NOTE ON GALOIS SUBRINGS OF PRIME GOLDIE RINGS

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Recently, in his paper [2], C. Faith proved the following: *Let D be a division ring, and R a left order of D. If G is a finite group of (ring) automorphisms in D such that $RG \subseteq R$, then $\{r \in R \mid r\sigma = r$ for all $\sigma \in G\}$ is a left order of $J(G) = \{d \in D \mid d\sigma = d$ for all $\sigma \in G\}$.* In this note, we shall extend partially the above result to (Artinian) simple rings.

Throughout the present note, A will represent a simple ring, and G a DF-group in A, i.e., a finite group of automorphisms in A such that the subring of A generated by all the units $v \in A$ with $\tilde{v} = v_L v_R^{-1} \in G$ is a division ring. Then, the subring coincides with the centralizer $V_A(B)$ of $B = J(G)$ in A, every intermediate ring of A/B is simple, and $[A:B] \leqslant |G|$. In particular, if $[A:B] = |G|$ then $t_G(A) = \{t_G(a) = \sum_{\sigma \in G} a\sigma \mid a \in A\}$ coincides with B. (See [3; § 7].) Needless to say, if $|G|$ is not divisible by the characteristic of A then $t_G(A) = B$.

Applying the argument used in the proof of [2; Lemma 5], we can easily see that if A is of characteristic $p \neq 0$ and G consists of only inner automorphisms then the commutator subgroup G' of G contains no elements of order p. Moreover, as is seen in the proof of [3; Th. 13.2], if A is of characteristic $p \neq 0$ and $|G| = p$ then $[A:B] = p$.

Henceforth, R will represent a (left and right) order of A such that $RG \subseteq R$. As is well-known, R is a prime Goldie ring. In what follows, we shall prove the following:

Theorem. *$R \cap B$ is an order of B.*

Now, let $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_r = 1$ be a composition series of G, and $A_i = J(G_i)$. Then, $\overline{G}_i = A_{i+1} \mid G_i$ (the contraction of G_i to A_{i+1}) is a DF-group of A_{i+1}/A_i and homomorphic to G_i/G_{i+1}. Since G_i/G_{i+1} is a simple group, the remaks mentioned above enable us to see that either $|\overline{G}_i| = [A_{i+1} : A_i]$ or $|\overline{G}_i|$ is not divisible by the characteristic of A, so that $t_{\overline{G}_i}(A_{i+1}) = A_i$. Therefore, the proof of our theorem will be completed by proving the following:

Lemma. *If $t_G(A)=B$ then $R\cap B$ is an order of B.*

Proof. Let $B=\sum_1^l B'f_{hk}$, where $F=\{f_{hk}\text{'s}\}$ is a system of matrix units, and $B'=V_B(F)$ a division ring. We shall distinguish here between two cases.

Case 1. $l=1$: Given an arbitrary element b of B, there exist a regular element r_1 of R and an element r_2 of R such that $b=r_1^{-1}r_2$. Now, let $A=\sum_1^n u_iB$. Then, as is well-known, there exists a regular element r_0 of R such that $w_i=r_0u_i\in Rr_1$ $(i=1, \cdots, n)$. If $t_G(Rr_1)=0$ then $B=t_G(A)=t_G(r_0A)$ $=t_G(\sum w_iB)=0$. This contradiction implies that we can find an element r of R such that $t_G(rr_1)\neq 0$. Since $(rr_1)b=rr_2$ and B is a division ring, we obtain $b=t_G(rr_1)^{-1}\,t_G(rr_2)$. Hence, $R\cap B$ is a left, and similarly right, order of B.

Case 2. $l>1$: We have $A=\sum A'f_{hk}$ with the simple ring $A'=V_A(F)$. Let $A'=\sum De'_{ij}$, where $E'=\{e'_{ij}\text{'s}\}$ is a system of matrix units, and $D=V_{A'}(E')$ a division ring. Then, $E=E'\cdot F=\{e'_{ij}\cdot f_{hk}\text{'s}\}$ is a system of matrix units and $D=V_A(E)$. Now, by Faith-Utumi theorem [1; Th. 2.3], $R\cap D$ contains an order K of D such that $R\supseteq\sum(\sum Ke'_{ij})f_{hk}$. Obviously, $R\cap A'$ $\supseteq\sum Ke'_{ij}$. Since $f_{hk}\sigma=f_{hk}$ and $(R\cap A')\sigma\subseteq R\cap A'$ for every $\sigma\in G$, $R\cap A'$ contains a subring R' such that $R'G\subseteq R'$ and $R\supseteq\sum R'f_{hk}\supseteq\sum(\sum Ke'_{ij})f_{hk}$. Noting that R' is an order of A' and $A'|G$ is a DF-group of A' such that $J(A'|G)=B'$ and $t_{A'|G}(A')=B'$, we see that $B'\cap R'$ is an order of B' (Case 1). It follows therefore $B\cap R$ $(\supseteq\sum(B'\cap R')f_{hk})$ is an order of $B=\sum B'f_{hk}$.

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(Received November 20, 1972)