$. So we find

(25) $$(K+1)\, nd \wedge \omega_i^0 = 0 .$$

Therefore we have the following cases:

(i) $K+1=0$. Using (23) we have on $P_e^{n+N}(\mathrm{R})$: $\gamma_{ii}^\alpha = 0$, and this means that V^n is totally geodesic. It is easy to see that if $N>1$ (i) is not consistent with (19).

(ii) $K+1 \neq 0$. Then we have $d \wedge \omega^i = 0$ and V^n is flat (isometric with an n-dimensional Euclidean space).

Hence we have the

Theorem 1. *Let V^n be a minimal hypersurface of $P_e^{n+1}(\mathrm{R})$ with* *constant Riemannian curvature. Then V^n is either totally geodesic or isometric with an n-dimensional Euclidean space.*

Let V^n be a minimal manifold of $P_e^{n+N}(\mathbf{R})$, $(N>1)$, with constant Riemannian curvature and trivial normal connexion. If there exists a second fundamental form whose eigenvalues with respect to ds^2 are all different, then V^n is isometric with an n-dimensional Euclidean space.

Remarks. a) From the previous theorem we may conclude that a minimal surface $V^2 \subset P_e^3(\mathbf{R})$ with constant Gauss curvature is either totally geodesic or a Clifford surface.

b) If we suppose the minimal manifold is of the type of Kenmotsu [13], i. e. if for the norm σ we have

$$\sigma = \sum_{\alpha,i,j} (\gamma_{ij}^{\alpha})^2 = n,$$

we obtain $K = -1 + \frac{1}{n-1}$. According to a theorem of Kenmotsu [13] V^n is then a minimal hypersurface of P_e^{n+1}. Moreover in that case V^n is an open submanifold of a Clifford minimal hypersurface $V_{k,n-k}$ and is locally a Riemannian direct product of spaces $\bar{V}_1$ and $\bar{V}_2$ of constant curvatures, dim $\bar{V}_1 = m \geqslant 1$ and dim $\bar{V}_2 = n-m \geqslant 1$. It follows from our hypothesis on the second fundamental form that this case only arise if $n=2$.

3. We consider as in §1 *a pseudo-umbilical manifold* $V^n \subset P_e^{n+N}(\mathbf{R})$, $(N>1)$, *with constant mean curvature, trivial normal connexion and constant Riemannian curvature.* Apart from (13), (14), (15) and (16), we also have

$$\alpha = \text{constant}. \tag{26}$$

Suppose $\alpha \neq 0$ ($\alpha = 0$ has already been considered, for then V^n is minimal.)

By an appropriate choice of the basic simplex we may diagonalize the second fundamental form φ_{n+2}; hence

$$\omega_i^{n+2} = -\gamma_{ii}^{n+2}\,\omega_i^0. \tag{27}$$

If $N>2$ we shall suppose

$$\gamma_{kk}^{n+2} - \gamma_{ll}^{n+2} \neq 0 \quad \text{for} \quad k \neq l. \tag{28}$$

Then it follows from (28) and $R_{\alpha\ kl}^{n+2} = 0$ that

$$\gamma_{kl}^{\alpha} = 0. \tag{29}$$

This proves that all the second fundamental forms φ_α are diagonal. We remark that if $N=2$ we may choose the basic simplex such that

$$\alpha = \text{constant} \Longleftrightarrow \omega_{n+1}^{n+2} = 0$$

[10], [16]. Consequently, the normal connexion in this case is trivial.

From (16) we readily get:

$$\text{(30)} \quad \begin{cases} \sum_r \gamma_{ii}^r \gamma_{jj}^r + K + 1 + \alpha^2 = 0, \\ \sum_r (\gamma_{ii}^r)^2 = (n-1)(K+1+\alpha^2). \end{cases}$$

On the other hand from (14) and (29) we have

$$\text{(31)} \quad \omega_i^{n+1} = \alpha\omega^i ,$$

$$\text{(32)} \quad \omega_i^r = -\gamma_{ii}^r \omega_i^0 .$$

Since α is constant, exterior differentiation of (31) gives:

$$\text{(33)} \quad \sum_r \gamma_{ii}^r \omega_r^{n+1} \wedge \omega_i^0 = 0 ,$$

while exterior differentiation of (32) leads to the exterior equation:

$$\text{(34)} \quad d\gamma_{ii}^r \wedge \omega_i^0 + \gamma_{ii}^r d \wedge \omega_i^0 - \alpha\omega_i^0 \wedge \omega_{n+1}^r + \omega_i^s \wedge \omega_s^r - \gamma_{jj}^r \omega_i^j \wedge \omega_j^0 = 0 .$$

On multiplying γ_{ii}^r and on summing over r we get

$$\text{(35)} \quad \begin{aligned} \sum_r \gamma_{ii}^r d\gamma_{ii}^r \wedge \omega_i^0 + \sum_r (\gamma_{ii}^r)^2 d \wedge \omega_i^0 - \alpha \sum_r \gamma_{ii}^r \omega_i^0 \wedge \omega_{n+1}^r \\ + \sum_r \gamma_{ii}^r \omega_i^s \wedge \omega_s^r - \sum_r \gamma_{ii}^r \gamma_{jj}^r \omega_i^j \wedge \omega_j^0 = 0 . \end{aligned}$$

From (30), (33) and $\omega_r^s + \omega_s^r = 0$, (35) becomes

$$\text{(36)} \quad (n-1)(K+1+\alpha^2) d \wedge \omega_i^0 + (K+1+\alpha^2)\omega_i^j \wedge \omega_j^0 = (K+1+\alpha^2) d \wedge \omega_i^0 = 0.$$

From the above we have the following cases:

(i) $K+1+\alpha^2 = 0$. By virtue of (30) this means

$$\text{(37)} \quad \gamma_{ii}^r = 0 .$$

If $N > 2$ (37) is not consistent with (28). If $N=2$ (37) proves that V^n is an umbilical manifold of a P_e^{n+1}.

(ii) $K+1+\alpha^2 \neq 0$. Then

$$\text{(38)} \quad \mathrm{d} \wedge \omega^i = 0,$$

and V^n is isometric with an n-dimensional Euclidean space.

We may now formulate the following

Theorem 2. *Let $V^n \subset P_e^{n+2}(\mathrm{R})$ be an n-dimensional pseudo-umbilical manifold with constant mean curvature and constant Riemannian curvature. Then V^n is either an umbilical manifold of a P_e^{n+1} or is isometric with an n-dimensional Euclidean space.*

Let $V^n \subset P_e^{n+N}(\mathrm{R})$, $(N > 2)$, be an n-dimensional pseudo-umbilical manifold with constant mean curvature, constant Riemannian curvature and trivial normal connexion. If there exists a second fundamental form all whose eigenvalues with respect to ds^2 are different, then V_n is isometric with an n-dimensional Euclidean space.

4. Consider now the case of a pseudo-umbilical $V^n \subset P_e^{n+2}(\mathrm{R})$ of constant mean and constant Riemannian curvature. Assuming $\pi: V^n \subset P_e^{n+2}$ is not umbilical we shall first investigate the mean curvature manifold V_H defined in § 1.

Put

$$Z_0 = X_{n+1},\ Z_i = X_i,\ Z_{n+1} = X_{n+2},\ Z_{n+2} = X_0 . \tag{39}$$

From

$$dZ_A = \alpha_A^B Z_B \tag{40}$$

we readily get:

$$\begin{cases} \alpha^i = \omega_{n+1}^i = -\alpha\omega^i, & \alpha_i^j = \omega_i^j , \\ \alpha_{n+1}^i = -\dfrac{1}{\alpha}\gamma_{ii}^{n+2}\alpha_i^0 , & \alpha_{n+2}^i = \dfrac{1}{\alpha}\alpha_i^0 . \end{cases} \tag{41}$$

From (41) we deduce that V_H is also a pseudo-umbilical manifold of constant mean curvature $\dfrac{1}{\alpha}$ and X_0 is *the mean curvature point of this manifold associated with* Z_0.

Further, from

$$d \wedge \alpha_i^j = \alpha_i^k \wedge \alpha_k^j + \Omega^{*j}_i,\quad \Omega^{*j}_i = \frac{1}{2} R^{*j}_{i\ kl}\alpha^k \wedge \alpha^l , \tag{42}$$

it follows

$$\Omega^{*j}_i = -\left(1 + \frac{1}{\alpha^2} + \frac{1}{\alpha^2}\gamma_{ii}^{n+2}\gamma_{jj}^{n+2}\right)\alpha_i^0 \wedge \alpha_j^0 \tag{43}$$

and so

$$R_i^{*j}{}_{kl} = \left(1 + \frac{1}{\alpha^2} + \frac{1}{\alpha^2}\gamma_{kk}^{n+2}\gamma_{ll}^{n+2}\right)(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) . \tag{44}$$

By means of (30) we get

$$R_i^{*j}{}_{kl} = -\frac{K}{\alpha^2}(\delta_{il}\,\delta_{jk} - \delta_{ik}\,\delta_{jl}) \tag{45}$$

or

$$R_i^{*j}{}_{kl} = -K^*(\delta_{il}\,\delta_{jk} - \delta_{il}\,\delta_{jk}) \tag{46}$$

where

$$K^* = \frac{K}{\alpha^2} . \tag{47}$$

Consequently V_H also has constant Riemannian curvature.

Now, we investigate the manifold generated by the point $\boldsymbol{X}_{n+2}$, which is the rectangular point of $\boldsymbol{H}$ on the principal quasi-normal $N_0 = \boldsymbol{X}_{n+1}\boldsymbol{X}_{n+2}$ of V^n [14]. Therefore we put

$$\boldsymbol{Z}_0 = \boldsymbol{X}_{n+2},\ \ \boldsymbol{Z}_i = \boldsymbol{X}_i,\ \ \boldsymbol{Z}_{n+1} = \boldsymbol{X}_0,\ \ \boldsymbol{Z}_{n+2} = \boldsymbol{X}_{n+1} \tag{48}$$

and from the connexion

$$\mathrm{d}\,\boldsymbol{Z}_A = \alpha_A^B\,\boldsymbol{Z}_B \tag{49}$$

we obtain

$$\begin{cases} \alpha^i = \gamma_{ii}^{n+2}\,\omega_i^0, \quad \alpha_i^j = \omega_i^j , \\ \alpha_{n+1}^i = \dfrac{1}{\gamma_{ii}^{n+2}}\,\alpha_i^0, \quad \alpha_{n+2}^i = -\dfrac{\alpha}{\gamma_{ii}^{n+2}}\,\alpha_i^0 . \end{cases} \tag{50}$$

From (50) it follows that the arithmetic invariant of Chern [12] is 1, and so the manifold $V(\boldsymbol{X}_{n+2})$ belongs to a P_e^{n+1}-space. Further as turns out the $(n-1)$-th mean curvature of $V(\boldsymbol{X}_{n+2})$ [11] is trivial.

From

$$\mathrm{d}\wedge\alpha_i^j = \alpha_i^k\wedge\alpha_k^j + \bar{\Omega}_i^j , \quad \bar{\Omega}_i^j = \frac{1}{2}\,\bar{R}_i{}^j{}_{kl}\,\alpha^k\wedge\alpha^l \tag{51}$$

it follows

$$\bar{\Omega}_i^j = -\frac{1+\alpha^2+\gamma_{ii}^{n+2}\,\gamma_{jj}^{n+2}}{\gamma_{ii}^{n+2}\,\gamma_{jj}^{n+2}}\,\alpha^i\wedge\alpha^j . \tag{52}$$

So we have

(53) $$\bar{R}_{i\ kl}^{\ j} = -\bar{K}(\delta_{il}\,\delta_{jk} - \delta_{ik}\,\delta_{jl}),$$

where

(54) $$\bar{K} = 1 + \frac{1+\alpha^2}{\gamma_{kk}^{n+2}\ \gamma_{ll}^{n+2}}$$

or

(55) $$\bar{K} = \frac{K}{1+\alpha^2+K}.$$

The latter manifold also has constant Riemannian curvature. On the other hand since

(56) $$\alpha_{n+1}^{n+2} = 0\,,$$

it follows that the normal connexion is trivial. Hence we have the

Theorem 3. *Let $V^n \subset P_e^{n+2}$ be a pseudo-umbilical manifold of constant mean and Riemannian curvature. The mean curvature point $\boldsymbol{H}$ of V^n generates a manifold V_H^n which is also pseudo-umbilical and has a constant mean and Riemannian curvature. The rectangular point of $\boldsymbol{H}$ on the principal quasi-normal N_0 associated at each point of V^n generates an n-dimensional hypersurface of a P_e^{n+1} with constant mean and Riemannian curvature and with trivial connexion. The $(n-1)$-th mean curvature of this manifold is trivial.*

Remarks. a) For $n=2$ the manifold $V(\boldsymbol{X}_{n+2})$ is a minimal surface with constant Gauss curvature of a P_e^3. Consequently it is a Clifford surface.

b) By making use of (30) for $n=3$, it can easily be proved that the pseudo-umbilical manifolds $V^3 \subset P_e^5$ of constant mean and Riemannian curvature satisfy $K+1+\alpha^2=0$. We conclude that the V^3 are umbilical in P_e^4. In that case the rectangular point of $\boldsymbol{H}$ on the principal quasi-normal is a fixed point. Therefore we have: *The 3-dimensional pseudo-umbilical manifolds of a P_e^5 of constant mean and constant Riemannian curvature are umbilical manifolds of a P_e^4.*

5. Let us go back to the considerations of §2 and put $n=2$. With the help of (20) we obtain:

(57) $$\mathrm{d} \wedge \omega_3^4 = 0\,.$$

Hence the torsion form ω_3^4 is closed and consequently it is always possible to choose the basic simplex such that

(58) $$\omega_3^4 = 0\,.$$

The equations (21) now become:

(59) $$\begin{cases} d\ln\gamma_{11}^3 \wedge \omega^1 = 2\omega_2^1 \wedge \omega^2, & d\ln\gamma_{11}^3 \wedge \omega^2 = 2\omega_1^2 \wedge \omega^1\,, \\ d\ln\gamma_{11}^4 \wedge \omega^1 = 2\omega_2^1 \wedge \omega^2, & d\ln\gamma_{11}^4 \wedge \omega^2 = 2\omega_1^2 \wedge \omega^1\,. \end{cases}$$

From this it follows:

(60) $$\gamma_{11}^4 = c\gamma_{11}^3\,, \quad c = \text{constant},$$

and taking account of (23) we obtain

(61) $$\gamma_{11}^4 = c\gamma_{11}^3 = c', \quad c' = \text{constant}\,.$$

This yields, as could be expected, that $\omega_1^2 = 0$. So V^2 is a Clifford surface with arithmetic invariant of Chern 1 and consequently V^2 belongs to a P_e^3.

As in [15] we can easily show that the points X_3 and X_4 of the principal quasi-normal N_0 generate pseudo-umbilical surfaces with constant mean and constant Riemannian curvature. We thus obtain a certain inverse property of the one mentioned in 4 a) and we formulate it in the following:

Theorem 4. *Let V^2 be a minimal surface of P_e^4 with constant Gauss curvature and trivial normal connexion and such that V^2 has a second fundamental form which is not umbilical. Then V^2 is a Clifford surface of a P_e^3. There exist two rectangular points on the principal quasi-normal N_0 that generate pseudo-umbilical surfaces of constant mean and constant Gauss curvature and such that each of them is the mean curvature manifold of the other.*

6. As in §4 we consider a pseudo-umbilical V^n with constant mean and constant Riemannian curvature. Suppose now that codimension $N>2$. Moreover we assume that the normal connexion forms ω_α^β satisfy

(62) $$\omega_\alpha^\beta = 0$$

and that the relations (27) and (28) are valid. Exterior differentiation of $\omega_{n+1}^\alpha = 0$ gives (29) and consequently we have (32). We remark that exterior differentiation of (62) shows that the normal connexion is trivial.

We now investigate the manifold V^{*n} generated by the mean curvature point $\boldsymbol{H}$. Put

$$Z_0 = X_{n+1}, \quad Z_i = X_i, \quad Z_{n+1} = X\,, \quad Z_r = X_r, \tag{63}$$

and from the connexion

$$dZ_A = \alpha_A^B Z_B \tag{64}$$

it follows

$$\begin{cases} \alpha^i = \omega_{n+1}^i, & \alpha_i^j = \omega_i^j, \quad \alpha_\alpha^\beta = 0\,, \\ \alpha_{n+1}^i = \omega^i, & \alpha_r^i = \omega_r^i\,. \end{cases} \tag{65}$$

Hence:

$$\alpha_{n+1}^i = -\frac{1}{\alpha}\alpha^i\,, \quad \alpha_r^i = -\frac{1}{\alpha}\gamma_{il}^r\,\alpha_l^0. \tag{66}$$

This shows:

(i) V^n is the mean curvature manifold of V^{*n};

(ii) V^{*n} has constant mean curvature;

(iii) V^{*n} has trivial normal connexion forms.

From

$$d \wedge \alpha_i^j = \alpha_i^k \wedge \alpha_k^j + \Omega^{*j}_i, \quad \Omega^{*j}_i = \frac{1}{2}R_i^{*j}{}_{kl}\,\alpha^k \wedge \alpha^l \tag{67}$$

it follows

$$\Omega^{*j}_i = -\left(1 + \frac{1}{\alpha^2} + \frac{1}{\alpha^2}\sum_r \gamma_{il}^r\gamma_{jl}^r\right)\alpha_i^0 \wedge \alpha_j^0 \tag{68}$$

and consequently

$$R_i^{*j}{}_{kl} = -\frac{K}{\alpha^2}(\delta_{il}\,\delta_{jk} - \delta_{ik}\,\delta_{jl})\,. \tag{69}$$

Thus V^{*n} has constant Riemannian curvature and

$$K^* = \frac{K}{\alpha^2}\,. \tag{70}$$

Thus we have

Theorem 5. *Let V^n be a pseudo-umbilical manifold with constant mean and constant Riemannian curvature and whose normal connexion forms are all trivial. The mean curvature point of V^n generates a manifold V^{*n} of the same type and V^n is the mean curvature manifold of V^{*n}.*

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KATHOLIEKE UNIVERSITEIT TE LEUVEN

WISKUNDIG INSTITUUT

CELESTIJNENLAAN 200B

B—3030 HEVERLEE (BELGIUM)

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