ON TORSION FREE MODULES OVER REGULAR RINGS

KIYOICHI OSHIRO

Throughout this paper, all rings have identity and all modules are unital.

Let M be a left R-module. We denote its torsion submodule in the sense of Gentile [2] by $T(M)$, i.e.,

$$T(M) = \{x \in M \mid \mathrm{Hom}_R(Rx,\ I(R)) = 0\}$$

where $I(R)$ is the injective hull of R as a left R-module.

A left R-module M is said to be torsion free if $T(M)=0$. For the properties of torsion free modules, the reader is refered to [6].

In section 2, we show that a regular ring R is left continuous if and only if every cyclic torsion free R-module is projective.

In section 3, we obtain module theoretic characterizations of a commutative regular ring R whose maximal ring of quotients coinsides with the Baer hull of R. In connection with this, Examples A and C are given which answer Pierce's questions (4) and (11) of [6] in the negative, respectively.

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1. Preliminaries

Let Q be a ring of left quotients of R. It is well known that a torsion free injective left R-module can be turned into a left Q-module (see [8]). So, if M is a torsion free left R-module, M is embedded in the Q-submodule QM of the injective hull of M. Note that for any two elements $\sum_{i=1}^{n} p_i a_i$ and $\sum_{j=1}^{m} q_j b_j$ in QM, $\sum_{i=1}^{n} p_i a_i = \sum_{j=1}^{m} q_j b_j$ if and only if, for any $r \in \bigcap_{i=1}^{n} (R : p_i)$[1] and $0 \neq r' \in R$, there exists $r'' \in \bigcap_{j=1}^{m} (R : rq_j)$ such that $r'' r' \neq 0$ and $\sum_{i=1}^{n} (r'' rp_i)a_i = \sum_{j=1}^{m} (r'' rq_j)b_j$.

In latter section 3, we need the following lemma, the proof of which is straightforward, and will be omitted.

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1) $(R:p) = \{r \in R \mid rp \in R\}$.

Lemma 1.1. *Let M be a torsion free left R-module and let $\{M_i\}$ a collection of R-submodules of M indexed by A. If $M = \sum_{i \in A} \oplus M_i$ then $QM = \sum_{i \in A} \oplus QM_i$.*

2. Continuous regular rings

We recall that a regular ring R is said to be left continuous if the lattice $L(R)$ of principal left ideals of R is continuous (see [9]).

Theorem 2.1. *Let R be a regular ring and Q its maximal ring of left quotients. Then the following conditions are equivalent :*

(*a*) *R is left continuous.*

(*b*) *Every idempotent in Q is contained in R.*

(*c*) *$A = Q(A \cap R)$ for any left ideal A of Q.*

(*d*) *Every principal left ideal of Q is generated by an idempotent of R.*

(*e*) *Every cyclic torsion free left R-module is projective.*

Proof. (a) $\Longrightarrow$ (b) By [10, Lemma 8].

(b) $\Longrightarrow$ (c) Since Q is a regular ring, this is evident.

(c) $\Longrightarrow$ (d) Let Qx be a principal left ideal of Q. Since $Qx = Q(Qx \cap R)$, we have $x = \sum_{i=1}^{n} q_i r_i$ for some $q_1, q_2, \cdots, q_n \in Q$ and $r_1, r_2, \cdots, r_n \in Qx \cap R$. Since R is a regular ring, there exists an idempotent e in $Qx \cap R$ such that $\sum_{i=1}^{n} Rr_i = Re$. Then we have $Qx = Qe$.

(d) $\Longrightarrow$ (a) By [9, Theorem 2.1], Q is left continuous. It is easily seen from (d) that $L(Q)$ is lattice isomorphic to $L(R)$. Hence R is left continuous.

(e) $\Longrightarrow$ (b) Let e be an idempotent of Q. Then, $R + Re$ is projective, since $R + Re = Re \oplus R(1-e)$. Therefore, by [3, Lemma 4], R is a direct summand of $R + Re$. However, since R is an essential submodule of $R + Re$, this implies that $Re + R = R$ and $e \in R$.

(c) $\Longrightarrow$ (e) Let Rx be a cyclic torsion free left R-module. By [1, Theorem 2.1], Qx is Q-projective. Here we claim that $Q \otimes_R Rx \simeq Qx$ canonically. To this end, we consider the exact sequence

$$0 \longrightarrow \mathrm{Ann}_R(x) \longrightarrow R \longrightarrow Rx \longrightarrow 0$$

where $\mathrm{Ann}_R(x) = \{r \in R \mid rx = 0\}$. Since Q is flat as a right R-module, the induced sequence of left Q-modules

$$0 \longrightarrow Q\mathrm{Ann}_R(x) \longrightarrow Q \longrightarrow Q \otimes_R Rx \longrightarrow 0$$

is exact. Now to show that $Q\otimes Rx \simeq Qx$, it is sufficient to show that $Q\mathrm{Ann}_R(x) = \mathrm{Ann}_Q(x)$. Clearly $Q\mathrm{Ann}_R(x) \subseteqq \mathrm{Ann}_Q(x)$. Conversely if $q \in Q$ such that $qx=0$, then it is easily seen from $Q(Qq \cap R) = Qq$ that $q \in \mathrm{Ann}_Q(x)$. Hence we have $Q\mathrm{Ann}_R(x) = \mathrm{Ann}_Q(x)$ and $Q \otimes_R Rx \simeq Qx$. So, $Q\otimes_R Rx$ is Q-projective and therefore, by [7, Theorem 2.8], Rx is R-projective.

3. Baer ring of quotients

Throughout this section, we assume that a ring R is commutative.

Let M be a torsion free R-module. In the following we shall consider the following conditions:

(α) M is a direct sum of cyclic R-modules.

(β) M is isomorphic to an essential submodule of a direct sum of cyclic torsion free R-modules.

(γ) M is isomorphic to a submodule of a direct sum of cyclic torsion free R-modules.

Remark 1. The following question has been asked by Pierce [6, p. 109]: "Let M be an R-module (where R is a regular ring) which is a finite direct sum of cyclic R-modules. Let N be a finitely generated R-submodule of M. Is N a direct sum of cyclic R-modules?" However there exists an example of a module over a Boolean ring which satisfies the condition (β) but not the condition (α). Therefore Pierce's question has a negative answer.

For constructing such an example, we need a lemma.

Lemma 3.1. *Let R be a Boolean ring, and Q its maximal ring of quotients. For e and f in Q, the following conditions are equivalent:*

(a) *$Re + Rf$ satisfies the condition (α).*

(b) *ef is contained in $Re + Rf$.*

Proof. (a) $\Longrightarrow$ (b) By the assumption, $Re+Rf$ is decomposed into a direct sum of cyclic R-submodules, say $Re+Rf = Rg_1 \oplus Rg_2 \oplus \cdots \oplus Rg_n$. Then, by Lemma 1.1, $Qg_1 + Qg_2 + \cdots + Qg_n = Qg_1 \oplus Qg_2 \oplus \cdots \oplus Qg_n$, i.e., $\{g_i \mid i = 1, 2, \cdots, n\}$ is a set of orthogonal elements. Hence, when we view Q as a partially ordered set, $g_1 + g_2 + \cdots + g_n$ is the supremum of $Rg_1 + Rg_2 + \cdots + Rg_n$. On the other hand, the supremum of $Re+Rf$ is $e+f-ef$. Hence we have $e+f-ef = g_1 + g_2 + \cdots + g_n$ and $ef \in Re + Rf$.

(b) $\Longrightarrow$ (a) If $ef \in Re + Rf$, then, clearly, $Re+Rf = Re(f-1) \oplus$

$Rf(e-1) \oplus Ref$.

Example A. Let S be an infinite set. Let Q be the set of all subsets of S, and R the set of all finite subsets and all cofinite subsets (i. e., complements of finite subsets) of S. Q become a Boolean ring by the following definition: for a, b in Q.

$$a+b=(a \cup b) \cap (a \cap b)^c$$

where $(a \cap b)^c$ denotes the complement of $a \cap b$ in S,

$$ab = a \cap b.$$

Note that the empty set is the zero element and S is the identity. Moreover R is a subring of Q and its maximal ring of quotients coincides with Q (see [4, p. 45]).

Now we can select e and f in Q such that e, f, $e \cap f$, $e \cap (e \cap f)^c$ and $f \cap (e \cap f)^c$ are infinite sets. Put $e' = e \cap (e \cap f)^c$ and $f' = f \cap (e \cap f)^c$. Then $Re' + Rf' + Ref = Re' \oplus Rf' \oplus Ref$ and it contains $Re + Rf$ as an essential submodule. However $Re + Rf$ can not be decomposed into a direct sum of cyclic R-modules. For, if it is decomposed into a direct sum of cyclic R-modules, then, by Lemma 3.1, $ef = re + sf$ for some r, s in R. It is easily seen that $e' \subseteq r^c$ and $f' \subseteq s^c$. If both r and s are finite sets, then ef is a finite set. If r is a cofinite set, then e' is a finite set. Similarly if s is a cofinite set, then f' is a finite set. At any rate we have a contradiction. Therefore $Re + Rf$ can not be decomposed into a direct sum of cyclic R-modules.

Lemma 3.2. *Let R be a ring and Q its ring extension with the same identity. If $\{e_i \mid i = 1, 2, \cdots, n\}$ is a set of idempotents of Q, then there exists a set $\{f_i \mid i = 1, 2, \cdots, m\}$ of orthogonal idempotents in Q such that*

$$\sum_{i=1}^{n} Re_i \subseteq \sum_{i=1}^{m} Rf_i \text{ and } \sum_{i=1}^{n} Qe_i = \sum_{i=1}^{m} Qf_i .$$

Proof. We proceed by induction on the number n of elements of $\{e_i \mid i = 1, 2, \cdots, n\}$.

$n = 1$. Obvious.

Assume $n > 1$ and the lemma is true on $n = k-1$.

$n = k$. Let $\{e_i \mid i = 1, 2, \cdots, k\}$ be a set of idempotents of Q. By the induction assumption, there exist orthogonal idempotents $f_1, f_2, \cdots, f_m$ such that

$$\sum_{i=1}^{k-1} Re_i \subseteq \sum_{i=1}^{m} Rf_j \text{ and } \sum_{i=1}^{k-1} Qe_i = \sum_{i=1}^{m} Qf_i.$$

Then we have

$$\sum_{i=1}^{k} Re_i \subseteq \sum_{i=1}^{m} \oplus Rf_i + Re_k$$
$$\subseteq Re_k(1 - \sum_{i=1}^{m} f_i) \oplus \sum_{i=1}^{m} \oplus R(1 - e_k) f_i \oplus \sum_{i=1}^{m} \oplus Re_k f_i$$

and

$$\sum_{i=1}^{k} Qe_i = Qe_k(1 - \sum_{i=1}^{m} f_i) + \sum_{i=1}^{m} Q(1 - e_k) f_i + \sum_{i=1}^{m} Qe_k f_i.$$

Hence the lemma is true on $n=k$, and our proof is complete.

We recall that if R is a semi-prime ring, then R has a unique minimal Baer ring of quotients which is called by Mewborn the Baer hull of R. And it is known in [5, Proposition 2. 5] that the Baer hull of R coincides with the ring generated by R and all idempotents of the maximal ring of quotients of R. Hence it follows from Theorem 2. 1 that a commutative regular ring R is a Baer ring if and only if it is continuous.

Proposition 3.3. *If R is a regular ring, then its Baer hull is a continuous regular ring.*

Proof. It is easily seen from Lemma 3. 2 that Baer hull of R is a regular ring. So, it is a continuous regular ring.

Lemma 3.4. *Let R be a regular ring and Q its maximal ring of quotients. If, for any x in Q, $Rx + R$ satisfies the condition (γ), then Q coincides with the Baer hull of R.*

Proof. First we assume that R is a continuous regular ring and show that $Q=R$. Let x be in Q. Since $R+Rx$ satisfies the condition (γ), by (e) of Theorem 2. 1, $R + Rx$ is isomorphic to a submodule of a projective module. Hence, by [3, Lemma 4], R is a direct summand of $R+Rx$ and it follows that $R + Rx = R$ and $Q = R$.

Next let R be an arbitrary regular ring. Then, by Proposition 3. 3, the Baer hull of R, say P, is a continuous regular ring. We claim that $Q=P$. If $Q \neq P$, then by the above observation, there exists x in Q such that $P + Px$ does not satisfy the condition (γ). But this provides that $R + Rx$ does not satisfy the condition (γ), a contradiction. Thus Q must coincide with P as desired.

Lemma 3.5. *Let R be a semi-prime ring and Q its maximal ring of quotients. Then every finitely generated torsion free R-module can be* *embedded in an external direct sum of finitely generated R-submodules of Q as an essential submodule.*

Proof. See [11, Corollary 5].

Lemma 3.6. *Let R be a semi-prime ring, and Q its maximal ring of quotients. If Q coincides with the Baer hull of R, then every finitely generated torsion free R-module satisfies the condition* (β).

Proof. In view of Lemma 3. 5, it is sufficient to show the statement for finitely generated R-submodules of Q.

First, for a cyclic R-submodule $Rq \subseteq Q$, we show that there exist idempotents e_i, $i=1,2,\cdots,m$, in Q such that $\sum_{i=1}^{m} Re_i$ contains Rq as an essential submodule. Since Q coincides with the Baer hull of R, by Lemm 3. 2, there exist r_i, $i=1,2,\cdots,m$, in R and orthogonal idempotents g_i, $i=1, 2, \cdots, m$, in Q such that $q=\sum_{i=1}^{m} r_i g_i$. Then Rq is an essential submodule of $\sum_{i=1}^{m} Rr_i g_i$. Since Q is a regular ring, it is easily seen that we can find idempotents e_i, $i=1,2,\cdots,m$, in Q such that $Rr_i g_i \subseteq Re_i$ and $Qr_i g_i = Qe_i$, $i=1,2,\cdots,m$. Hence Rq is an essential submodule of $\sum_{i=1}^{m} Re_i$.

Now let $M=\sum_{i=1}^{n} Rq_i$ be any finitely generated R-submodule of Q. For each i, $i=1,2,\cdots,n$, there exist idempotents e_{ij}, $j=1,2,\cdots,n_i$, in Q such that $\sum_{j=1}^{n_i} Re_{ij}$ contains Rq_i as an essential submodule. Hence $\sum_{i=1}^{n}\sum_{j=1}^{n_i} Re_{ij}$ contains $\sum_{i=1}^{n} Rq_i$ as an essential submodule. On the other hand, by Lemma 3. 2, there exist orthogonal idempotents f_i, $i=1,2,\cdots,s$, in Q such that $\sum_{i=1}^{s} \oplus Rf_i$ contains $\sum_{i=1}^{n}\sum_{j=1}^{n_i} Re_{ij}$ as an essential submodule. Consequently $\sum_{i=1}^{s} \oplus Rf_i$ contains $\sum_{i=1}^{n} Rq_i$ as an essential submodule.

By Lemmas 3. 4 and 3. 6, we have

Theorem 3.7. *Let R be a commutative regular ring, and Q its maximal ring of quotients. Then the following conditions are equivalent:*

(a) *Every finitely generated torsion free R-module satisfies the condition* (β).

(b) *Every finitely generated torsion free R-module satisfies the condition* (γ).

(c) *Q coincides with the Baer hull of R.*

Remark 2. (i) Reviewing our observation, the following conditions are also equivalent to the conditions in Theorem 3. 7:

(a′) *For any x in Q, $R+Rx$ satisfies the condition* (β).

(b′) *For any x in Q, $R+Rx$ satisfies the condition* (γ).

However, by Example A, the following condition is not equivalent to the above conditions:

(d) *Every finitely generated torsion free R-module satisfies the condition* (α).

(ii) As is seen from the following example, in order that a regular ring satisfies the conditions of Theorem 3. 7, it need not be self-injective or Boolean.

Example B. Let D be a finite field. Take an infinite index set A, and let

$$Q = \prod_{i \in A} D_i, \ D_i = D \quad \text{for all } i \in A$$

and

$$R = \sum_{i \in A} \oplus D_i + 1 \cdot D,$$

where 1 is the identity of Q. Then R is a regular ring and its maximal ring of quotients is Q. Since D is a finite field, it is easily seen that Q coincides with the ring generated by R and all idempotents of Q, i. e., Q is the Baer hull of R.

The proof of the following lemma is easy.

Lemma 3.9. *Let R be a ring and Q its maximal ring of quotients. For any idempotent e in Q, Re is R-injective if and only if Re is a self-injective ring.*

By Lemmas 3. 6 and 3. 9, we have

Theorem 3.10. *Let R be a commutative semi-prime ring such that its maximal ring of quotients coincides with its Baer hull. Then a torsion free R-module M is finitely generated and injective if and only if $M \simeq R/J_1 \oplus R/J_2 \oplus \cdots \oplus R/J_n$ (as a module), where J_i is an ideal of R such that R/J_i is self-injective for $i = 1, 2, \cdots, n$.*

Remark 3. In case R is a Boolean ring, Theorem 3. 10 was shown by Pierce [6, p. 104] without the assumption that M is torsin free, and he asked if this is valid for an arbitrary regular ring ([6, p. 110]). However we can answer this question in the negative by giving the following example.

Example C. Let D be a finite field, say $D=\{a_1, a_2, \cdots, a_n\}$, and let D' be a proper subfield of D. Take an infinite index set A and let

$$Q=\prod_{i\in A} D_i,\ D_i=D \text{ for all } i\in A.$$

Here we denote by R the subring of Q consisting of all elements such that all but a finite number of whose components belong to D'. Then R is a continuous regular ring and its maximal ring of quotients coincides with Q (cf. [10, Example 3]). Moreover if we denote by x_i the element of Q such that all components are a_i, $i=1, 2, \cdots, n$, then we have $Q=Rx_1+Rx_2+\cdots+Rx_n$. Hence Q is a finitely generated injective R-module, but not a direct sum of cyclic R-modules. Because, if it is a direct sum of cyclic R-modules, then it is projective, since by Theorem 2.1 each component is so. By [3, Lemma 4], this implies that $Q=R$, a contradiction.

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DEPARTMENT OF MATHEMATICS,
YAMAGUCHI UNIVERSITY

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