# ON THE FIXED POINT SETS OF DIFFERENTIABLE $G_2$ ACTIONS ON A EUCLIDEAN SPACE

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Recently, W. C. Hsiang and W. Y. Hsiang [5] investigated the fixed point sets of differentiable actions of compact simple Lie groups on Euclidean spaces and dealt with some cases such that the fixed point sets are non-empty. In this paper we deal with the case of the compact exceptional simple Lie group $G_2$ of rank 2, which was left out in the above.

## 1. Subgroups of $G_2$

Let $G$ be a compact connected Lie group and $H$ a closed connected maximal rank subgroup of $G$. We denote the Weyl groups of $G$ and $H$ by $W(G)$ and $W(H)$ respectively. Then we have $W(G)=N_T/T$ and $W(H)=N_T\cap H/T$ where $T$ is a maximal torus of $H$ and $N_T$ is the normalizer of $T$ in $G$.

**Proposition 1.1.** *Let $g$ be an element of $G$. If $g\in N_T$ and $(gT)\ W(H)\ (gT)^{-1}=W(H)$ then $gHg^{-1}=H$.*

*Proof.* Let $\mathfrak{G}$, $\mathfrak{H}$ and $\mathfrak{T}$ be the Lie algeblas of $G$, $H$ and $T$ respectively and $\mathfrak{G}^c$, $\mathfrak{H}^c$ and $\mathfrak{T}^c$ their complexifications (i. e. $\mathfrak{G}^c=\mathfrak{G}+\sqrt{-1}\mathfrak{G}$ etc.). We denote the sets of non-zero roots of $\mathfrak{G}^c$ and $\mathfrak{H}^c$ with respect to $\mathfrak{T}^c$ by $\Delta$ and $\Delta'$ respectively. For any $\alpha\in\Delta$ we define $H_\alpha\in\mathfrak{T}^c$ by the relation $(H, H_\alpha)=\alpha(H)$ for all $H\in\mathfrak{T}^c$, where the inner product is the Killing form of $\mathfrak{G}^c$. Then, $\sqrt{-1}H_\alpha$, $\alpha\in\Delta'$ generate $\mathfrak{T}$. Let $s_\alpha$ be the reflexion of $\mathfrak{T}$ with respect to the hyperplane orthogonal to $\sqrt{-1}H_\alpha$. Then we can identify the Weyl group $W(G)$ with the group generated by $s_\alpha$, $\alpha\in\Delta$ and similary the Weyl group $W(H)$ also has the same property. Now we consider the automorphism $\mathrm{Ad}_{\mathfrak{G}}(g)$ of $\mathfrak{G}$. By the assumption we have $A=\mathrm{Ad}_{\mathfrak{G}}(g)\,|\,\mathfrak{T}\in W(G)$ and $As_\alpha A^{-1}\in W(H)$, $\alpha\in\Delta'$. Since any reflexion of $W(H)$ has a form of $s_\alpha$, $\alpha\in\Delta'$, there exists $\beta\in\Delta'$ such that $As_\alpha A^{-1}=s_\beta$. Let $H\ (\in\mathfrak{T})$ be orthogonal to $\sqrt{-1}H_\beta$. Then $s_\alpha A^{-1}(H)=A^{-1}(H)$, that is, $A^{-1}(H)$ is orthogonal to $\sqrt{-1}H_\alpha$. Since $A$ is an isometry, this implies that $A(\sqrt{-1}H_\alpha)$ is orthogonal to $H$ and hence $A(\sqrt{-1}H_\alpha)=\sqrt{-1}\ cH_\beta$ for some real number $c$. Thus $\alpha A^{-1}=c\beta$ and since $\alpha A^{-1}$ and $\beta$ are roots in $\Delta$ we have $c=\pm 1$. Then it follows that $H$ is invariant under $\mathrm{Ad}_{\mathfrak{G}}(g)$ and hence $gHg^{-1}=H$. q. e. d.

For the later we note the following

**Proposition 1.2.** *Let $G$, $H$ and $T$ be as above. If $K$ is the normalizer of $W(H)$ in $W(G)$, then $N_H/H \cong K/W(H)$ where $N_H$ is the normalizer of $H$ in $G$.*

*Proof.* Let $\tilde{K}=\{g\in N_T \mid gHg^{-1}=H\}$. There is an exact sequence: $\{1\}\longrightarrow \tilde{K}\cap H \longrightarrow N_H/H \longrightarrow \{1\}$. Hence $N_H/H\cong \tilde{K}/\tilde{K}\cap H \cong (K/T)/W(H)$. Therfore it is sufficient to show that $\tilde{K}/T=K$. Clearly $\tilde{K}/T\subseteq K$. On the orther hand $g\in N_T$, $(gT)W(H)(gT)^{-1}=W(H)$ for any $gT\in$ K. Then $gHg^{-1}=H$ by the proposition 1. 1. Thus $g\in\tilde{K}$, that is $\tilde{K}/T\supseteq K$. q. e. d.

In the remander of this section we denote the exceptional compact Lie group of rank 2 by $G$. The maximal subgroups of maximal rank of $G$ are known to be isomorphic to $SO(4)$ or $SU(3)$ and the subgroups, which are isomorphic, are conjugate [2].

**Proposition 1.3.** *(a) Let $L$ be the normalizer of $SU(3)$ in $G$. Then $L/SU(3)\cong Z_2$. (b) The normalizer of $SO(4)$ coincides with $SO(4)$ itself. (c) Let $H$ be a subgroup of $G$ isomorphic to $SU(2)$. Then the normalizer of $H$ is conjugate to $SO(4)$.*

*Proof.* The Weyl group $W(G)$ is a dyhedral group of order 12, and $W(SU(3))$ and $W(SO(4))$ are isomorphic to the permutation group $S_3$ of 3 letters and $Z_2\oplus Z_2$ respectively. Then $W(SU(3))$ is a normal subgrnup of $W(G)$ and the normalizer of $W(SO(4))$ in $W(G)$ coinsides with $W(SO(4))$ itself. Hence (*a*) and (*b*) follow from the proposition 1. 2.

Now we consider the case (*c*). Let $a$ $(\neq 1)$ be the element of the center of $H$. Then $a^2=1$. On the orther hand the elements of order 2 in $G$ are conjugate. In fact an element of order 2 is contained in a torus $T$ $(SU(3))$ and the elements of order 2 in $SU(3)$ are conjugate. Then, since subgroups which are isomorphic to $SU(3)$ are conjugate in $G$, it follows that the elements of order 2 in $G$ are conjugate. Let $K$ be the normalizer of $a$. Then clearly $H\subseteq K$ and $K\neq G$, since $G$ has no center. The center of $SO(4)$ is $Z_2\oplus Z_2$ and hence $K$ contains a subgroup isomorphic to $SO(4)$. Then it is clear that $H$ is a normal subgroup of $K$, since $H$ is isomorphic to $SU(2)$. Hence the normalizer of $H$ in $G$ is conjugate to $SO(4)$. q. e. d.

**Remark.** It is easy to see that the subgroups of $G$ isomorphic to $SU(2)$ are conjugate.

## 2. A property of a differentiable action

For the later we prove a theorem with respect to a differentiable action of a compact Lie group $G$ of which a principal isotropy subgroup is a maximal torus of $G$.

**Proposition 2.1.** *Let $G$ be a compact Lie group, and $\varphi$ a real representation of $G$. If a principal isotropy subgroup of $\varphi$ is a maximal torus of $G$ and there is no exceptional orbit, then $G$ is connected.*

*Proof.* Let $G^0$ be the connected component of the identity of $G$. Then $\varphi \mid G^0 = \mathrm{Ad}_{G^0} \oplus$ trivial part [4]. Let $V$ be a representation space of $\varphi$ and $T$ a maximal torus of $G$. We denote the set $V-$(the singular set) by $V_0$. Since $V_0$ is the principal orbit bundle of $\varphi \mid G^0$ it is easily seen that $V_0$ is $G^0$-equivariantly homeomorphic to $G^0/T \times V_0/G^0$. $G/G^0$ acts naturally on $V_0/G^0$. Let $gG^0 \in G/G^0$ be a prime order element. By the theorem of P. A. Smith in [1], $gG^0$ has a fixed point in $V_0/G^0$, since $V_0/G^0$ is homemorphic to a Weyl chamber and hence $V_0/G^0$ is contractible. Hence there is a point $x \in V_0$ such that $gG^0x = G^0x$. Now by taking an element $g_0 \in G^0$ satisfing $gx = g_0x$ we have $g_0^{-1}g \in G_x$. Since $G_x$ is a maximal torus of $G$ by the assumption we get $g \in G^0$. This is a contradiction. q. e. d.

**Theorem 2.2.** *Let $\phi$ be a differentivble action of a compact connected Lie group $G$ on a simply connected differentiable manifold $M$. If the connected component of the identity of a principal isotropy subgrop is a maximal torus of $G$, then the isotropy subgroups of $\phi$ are connected.*

*Proof.* Let $T$ be a maximal torus of $G$, $A = F(T, M)$[1] and $M_0 = M-$(the singular set). Then the Weyl group $W(G)$ acts on $A$. We easily see $M_0 = G/T \times_{W(G)} (A \cap M_0)$ and $\pi_1(M_0) = 0$, since the singular set has at least codimension 3. By the homotopy exact sequence of a fiblation: $A \cap M_0 \longrightarrow M_0 \longrightarrow G/N_T$ we know that the number of the components of $A \cap M_0$ is equal to order of $W(G)$. Then, since $M_0$ is connected, $W(G)$ acts simply transitively on the components of $A \cap M_0$. Hence $\phi$ has no exceptional orbit and a pricipal isotropy subgroup is a maximal torus of $G$. Let $x \in M$. Then the slice representation at $x$ of $G_x$ satisfies the assumption of the proposition

1) $F(T, M)$ is the set of fixed points of $T$ in $M$.

2. 1 and hence $G_x$ is connected. q. e. d.

### 3. Weight systems

Let $\phi$ be a differentiable $G$ action on a Euclidean space and $T$ be a maximal torus of $G$. Then by the theorem of P. A. Smith in [1] the local representation of $T$ at a fixed point of $T$ is well defined. The weight system of the local representation of $T$ is defined to be the weight system of $\phi$ and denoted by $\Sigma\phi$. We see in [5] that for each simple Lie group, weight systems of actions with a principal isotropy subgroup of a positive dimension are classified and also the fixed point sets are determined with few exceptions.

Now on we consider the case where $G$ is the exceptional compact simple Lie group of rank 2. Then the non-zero root system of $G$ is given by

$$\Delta'(G)=\{\pm\theta_i,\ \pm(\theta_i-\theta_j),\ i<j,\ i,j=1,2 \text{ and } 3\}.$$

Let $\phi$ be a differentiable $G$ action on a Euclidean space $E^m$ with a principal isotropy subgroup $H_\phi$ of positive dimension. Then it is known by [5] that

(1) $\Sigma'(\phi)=\Delta'(G)$ and $H_\phi^0=$ a maximal torus of $G$,

(2) $\Sigma'(\phi)=\{\pm\theta_i,\ i=1,2$ *and* $3\}$ and $H_\phi^0=SU(3)$ or

(3) $\Sigma'(\phi)=\{\pm\theta_i,\ i=1,2$ *and* $3$: *each weight has the multiplicity* $2\}$ and $H_\phi^0=SU(2)$.

In the following sections we investigate the fixed point set for each of those cases.

### 4. Fixed point sets

First we consider the case where $H_\phi^0$ is a maximal torus of $G$.

**Propositon 4.1.** *$F(G, E^m)$ is $Z_p$—acyclic for $p=2$ and 3.*

*Proof.* We suppose $F(G, E^m)$ is empty and then show that we arrive at a contradiction. Since the isotropy subgroups of $\phi$ are connected by the theorem 2. 1, the possible isotropy subgroups are maximal tori and subgroups which are isomorphic to $U(2)$, $SO(4)$ or $SU(3)$. Let $T$ be a maximal torus of $G$ and $A=F(T, E^m)$. Then the Weyl group $W(G)$ acts on $A$ and $W(G)$ is a group of order 12 defined by the relations: $t^6=1$, $s^2=1$ and $sts=t^{-1}$. Since $W(G)_a=W(G_a)$ for any $a\in A$, the possible isotropy subgroups of $W(G)$-action on $A$ are isomorphic to 1, $Z_2$, $Z_2\oplus Z_2$ or $S_3$. Because $A$ is

$Z$-acyclic, we see by the theorem of P. A. Smith that they are exactly isotropy subgroups. Let $C=\{a\in A \mid W(G)_a \cong Z_2\oplus Z_2\}$. Then if $c\in C$, $G_c\cong SO(4)$ and the slice representation at $c$ is $\mathrm{Ad}_{G_c}\oplus(m-14)\,\theta$, where $\theta$ is a trivial 1-dimensional representation of $G_c$ [4]. Hence $C$ is a submanifold of $A$ of codimension 2. On the orth·r hand there are three subgroups isomorphic to $Z_2\oplus Z_2$ and generated by $\{s, st^3\}$, $\{st, st^4\}$ and $\{st^2, st^5\}$ respectively. Morever they are conjugate. Therefore we see that $C$ is the disjoint union of three $Z_2$-acyclic submanifolds of the same dimension. Also $C$ is the fixed point set of $t^3$ and hence $Z_2$-acyclic. This contradicts the above. Thus we see $F(G, E^m)$ is non-empty.

Now let us take the subgroup $Z_3$ of $W(G)$ and consider a $Z_3$-acyclic submanifold $F(Z_3, A)$. Then the possible isotropy subgroups on $F(Z_3, A)$ are $SU(3)$ and $G$. Considering the slice representation of $G$ we see that the fixed point set of $G$ is open and closed in $F(Z_3, A)$. Thus, because of the connectedness of $F(Z_3, A)$, we see that $SU(3)$ is not an isotropy subgroup. Hence $F(G, E^m)=F(Z_3, A)$. Similarly $SO(4)$ is not an isotropy subgroup and hence we have $F(G, E^m)=F(Z_2\oplus Z_2, A)$ which is $Z_2$-acyclic. q. e. d.

Next we consider the case where $H_\phi^0$ is isomorphic to $SU(3)$. Then $\phi$ has a fixed point of $G$, since, if not, $\phi$ has the uniform dimensional orbits, but this is impossible [3]. Let $E_0=E^m-$(the fixed set of $G$) and $F=F(SU(3), E^m)\cap E_0$. Then we see $E_0=G/SU(3)\times_{L/SU(3)}F$, where $L$ is the normalizer of $SU(3)$ and $L/SU(3)\cong Z_2$ by the proposition 1. 3. $E_0$ admits a fibering $F\longrightarrow E_0\longrightarrow G/L$. It is well known that $G/SU(3)=S^6$ and hence $G/L=P^6$ (real projective space). Then from the homotopy exact sequence of the fibering we know that $F$ has 2 connected cnmponents, since $E_0$ is simply connected. Thus $Z_2$ acts simply transitively on the components of $F$ and hence there is no exceptional orbit. Then as in the proof of the proposition 4. 1 we have the following

**Proposition 4.2.** *$F(G, E^m)$ is $Z_2$-acyclic.*

## 5. The case $H_\phi^0=SU(2)$

Let $G_x$, $x\in E^m$ be an isotropy subgroup of the rank 2. Then the set of the complementary weights of $G_x$ i. e. $\Delta'(G)-\Delta'(G_x)$ is contained in $\Sigma'(\phi)$. Thus $\Delta'(G)-\Delta'(G_x)\subseteqq\{\pm\theta_i: i=1, 2 \text{ and } 3\}$ and hence $\Delta'(G_x)\supseteqq\{\pm(\theta_i-\theta_j): i<j\}$. Hence $G_x$ has at least dimension 8. Therefore the possible isotropy subgroups of rank 2 of $\phi$ are $G$, $L$(=the normalizer of $SU(3)$ and $SU(3)$).

Let $T$ be a maximal torus of $SU(3)\subset G$. Then we have

**Proposition 5.1.** *$F(SU(3), E^m)$ is identical with $F(T, E^m)$.*

*Proof.* It is sufficient to show that $G_a\supseteqq SU(3)$ for any $a\in F(T.E^m)$. It is clear if $G_a=G$. Hence we may suppose that $G_a^0=gSU(3)g^{-1}$ for some $g\in G$. Since $G_a^0\supseteqq T$ and $W(SU(3))$ is normal in $W(G)$ we can assume that $g\in N_T$ and $W(G_a^0)$ then equals $(gT)\ W(SU(3))\ (gT)^{-1}$. Hence by the proposition 1.2 we have $G_a^0=SU(3)$.

From now on we assume $F(G, E^m)$ is empty and show that we then arrive at a contradiction. Let us denote the singular set by $E_s$ and put $E_0=E^m-E_s$. Then we have $E_s=G/SU(3)\times_{Z_2}F(T, E^m)$ by the proposition 5.1. Now we prove the following

**Proposition 5.2.** *$H_c^{p+m-12}(E_s;\ Z_2)=Z_2$ if $0\leqq p\leqq 6$, and 0 ortherwise*[2].

*Proof.* Since $E_s$ admits a fibering $F(T, E^m)\longrightarrow E_s\longrightarrow G/L=P^6$, there is a spectral sequence which converges to $H_c^*(E_s;\ Z_2)$ and whose $E_2$ terms are $E_2^{p,q}(=H_c^p(P^6;\ H_c^q(F(T, E^m);\ Z_2)))$. $F(T, E^m)$ is an $(m-12)$ dimensional acyclic manifold and hence $H_c^q(F(T, E^m);\ Z_2)=0$ if $q\neq m-12$, and $Z_2$ if $q=m-12$. Thus $E_2^{p,m-12}=Z_2$ for $0\leqq p\leqq 6$ and otherwise $E_2^{p,q}=0$. This proves the proposition. q. e. d.

Then, by the exact sequence of the pair $(E^m,\ E_s)$

$$\cdots\longrightarrow H_c^i(E^m;\ Z_2)\longrightarrow H_c^i(E_s;\ Z_2)\longrightarrow H_c^{i+1}(E_0;\ Z_2)\longrightarrow\cdots$$

we have

**Proposition 5.3.** *$H_c^i(E_0;\ Z_2)=Z_2$ if $i=m$ or $m-11\leqq i\leqq m-5$, and 0 ortherwise.*

Let us put $A=F(SU(2),\ E^m)$ where $SU(2)$ is considered as a subgroup of $SU(3)$. Then we have the following

**Proposition 5.4.** *$A\cap E_s=S^1\times_{Z_2}F(T,\ E^m)$, where $Z_2$ acts on $S^1$ antipodally.*

*Proof.* Since $E_s=G/SU(3)\times_{Z_2}F(T,\ E^m)$ it is clear that $A\cap E_s=F(SU(2),\ G/SU(3))\times_{Z_2}F(T, E^m)$. $G/SU(3)=S^6$ and $G$ acts orthogonally on $S^6$. Then, since the isotropy representation of $SU(3)$ is the standard representation $\mu_3$ of $SU(3)$, it follows that $F(SU(2),\ G/SU(3))=S^1$.

2) We use the Alexander-Spanier cohomology with compact supports.

Let $S$ be a torus of $SU(2)$. Then $A \subseteqq F(S, E^m)$. Since the local representation of $SU(3)$ at a fixed point of $SU(3)$ is $2(\mu_3)_R \oplus (m-12)$ we have $\dim A = \dim F(S, E^m) = m-8$. Thus $\bar{A} = F(S, E^m)$ since $F(S, E^m)$ is connected. Hence $A$ is acyclic.

Let $A_0 = A - E_s$. Then we have the following

**Proposition 5.5.** *$H_c^i(A_0; Z_2) = Z_2$ if $i = m-8$, $m-10$ or $m-11$ and 0 ortherwise.*

*Proof.* As in the proposition 5.2 we have $H_c^{p+m-12}(A \cap E_s; Z_2) = Z_2$ for $p=0$ or 1. and 0 otherwise. Hence by the exact sequence of the pair $(A, A \cap E_s)$ we get the propositicn. q. e. d.

Now we show that our assumption i. e. $F(G, E_m) \neq \phi$ leads to a contradiction. Let $N$ be the normalizer of $SU(2)$. Then we have $E_0 = G/SU(2) \times_{N/SU(2)} A_0$ and hence $E_0$ admits a fibering $A_0 \longrightarrow E_0 \longrightarrow G/N$. Now consider the spectral sequence of the fibering whose $E_2$ terms are $E_2^{p,q} = H_c^p(G/N; H_c^q(A_0; Z_2))$ and that converges to $H_c^{p+q}(E_0; Z_2)$. Since $N$ is isomorphic to $SO(4)$ by the propcsition 1. 3 it is known that the Poincaré polynomial of mod 2 of $G/N$ is $1+t^2+t^3+t^4+t^5+t^6+t^8$. Let us denote $s = \max\{q \mid H_c^q(A_0; Z_2) \neq 0\}$. Then $E_2^{8,s} = H_c^8(G/N; H_c^s(A_0; Z_2)) \cong H_c^s(A_0; Z_2) \neq 0$. We have $H_c^s(A_0; Z_2) \cong H_c^{8+s}(E_0; Z_2)$, since $E_2^{8,s} (\cong E_3^{8,s} \cong \cdots \cong E_\infty^{8,s})$ is the only non-zero term in degree 8+s. Since 8+s is the highest dimension with non zero cohomology, it must be $8+s=m$ by the proposition 5. 3 and $H_c^{m-8}(\bar{A}_0; Z_2) = Z_2$. Now we consider the differential $d_2$: $E_2^{6,s} \longrightarrow E_2^{8,s-1}$. If $E_2^{8,s-1} = 0$ then $E_2^{6,s} \cong E_3^{6,s} \cong \cdots \cong E_\infty^{6,s} \cong H_c^s(A_0; Z_2) \neq 0$ and this implies that $H_c^{m-2}(E_0; Z_2) \neq 0$. By the proposition 5. 4 this is impossible and hence $E_2^{8,s-1} \neq 0$. Hence we have $H_c^{m-9}(A_0; Z_2) \neq 0$, which contradicts the proposition 5. 5. We see from this that $F(G, E^m)$ is not empty.

Nexst we prove the following

**Proposition 5.6.** *$F(G, E^m)$ is $Z_2$-acyclic.*

*Proof.* Let $T$ be a maximal torus of $G$ and $F = F(T, E^m)$. Then the Weyl group $W(G)$ acts on $F$. Take a subgroup $Z_2 \oplus Z_2$ of $W(G)$ and consider $F(Z_2 \oplus Z_2, F)$. Then the pcssible isotropy subgroups of $\phi$ on $F(Z_2 \oplus Z_2, F)$ are $G$ and the normalizer $L$ of $SU(3)$. Considering the slice representation of $G$, it is easily known that $L$ is not an isotropy subgroup. Hence $F(G, E^m) = F(Z_2 \oplus Z_2, F)$ and $Z_2$-acyclic. q. e. d.

We thus get our main theorem by summarizing the above as follows

**Theorem 5.7.** *Let $G$ be the exceptional compact Lie group of rank 2 and $\phi$ a differentiable $G$ action on a Euclidean space with a pricipal isotropy subgroup $H_\phi$ of a positive dimension. Then, if $H_\phi^0$ is a maximal torus of $G$, the fixed point set of $G$ is $Z_p$-acyclic for $p=2$ and 3 and, if $H_\phi^0$ is $SU(3)$ or $SU(2)$, the fixed point set of $G$ is $Z_2$-acyclic.*

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